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Functional Sunspot Equilibria

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M E X I C O

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Abstract

Consider a one step forward looking model where agents have beliefs about the law of motion of the system, and their forecast for the next period is made with knowledge of the past values of the state variable but not the current value, and is allowed to depend on the current realization of an extrinsic random process. The paper provides (and characterizes) the conditions for the existence of sunspot equilibria for the model described.

KEYWORDS: extrinsic uncertainty, stochastic equilibria

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1. Introduction

This paper is about the existence of sunspot equilibria, i.e., stochastic equilibria driven purely by extrinsic uncertainty. We specify a class of stochastic equilibria that we name *functional sunspot equilibria* (henceforth, FSE); these are self-fulfilling equilibria in which expectation formation is assumed to be determined by functions that agents believe map the value of the state variable at a date to its value at the subsequent date.[3] One of a finite number of functions is chosen depending on the realization of a finite state stationary Markov process that models extrinsic uncertainty, and, in every Markov state, the actual function linking the state variable across two successive periods is required to coincide with the agents' believed function.[4] We show that this formulation, whereby agents form forecasts based on a systematic relationship between past and present prices with the proviso that their beliefs about the functioning of the economy may be affected by extrinsic uncertainty, has important implications for the existence of equilbria. FSE turn out to be the stochastic counterparts of nonstationary perfect foresight paths; they exist quite generally and provide a simple route to very rich stochastic dynamic behaviour with a parsimonious parameterization. Evidently, our work adds to the scope for multiplicity of rational expectations equilibria already identified in the literature.

Much of the existing literature on sunspot equilibria in dynamic economic models provides results on a narrow class of equilibria that have been called finite state stationary sunspot equilibria (henceforth, FSSE). These are equilibria in which the agents' belief about the value of the state variable is a function of only the current realization of the extrinsic state where the extrinsic state is assumed to follow a finite state stationary Markov process. Most studies on FSSE are in the framework of the basic Overlapping Generations model (henceforth, OLG) with two period lived agents, one consumption good and a constant stock of money (see, e.g., Azariadis (1981), Azariadis and Guesnerie (1986), Grandmont (1986), Guesnerie (1986) and Peck (1988)). It is well known that the existence of FSSE is intimately tied to the existence of a certain kind of invariant set in the perfect foresight dynamics of the deterministic model, a condition that can be related to the indeterminacy of the steady state; in particular, in the basic OLG model, FSSE do not exist when demand functions have the gross substitutes property.

Evidently, one can think of more general definitions of sunspot equilibrium. Woodford (1986) studies sunspot equilibria that are stationary stochastic processes

[3] The sunspot equilibria for a linear OLG model discussed in Shell (1977) had a similar feature.

[4] So FSE are stationary Markov equilibria.

and, as an application, he shows that they exist in the multigood OLG model where FSSE exist only in nongeneric cases, while Woodford (1994) considers equilibria that need not even be stationary.[5] Such analyses have had less impact when compared to FSSE probably because of the simplicity of FSSE. A different equilibrium notion has been considered by Chiappori and Guesnerie (1993) under the name "random walk" equilibria. These are equilibria in which the state variable follows a random walk on a countable state space and the limits of the trajectories, taken with respect to the index of the elements of the state space, are steady states of the model. Most of their analysis is geared towards the basic OLG model and they show that such equilibria can exist in the gross substitutes case. Again, random walk equilibria can be quite complicated; furthermore, they require the existence of two or more steady states.

In an FSE, the agents' belief at date t about the value of the state variable at date t is determined by both the current realization of the sunspot state (denoted s_t) and by the value of the state variable at date $t-1$ (denoted x_{t-1}), i.e., $x_t^e = h(x_{t-1}; s_t)$. These beliefs are extrapolated one step forward to predict x_{t+1} which, in conjunction with the market clearing conditions, determine x_t, the market clearing value of the state variable at date t. That expectations be self-fulfilling requires that, for each state s, the induced market clearing value x_t equal $h(x_{t-1}; s)$.

Evidently, the building block for FSE are belief functions that are "backward looking." Our motivation for such a specifiaction is two fold. Firstly, the formulation we consider arises naturally in markets where agents are required to act (place demands) before the actual prices at which trade is carried out is available and where the market clearing price is determined on the basis of the expressed demands, e.g., trading in mutual funds provides an instance where the actual trade often takes place at prices quoted at the end of the trading day after the agents have placed their demands. The simplest way to close such a model is to postulate that agents beliefs about the functioning of the economy are summarized by a functional relationship between the past and the current prices, i.e., a backward looking belief function, which is then extrapolated one period ahead to generate forecasts; also, this is

[5] A important strand in the literature started with Woodford (1986) and focuses on the relation between determinacy of the steady state and the existence of Markovian stationary sunspot equilibria without the restriction that the support be finite, i.e., a generalization of FSSE, in general models with many goods with predetermined variables; Bloise (2003) is a recent example of such work.

In the case of the basic OLG model, a generalization of FSSE to Markov processes with a compact state space and strictly positive and continuous transitions in the OLG model is available in Grandmont (1986); the nature of the existence result, however, remains the same as in the finite state case.

the route followed in earlier work that studies the issue of convergence to rational expectations equilibria in such models.[6] The second motivation is somewhat more conceptual and stems from the manner in which one chooses to impose the axiom of self-fulfilling expectations on the economy. In principle, one may impose the axiom of self-fulfilling expectations directly on sequences of forecasts without necessarily specifying where these forecasts come from; FSSE is a case in point where the consistency condition one imposes on the agents' beliefs x_s is that it clears the market in period t, where s is the state in period t, and the story accompanying the formation of beliefs is: Before going to the market at date t, agents observe s_t, the sunspot realization at date t, and predict the next period's price distribution conditional on this observation. Our formulation is conceptually very different since it looks at cases where these forecasts can be attributed to a "theory" that agents have about the functioning of the economy, and then requires the theory to be self-fulfilling. The gain from such a method is that one can then understand the axiom of self-fulfilling expectations in terms of a correct theory that agents have about the process generating the dynamics of the economy. We show that this rudimentary way of incorporating the fact that agents form forecasts based on systematic relationship between the past and present has important implications when one considers the possibility that these theories might be affected by extrinsic uncertainty.

We concentrate on the specific case of nonlinear one step forward looking self-referential models with a one dimensional state variable, a specification which includes the basic OLG model. We provide an existence result for FSE after which

[6] The postulate that belief functions be "backward looking" is essential to obtaining well defined sequences of temporary equilibria with self-fulfilling expectations in situations where the agents use a functional representation to form expectations. In order to obtain a unique temporary equilibrium at date t, it is imperative to "tie down" the forecast x^e_{t+1} and a well documented feature of temporary equilibrium models shows that the only way to do so is to look backwards and specify x^e_{t+1} as a function of the past of the economy. We recall an argument from Grandmont and Laroque (1991) to illustrate this point. Assume that forecasts are point forecasts, that there is no uncertainty, and that the market clearing relation is given by $T(x_t, x^e_{t+1}) = 0$ with $T(\bar{x}, \bar{x}) = 0$. Under the regularity condition that the partial derivative of T with respect to the first argument evaluated at $\bar{x}$ does not vanish, one can find a belief function $h(x)$ that is well defined and unique around $\bar{x}$ and which satisfies $T(h(x), x) = 0$ for *all* x in some neighborhood of $\bar{x}$. Hence, it is impossible to define a unique temporary equilibrium fully determined by the current value of the state variable x_t since, at date t, each x in an open set can serve as an temporary equilibrium with self-fulfilling expectations. So it is natural to set $x^e_{t+1} = h(h(x_{t-1}))$, as in models of learning, e.g., Marcet and Sargent (1989), Grandmont and Laroque (1991), obtaining thereby a unique temporary equilibrium at date t, since x_{t-1} is a predetermined variable, and generate well defined sequences of temporary equilibria with self-fulfilling expectations. The problem outlined above does not show up in the case of FSSE since there the equilibrium notion associates to each state s a unique constant x_s thus tying down x^e_{t+1} via $x_{s(t)}$. Since we consider more general state dependent self-fulfilling functions, we run into the difficulty sketched above unless the belief functions are "backward looking".

we examine them in greater detail in the canonical model in which, traditionally, sunspot equilibria have been analyzed—the basic OLG model. We find that a necessary and sufficient condition for the existence of FSE is that the Markov transition matrix be singular and that there exist a set of values for the state variable in which the perfect foresight dynamics are well defined and the set is invariant in the "forward" dynamics.[7] When the existence result is applied to the OLG case it shows that FSE exist for *all* parameter configurations; in particular, we obtain existence in the gross substitutes case where, as noted earlier, FSSE do not exist. We also obtain FSE when a critical cycle of period two exists, a case where FSSE need not exist.[8] For pedagogical reasons we also study FSE in a linear formulation which, because of its simplicity, gives insight into the nature of the more general problem (FSSE have also been studied in a linear framework by, e.g., Chiappori, Geoffard and Guesnerie (1991)). Of course, in the linear case too we find that FSE exist for *all* parameter configurations, in particular, for configurations under which the perfect foresight dynamics diverge from the steady state so that the steady state is determinate, a case where FSSE do not exist. We stress, however, that in those cases where FSE exist but FSSE do not, e.g., the gross substitutes case of the basic OLG model, we are unable to show the existence of permanent oscillations with a uniform lower bound on the amplitude, i.e., the FSE trajectories are random but appear to converge to a constant value with probability one.

In comparing FSE and FSSE, note that the latter appear as special cases of our formulation where the state dependent belief functions reduce to statewise constants; naturally, conditions under which FSSE exist are more demanding than those for FSE. FSE exist much more generally and display substantially richer stochastic dynamic behaviour and, in the absence of extrinsic uncertainty, FSE coincide with possibly nonstationary perfect foresight solutions of the model (while FSSE coincide with stationary solutions). Also, since the indeterminacy of a steady state implies the existence of an invariant set in the forward perfect foresight dynamics, our analysis provides a direct route to the conjecture that if the steady state is indeterminate then simple sunspot equilibria do exist even though FSSE need not exist. In effect, we provide a generalization of FSSE and, as a consequence, we show that

[7] The relevance of some kind of invariance property for the perfect foresight dynamics for the existence of sunspot equilibria is a common feature and has appeared in the literture. A well known sufficient condition for the existence of FSSE and its generalization to an arbitrary number of states is that the invariant set include the steady state in its relative interior, a property that is guaranteed when the steady state is indeterminate.

[8] See Azariadis and Guesnerie (1986) or Grandmont (1986). The existence of a critical cycle of period two is a nongeneric phenomenon.

the problem of multiplicity of rational expectations equilibria may be considerably more severe than believed based on the analysis of FSSE.

In the OLG framework, FSE can be thought of as a simpler way of obtaining the behaviour that Chiappori and Guesnerie (1993) induced through their random walk equilibria though the two existence results are quite different at first sight.[9] Also, in work done independently of the research reported in this paper, Magill and Quinzii (2003) propose an equilibrium concept that, in the special case of extrinsic uncertainty, coincides with FSE.[10] Our focus is different from theirs and so is our existence result which brings out the importance of the singularity of the transition matrix when uncertainty is extrinsic; when comparing our results with the analysis in Magill and Quinzii (2003), one can interpret our result on the role of the singularity of the transition matrix as yet another way in which sunspot equilibria are singular cases.

We remark that an infinite time horizon is essential to obtaining FSE since the dynamics of the state variable inherently depend on the past value of the state variable through functions that are state dependent and self-fulfilling; it follows that the sunspot equilibria in the canonical two period model (Cass and Shell 1983) cannot be obtained as FSE. We mention two issues that require further investigation. The welfare properties of FSSE are easily studied and it is known that they are ex-ante inefficient while their optimality properties under a weaker (conditional) notion of optimality is easily checked using the "unit root property" due to Aiyagari and Peled (1991); in the case of FSE, while it is easy to show that they too are ex-ante inefficient, their welfare properties under a conditional notion of optimality are not clear since they induce nonstationary paths and an answer requires the use of the criterion for optimality developed by Chattopadhyay and Gottardi (1999) and poses an interesting challenge. Also, the precise relation between FSE and the more general class of sunspot equilibria studied by Woodford (1986) and Woodford

[9] In addition to requiring that there be two steady states, the existence of random walk equilibria also imposes conditions on the transition probabilities of a random walk money supply where money transfers are proportional. Chiappori and Guesnerie (1993) do not discuss the possibility of obtaining random walk equilibria with a deterministic money supply and "sunspot" induced random prices. This should be possible but it is not obvious.

[10] They consider a one good OLG model with stochastic endowments and the gross substitutes case. They argue in favour of equilibria built around a stationary expectation function and derive the invariance requirement as a necessary condition. They show that such equilibria exist in large numbers and they provide a condition under which all trajectories converge to one or the other strongly stationary equilibrium which is the analogue of the two steady states in the deterministic case. They note that their equilibrium concept also applies to the case of extrinsic uncertainty proving existence even though FSSE do not exist in their specification due to the gross substitutes property.

(1994) and the random walk equilibria studied by Chiappori and Guesnerie (1993) pose interesting questions.

Section 2 of the paper introduces FSE, provides an existence result, and relates FSE to SSE and other equilibrium concepts of the sunspot type. Section 3 applies the analysis to the OLG setting and to a linear model. Section 4 concludes the paper.

2.1 Functional Sunspot Equilibria

Consider a deterministic economic model where the state variable is one dimensional. x_t, the equilibrium value of the state variable at date t, is determined given x^e_{t+1}, the value of the state variable expected to prevail the next period, according to a functional relationship:

$$V(x^e_{t+1}) = U(x_t), \tag{1}$$

where $x_t \in X$, the state space with $X \subset R$, and $x^e_{t+1} \in R$. Let $\mathcal{H}(x)$ denote the set of values of x^e that solve (1) for each value of x; in words, $\mathcal{H}(x)$ is the set of beliefs that rationalize the choice x. As x varies in X, a correspondence $\mathcal{H}$ is generated.

In order to proceed further, we need to specify a rule by which x^e_{t+1} is determined. Let us assume that the information structure that agents are assumed to have access to does not contain the current value of the state variable. Hence, the forecasting scheme employed by agents cannot depend on the current value of the state variable. As discussed in the introduction, this leads us naturally to the use of a forecasting process that is "backward looking" as in earlier work that studies the issue of convergence to rational expectations equilibria in such models.[11]

So suppose that agents believe that the law of motion for the state variable takes the form $x_{t+1} = h(x_t)$ for $x_t \in D$ where $D \subset X$. In our framework, this induces beliefs about future values of the state variable according to $x^e_{t+1} := h(x_t)$ for $x_t \in D$. Since x_t is not known when the forecast is made, one has

$$x^e_{t+1} = h(h(x_{t-1})). \tag{2}$$

We would like the beliefs of the agents to be "consistent", i.e., correctly specified. Since the agent uses (2) to form her forecast, and the true value of x_t is determined according to (1), the consistency condition is modelled by requiring that the believed law of motion be validated, that is, $x_t = h(x_{t-1})$ for $x_{t-1} \in D$, so that

[11] See, e.g., Marcet and Sargent (1989) for a linear model. There the parameter β in the equation $x_t = \beta x_{t-1}$ is estimated at date t using values of x observed upto and including date $t-1$ so that the estimate is independent of the x_t value that it determines. This simplifies considerably the dynamic law of the model and stems from a feature of temporary equilibrium models alluded to in the introduction, namely, that existence of equilibria and the analysis of dynamics is rather problematic when current values of the state variable affect the forecasts.

$V(h(h(x_{t-1}))) = U(h(x_{t-1}))$ for all $x_{t-1} \in D$.

We say that the forecasting function h is *self-fulfilling* on D if

$V(h(h(x))) = U(h(x))$ for all $x \in D$.

Evidently, the requirement that h be self-fulfilling is met if $h(x) \in D$ for all $x \in D$ and h is a selection from $\mathcal{H}$. Conversely, if h is self-fulfilling then the fact that $U(h(x))$ must be well defined implies that $h(x) \in X$ for all $x \in D$, while the fact that $V(h(h(x)))$ is well defined implies that $h(h(x))$ is well defined so that $h(x) \in D$ must also hold, and, in addition, h must be a selection from $\mathcal{H}$.

We say that D is *invariant* for h if $x \in D \;\Rightarrow\; h(x) \in D$.

We have shown that h is self-fulfilling on D if and only if D and h are such that

$$x \in D \;\Rightarrow\; h(x) \in D \quad \text{and} \quad V(h(x)) = U(x) \text{ for all } x \in D, \tag{3}$$

so that D is invariant for h and h is a selection from $\mathcal{H}$.

With the groundwork in the deterministic case behind us, we can turn to the subject matter of this paper, the stochastic case. We wish to permit the possibility that agents believe that the realizations of a random process affects the law of motion of the economy. Specifically, we assume that agents observe a finite N state Markov Chain taking values μ_s, $s = 1, \cdots, N$, with transition matrix Π (with typical entry π_{ij} the probability of transiting to state j next period conditional on being in state i in the current period). Without loss of generality, we set $\mu_s = s$.

Consider the information structure wherein, at date t, agents have information upto date $t-1$ on the state variable and *also know* the realization of the sunspot at date t, denoted s_t. They believe that the law of motion for the state variable takes the form $x_{t+1} = h(x_t; s_{t+1})$ so that the realization s_{t+1} affects the function taking x_t to x_{t+1}. The beliefs about future values of the state variable are now induced according to $x^e_{t+1} = h(x_t; s_{t+1})$ for $x_t \in D$. The method for forecasting is the natural extension of the deterministic case as described in (2): At date t, agents forecast x_{t+1} *before* they observe x_t and s_{t+1} and so the forecast is given by a random variable conditional on x_{t-1} and s_t

$$x^e_{t+1} = h(h(x_{t-1}; s_t); s_{t+1}) \text{ for } x_t \in D. \tag{4}$$

We wish to impose a natural consistency requirement on the beliefs of agents, namely, that the state dependent belief functions be self-fulfilling. The requirement is that when (4) is used to form forecasts, and when x_t is determined by solving

$$\sum_{s_{t+1}=1}^{N} \pi_{s_t s_{t+1}} V(h(h(x_{t-1}; s_t); s_{t+1})) = U(x_t),$$

the stochastic analogue of (1), then $x_{t+1} = h(x_t; s_{t+1})$ is validated for all $x_t \in D$ and every state $s = 1, \cdots, N$.

This leads us to say that the forecasting functions $h(\cdot; s)$, $s = 1, \cdots, N$, are *self-fulfilling* on D in the stochastic case if

$\sum_{j=1}^{N} \pi_{sj} V(h(h(x;s);j)) = U(h(x;s))$, for all $x \in D$ and for all $s = 1, \cdots, N$.

As in the deterministic case, the functions $h(\cdot;s)$, $s = 1, \cdots, N$, are self-fulfilling on D in the stochastic case if and only if

$$x \in D \;\Rightarrow\; h(x;s) \in D \;\text{ and }\; \sum_{j=1}^{N} \pi_{sj} V(h(x;j)) = U(x), \qquad s = 1, \cdots, N. \quad (5)$$

It is obvious that (5) can have trivial solutions in which uncertainty plays no role. We say that a set of forecasting functions $h(x;s)$, $s = 1, \cdots, N$, leads to a *nondegenerate solution* if

for some $x \in D$ and some $s = 1, \cdots, N$, $\quad V(h(x;s)) - U(x) \neq 0$.

We define a *finite state functional sunspot equilibrium (FSE)* as a nondegerate solution of the system (5).

Remark 1: In reading (5) one notices that the description given is time invariant. However, when we construct trajectories that satisfy (5), time subscripts necessarily enter and the correct way of specifying trajectories is by using the rule $x_t = h(x_{t-1}; s_t)$. One could consider an "alternative" specification in which the agents' model is that the law of motion for the state variable takes the form $x_{t+1} = h(x_t; s_t)$ in which case instead of (5) we get the condition

$$x \in D \;\Rightarrow\; h(x;s) \in D \;\text{ and }\; \sum_{j=1}^{N} \pi_{sj} V(h(x;s)) = U(x), \qquad s = 1, \cdots, N,$$

an equation whose only nontrivial stochastic solutions are randomizations over the set $\mathcal{H}(x)$. Our specification of the timing is the only one that permits interesting stochastic solutions.

2.2 Existence

With our equilibrium notion in place, we can state and prove an existence result. We provide a constructive proof to show that if we have a pair D and h where D is a non-degenerate interval that is invariant for h and h is a selection from $\mathcal{H}$, and if the image of D under the composition of V with h is a non-degenerate interval, then FSE exist whenever the transition matrix Π is singular.

Proposition 1: *Assume that there exists a pair D and h, where D is a non-degenerate interval, such that $x \in D \;\Rightarrow\; h(x) \in D$ and $V(h(x)) = U(x)$ for all $x \in D$, and that $V(h(D))$, the image of D under the composition of V with h, is a non-degenerate interval. Then, for any $N > 1$ and any Π which is singular, an FSE exists.*

Proof: Let $Z := \{z \in R^N : \Pi \cdot z = 0_N, \| z \| = 1\}$. Since Π is singular, $Z \neq \emptyset$. Suppose that $\tilde{z} \in Z$ is such that all the coordinates take the same value. Since

$\sum_{j=1}^{N} \pi_{sj} = 1$ for all $s = 1, \cdots, N$, we must have $\Pi \cdot \tilde{z} = \tilde{z}$. But then $\tilde{z} \in Z$ only if $\tilde{z} = 0_N$ contradicting the fact that $0_N \notin Z$. It follows that if $z \in Z$, then $z_i \neq z_j$ for some i, j. Furthermore, $\text{Max}\{z_1, \cdots, z_N\} > 0$ and $\text{Min}\{z_1, \cdots, z_N\} < 0$.

Set $\overline{V}_D := \sup_{x \in D} V(h(x))$ and $\underline{V}_D := \inf_{x \in D} V(h(x))$. By hypothesis $(\underline{V}_D, \overline{V}_D) \neq \emptyset$. Also set $\widetilde{D} := \{x \in D : V(h(x)) \in (\underline{V}_D, \overline{V}_D)\}$. Since $Z \neq \emptyset$ and $(\underline{V}_D, \overline{V}_D) \neq \emptyset$, for any $\xi : D \to Z$, there exists α such that

$$\frac{\overline{V}_D - V(h(x))}{\text{Max}\{\xi_1(x), \cdots, \xi_N(x)\}} > \alpha(x) > \frac{\underline{V}_D - V(h(x))}{\text{Max}\{\xi_1(x), \cdots, \xi_N(x)\}} \quad \text{for all } x \in \widetilde{D}$$

$$\frac{\underline{V}_D - V(h(x))}{\text{Min}\{\xi_1(x), \cdots, \xi_N(x)\}} > \alpha(x) > \frac{\overline{V}_D - V(h(x))}{\text{Min}\{\xi_1(x), \cdots, \xi_N(x)\}} \quad \text{for all } x \in \widetilde{D}.$$

$\alpha(x) \neq 0$ for all $x \in \widetilde{D}$,

$\alpha(x) = 0$ for all $x \in D/\widetilde{D}$.

Also, since $\xi(x) \in Z$ and $\alpha(x) \in R$, $\Pi \cdot \alpha(x)\xi(x) = 0_N$.

By construction,

$$\overline{V}_D > \alpha(x)\xi_s(x) + V(h(x)) > \underline{V}_D \quad \text{for all } x \in \widetilde{D} \text{ and } s = 1, 2, \cdots, N,$$

and at least one of the inequalities is strict if $x \in D/\widetilde{D}$.

Since $V(h(D))$ contains $(\underline{V}_D, \overline{V}_D)$, for $y \in (\underline{V}_D, \overline{V}_D)$ there exists $x \in D$ such that $V(h(x)) = y$; denote by $V_D^{-1}(y)$ such a value of $h(x)$. Define the belief functions

$$h(x; s) := V_D^{-1}(\alpha(x)\xi_s(x) + V(h(x))) \quad \text{for all } x \in D \text{ and } s = 1, 2, \cdots, N.$$

By construction, the functions $h(\cdot; s)$ are all well defined and for every $s = 1, \cdots, N$, $h(x; s) \in D$ for all $x \in D$ so that the invariance condition holds.

Also, since

$$V(h(x; s)) = \alpha(x)\xi_s(x) + V(h(x)) \quad \text{for all } x \in D \text{ and } s = 1, 2, \cdots, N,$$

and $\Pi \cdot \alpha(x)\xi(x) = 0_N$, it follows that

$$\sum_{j=1}^{N} \pi_{sj} V(h(x; j)) = \sum_{j=1}^{N} \pi_{sj}\alpha(x)\xi_j(x) + (\textstyle\sum_{j=1}^{N} \pi_{sj})V(h(x))$$
$$= V(h(x)) \quad \text{for all } x \in D \text{ and } s = 1, 2, \cdots, N. \tag{6}$$

We proceed to check that (5) holds for the functions $h(\cdot; s)$ as defined. Since we started with a pair D and h where D is invariant for h and h is a selection from $\mathcal{H}$, (3) holds, which, when combined with (5) leads to

$$\sum_{j=1}^{N} \pi_{sj} V(h(x; j)) = V(h(x)) \quad \text{for all } x \in D \text{ and } s = 1, \cdots, N$$

which is (6). The property $\alpha(x) \neq 0$ for all $x \in \widetilde{D}$ guarantees that the solution we have constructed is non-degenerate. This completes the proof of the proposition.

Remark 2: The construction in the proof above shows that FSE exist in abundance since there is a lot of flexibility in specifying the functions ξ and α. It is possible to extend the construction of the proof to cases where $V(h(D))$ is not an interval by specifying the functions ξ and α appropriately. Also, it is worth noting that differentiability assumptions play no role in our existence result.

We turn to a result which is in the nature of a necessary condition that is implied by the existence of an FSE. We do not emphasize the invariance properties that the forecasting functions must satisfy since these have been observed for related concepts in earlier literature (see, e.g., Grandmont (1986)); instead we focus on what is new about FSE.

Proposition 2: *If an FSE exists for* Π *then* Π *must be singular.*

The proof of Proposition 1 is trivial: Use the fact that Π is a stochastic matrix to conclude that, since an FSE is a nondegenerate solution of the system (5), $Z \neq \emptyset$ where $Z := \{z \in R^N : \Pi \cdot z = 0_N, \| z \| = 1\}$. It follows that Π is singular.

It is straight forward to show the following implication of Proposition 2.

Corollary 1: *If an FSE exists for* Π *then there exists an FSE for the same economy and* $\tilde{\Pi}$ *where the rows of* $\tilde{\Pi}$ *are identical; in particular, if* $N = 2$ *then the rows of* Π *are identical. One can therefore always sustain extrinsic uncertainty via an i.i.d. process.*

2.2 FSE in relation to FSSE

The most general notion of a sunspot equilibrium that one can consider is that of a stochastic process for x driven by present and past realizations of extrinsic uncertainty which satisfies an equation system that can be derived from (1). That, of course, is too general to tell us much unless we look at the very special case of low dimensional linear models. Therefore, particular emphasis has been placed on sunspot models in which the induced process for x is sufficiently simple which usually means, as noted in the introduction, that FSSE are considered so that the sunspot process is taken to be a time homogeneous Markov process with a finite number of states and the realizations of x depend only on the current realization of the sunspot. More formally, an FSSE for Π is a vector $(x_1, \cdots, x_N)$ such that

$$\sum_{j=1}^{N} \pi_{sj} V(x_j) = U(x_s), \qquad s = 1, \cdots, N.$$

It is evident that FSSE and FSE are very different objects even though, at a formal level, FSSE can be obtained as a special case of FSE by restricting h to take the form $h(x; s) = x_s$, a fixed value independent of x.

The formulation of FSE, and their existence under the fairly general conditions as established by Proposition 1, derives from the requirement that the agents are backward looking and at date t predict x_{t+1} on the basis of x_{t-1} and the sunspot variable. This corresponds to analysing a case where agents forecasts are not time invariant statewise constants but functions that map past states into future ones and the motivation behind such a formulation presented in the introduction was in terms of the information structure. Clearly, FSE and FSSE are conceptually very different and the difference can be traced to differences in the formulation of the information structure that lies behind each of them. The difference in the concepts shows up as a difference in the conditions for existence; in the case of FSSE, one needs an invariant set whose existence is guaranteed under very special circumstances (we postpone the details to the next section since most of the results for FSSE were obtained within the framework of the OLG model).

The random walk equilibria proposed by Chiappori and Guesnerie (1993) is yet another notion of a sunspot equilibrium which we discuss in the next section since most of their analysis is geared towards the OLG model.

3.1 An Overlapping Generations Formulation

In this section we apply our results to the basic OLG model. We also relate FSE to results on FSSE and random walk equilibria in the overlapping generations model.

Consider the standard OLG model with two period lived agents, one perishable commodity and a constant stock of fiat money. The utility function of an agent is denoted $u(c_1)+v(c_2)$ where c_1, c_2 are consumption when young and old respectively. Endowments are ω_1, ω_2 respectively. The stock of money is normalized to unity. The following standard assumptions will be made

A.1: *u and v are continuous on* $[0, +\infty)$ *and twice continuously differentiable on* $(0, +\infty)$ *with* $u' > 0$, $v' > 0$, $\lim_{z\to 0} u'(z) = +\infty$, $\lim_{z\to 0} v'(z) = +\infty$, $u'' < 0$, $v'' < 0$. *Furthermore,* $\omega_1 > 0$, $\omega_2 > 0$. *Finally,* $\frac{u'(\omega_1)}{v'(\omega_2)} < 1$ *so we are in the Samuelson case.*

We formulate the dynamics of the model in terms of the level of real balances, i.e., by the inverse of the price of the consumption good with the price of money normalized to one. This normalized price is denoted x.

Denote $V(x) := v'(\omega_2 + x)x$ and $U(x) := u'(\omega_1 - x)x$.

In the absence of sunspot activity in the model, the equilibrium price at date t, x_t, is determined, given x^e_{t+1}, the price that is expected to prevail at date $t+1$, according to the following first order condition:

$$V(x^e_{t+1}) = U(x_t). \tag{7}$$

As indicated in Section 2, the correspondence $\mathcal{H}$ can be generated by considering, for each x, the values of x^e_{t+1} such that (7) holds. Let $A := \{(x, \mathcal{H}(x))\}$; A is obtained by reflecting the agents' offer curve with respect to the vertical axis. Under A.1, U

is a monotone increasing function and both U and V are differentiable. This ensures that A is the graph of a differentiable function with the second coordinate as the independent variable. Furthermore, there is a unique $\bar{x} > 0$ such that $(\bar{x}, \bar{x}) \in A$, i.e., a positive steady state which happens to be unique under A.1. There is a "second" steady state, autaraky.

By applying Proposition 1, we show that under A.1 FSE *always* exist since the existence of a invariant self-fulfilling belief function is guaranteed. For ease of exposition we consider diferent classes of economies.

For the first class of economies, we assume that the offer curve is monotone. This requires that V be monotone increasing or that

$V'(x) = v'(\omega_2 + x) + xv''(\omega_2 + x) > 0.$

A.2: $1 \geq -\frac{xv''(\omega_2+x)}{v'(\omega_2+x)}$ for all $x > 0$.

It is well known that, under A.1 and A.2, the correspondence $\mathcal{H}$ is single valued hence a function which we denote h; $D := [0, \bar{x}]$, where $\bar{x}$ is the unique monetary steady state, with h is the required pair which gives an invariant self-fulfilling belief function. Note that if we were to set $D = [0, x]$, with $x > \bar{x}$, then the invariance property is lost; however, $D = [0, x]$ with $\bar{x} \geq x$ *does* satisfy the invariance requirement. Similarly, if we set $D = [a, x]$ for $0 < a < x$, the invariance property is lost.

Corollary 2: (gross substitutes economies) *Let A.1 and A.2 hold. For any* Π *which is singular, FSE exist with* $D := [0, x]$ *with* $\bar{x} \geq x$*, and* h *defined by, for each* $x \in D$*,* $h(x) =: \tilde{x}$ *where* $V(\tilde{x}) = U(x)$*.*

As we pointed out in the introduction, FSSE do not exist under A.2.

While Corollary 2 establishes the existence of FSE, it does not provide information about the possible shapes of the various $h(\cdot; s)$ functions that may appear as solutions to the system of equations that define FSE. In Example 1 below we specify a particularly simple type of FSE with $N = 2$ where, in the first state, the state variable moves towards the monetary steady state and in the second state, towards the autarkic one. Specifically, we take $D := [0, \bar{x}]$ and $N = 2$ and obtain solutions $h(\cdot; s)$, $s = 1, 2$, with the property that the two functions lie on either side of the 45 degree line through the origin so that under the action of $h(\cdot; 1)$, the state variable is pulled towards the monetary steady state $\bar{x}$, while under the action of $h(\cdot; 2)$, it is pulled towards the autarkic steady state $x = 0$. We obtain these solution functions in a straight forward constructive manner. It should be clear though that these are by no means the only possible specifications. The model admits other shapes of these functions that need not be monotone and hence the dynamic laws may be more complex than those outlined in Example 1.

Example 1: Assume A.1 and A.2. Fix $h(\cdot;1)$ defined on $D = [0,\bar{x}]$ satisfying $h(0;1) = 0$, $h(\bar{x};1) = \bar{x}$, $h(x;1) > x$ for all $x \in (0,\bar{x})$, $h(\cdot;1)$ is strictly increasing, and differentiable around 0 (from the right). It follows that $\lim_{x\to 0} h'(x;1) > 0$. h which solves (4) is also increasing, $x \geq h(x)$ for all $x \in [0,\bar{x}]$, and differentiable around 0 (from the right) with $\lim_{x\to 0} h'(x) > 0$ (since we are in the Samuelson case). It follows that $V(h(x)) < V(h(x;1))$ for all $x \in (0,\bar{x})$ since V is increasing under A.2. Choose $\pi \in (0,1)$ such that $V(h(x)) \geq \pi V(h(x;1))$ for all $x \in [0,\bar{x}]$. Clearly, such a π exists since

$$\lim_{x\to 0} \frac{V(h(x))}{V(h(x;1))} = \lim_{x\to 0} \frac{V'(h(x))h'(x)}{V'(h(x;1))h'(x;1)} = \lim_{x\to 0} \frac{h'(x)}{h'(x;1)} > 0.$$

It follows that

$$V(h(x)) \geq \frac{V(h(x)) - \pi V(h(x;1))}{1-\pi} \geq 0 \qquad \text{for all } x \in [0,\bar{x}].$$

Set $h(x;2) := V^{-1}\left(\frac{V(h(x)) - \pi V(h(x;1))}{1-\pi}\right)$ for all $x \in [0,\bar{x}]$; by the inequality above, $h(\cdot;2)$ is well defined. It is immediate that $h(0;2) = 0$ and $h(\bar{x};2) = \bar{x}$, since we know that $h(0) = h(0;1) = 0$ and $h(\bar{x}) = h(\bar{x};1) = \bar{x}$. From (6), $h(\cdot;1)$ and $h(\cdot;2)$ constitute an FSE if $\pi V(h(x;1)) + (1-\pi)V(h(x;2)) = V(h(x))$ for all $x \in [0,\bar{x}]$; we have constructed an FSE since $h(\cdot;2)$ was defined so as to satisfy the equation. We now show that $h(x;2) < h(x)$ for all $x \in (0,\bar{x})$. Since h and $h(\cdot;1)$ are increasing functions and $h(x;1) > h(x)$ for all $x \in (0,\bar{x})$ by construction, and V is an increasing function, one has $V(h(x;1)) > V(h(x))$; hence, by (6), $V(h(x;2)) < V(h(x))$ which implies in turn that $h(x;2) < h(x)$ for all $x \in (0,\bar{x})$. Finally, note that the construction imposes a very mild condition on $\lim_{x\to 0} h'(x;1)$; the derivative could exceed one thus ensuring nonconvergence to the autarkic steady state.

We turn to the case where A.2 does not hold. The offer curve must bend backwards and there exists $x^* \in (0,\omega_1)$ such that $V(x) \geq 0$ for all $x \in [0,x^*]$. $\mathcal{H}$ is no longer a function; instead one uses two (or more if necessary) functions, h^+ and h^-, where $h^+(x^*) = h^-(x^*)$. If $h^+(x^*) \geq x^*$ then Corollary 2 goes through unchanged (this is the case in which the offer curve bends "late" and the steady state is on the "lower" branch, $h^+(\bar{x}) = \bar{x}$). If instead $h^+(x^*) < x^*$, so that the offer curve bends "early" and the steady state is on the "upper" branch, $h^-(\bar{x}) = \bar{x}$, then one possibility is to modify the construction by setting $D := [0,x^*]$ and choosing a selection from $\mathcal{H}$ which is invariant—there are various possibilities, e.g. (i) choose

h^+, (ii) choose h^+ on $[0, \hat{x}]$ and h^- on $(\hat{x}, x^*]$ where $\hat{x}$ is such that $V(x^*) = U(\hat{x})$.

Corollary 3: (backward bending offer curves) *Let A.1 hold. For any* Π *which is singular, FSE exist with (i)* $D := [0, x]$ *with* $\bar{x} \geq x$*, if* $x^* \geq \bar{x}$*, or with (ii)* $D := [0, x^*]$*. In either case* h *can be specified to make* D *invariant.*

The solutions described in Corollaries 2 and 3 have the property that they cannot be bounded away from the autarchic steady state.[12] This leads us to ask if there are other solutions where the dynamics of the state variable are confined to some neighbourhood of the monetary steady state and bounded away from the autarkic steady state. The answer is yes: An invariant self-fulfilling belief function exists around the monetary steady state if and only if a cycle of period two exists. Here one can set $D := [x_1, x_2]$, where $0 < x_1 < x_2 < x^*$ are the values of the state variable along the two cycle. It is known that under an additional nondegenracy assumption regarding the two cycle, the stated condition is also necessary and sufficient for the existence of FSSE under A.1.[13] So FSE can exist when a critical cycle of period two exists, a case where FSSE need not exist.

Corollary 4: (backward bending offer curves) *Let A.1 hold. For any* Π *which is singular, FSE with the property that the dynamics of the state variable are confined to an interval around the monetary steady state and bounded away from the autarkic steady state exist if and only if a cycle of period two exists.*

We can now discuss the relation of FSE with earlier work on sunspots within the OLG framework. As indicated in our discussion at the end of Section 2.2, the existence of FSSE requires a particular kind of invariant set for the deterministic dynamics. The easiest way to meet the requirement is to posit that the steady state be indeterminate in the forward dynamics. Examples of such results are Azariadis and Guesnerie (1986), Grandmont (1986) and Guesnerie (1986). Corollary 4 shows that FSE also obtain under the same circumstances and for an additional nongeneric case to boot. However, FSE obtain far more generally as Corollaries 2 and 3 have shown.

The random walk equilibria introduced by Chiappori and Guesnerie (1993) are very different compared to FSSE and they too can exist in the gross substitutes case. Though they behave like the FSE in Example 1, the conditions for existence of random walk are somewhat different. More importantly, we feel that the constructive

[12]Many of the sunspot equilibria discussed in Woodford (1994) have the same feature.

[13]See Azariadis and Guesnerie (1986) or Grandmont (1986).

proof of our Proposition 1 together with the additional details provided in Example 1 are much easier to follow. We prefer to view FSE as a much simpler approach to obtaining random walk type behaviour.

3.2 The Linear Case

In this section we consider the special case that arises when the function that links the current value of the state variable with its forecasted value is linear. The simpler specification makes it easier to follow the analysis of Section 2.

It is convenient to work with a function in deviations of the state variable from its unique steady state value, which is denoted x. The function is accordingly expressed as

$$x_t = a x^e_{t+1} \tag{8}$$

where 0 is the unique steady state of the system. Set $k := a^{-1}$; k determines the stability of the perfect foresight dynamics associated with the map (8).

In the deterministic case, a typical agent's beliefs will be assumed to be described by a function $h(x) = \beta x$. Since the information available to the agent at date t includes all realizations of the state variable up to and including $t-1$, the agent iterates twice on the belief h to generate the forecast

$$x^e_{t+1}(x_{t-1}) = \beta^2 x_{t-1}. \tag{9}$$

By combining (8) and (9), the actual dynamics of the system are obtained

$$x_t = a\beta^2 x_{t-1}.$$

In this case *self-fulfilling* belief functions are the fixed points, in β, of the function $\Omega(\beta) = a\beta^2$. Ω has two roots, $\bar{\beta}_1 = 0, \bar{\beta}_2 = \frac{1}{a}$ which are the two self-fulfilling belief functions.

With extrinsic uncertainty described by a finite state Markov process, the belief functions take the form $h(x_{t-1}; s) = \beta_s x_{t-1}$. With our information structure, one obtains the following forecasting rule given that $s_t = s$

$$x^e_{t+1}(x_{t-1}; s) = \left[\sum_{j=1}^N \pi_{sj}\beta_j\beta_s\right] x_{t-1}, \qquad s = 1, \cdots, N. \tag{10}$$

The natural consistency requirement that the state dependent belief functions be self-fulfilling takes a very simple form obtained by combining (10) with (8):

$$\beta_s x_{t-1} = a\left[\sum_{j=1}^N \pi_{sj}\beta_j\beta_s\right] x_{t-1}, \qquad s = 1, \cdots, N,$$

which, because of the linear framework, can be expressed solely in terms of the beliefs β as

$$k = \left[\sum_{j=1}^N \pi_{sj}\beta_j\right], \qquad s = 1, \cdots, N.$$

Since Π is a stochastic matrix, $\sum_{j=1}^N \pi_{sj} = 1$ for all $s = 1, \cdots, N$. It follows that the system of equations which we wish to solve is

$$\Pi \cdot [k 1_N - \beta] = 0_N, \tag{11}$$

where, 1_N is a column vector of 1s, and 0_N is a column vector of 0s.

Propositions 1 and 2 apply since R is an invariant set and $k \neq 0$.

To see how FSE can be constructed, suppose that Π is singular. As in the proof of Proposition 1, there exists a vector $z \in R^N$, $z \neq 0_N$, such that $\Pi \cdot z = 0_N$ and $z_i \neq z_j$ for some i, j. Define $\beta^*_s := z_s + k$. β^* solves the system of equations (11) and has the property that $\beta_i \neq \beta_j$ for some i, j.

We can also construct solutions using Π as the variable that one solves for. Trivially, the perfect foresight root $\bar{\beta}_2 = k$ solves (11). Fix any collection of values of β_s, $s = 1, \cdots, N$, $\beta_i \neq \beta_j$, $i, j = 1, \cdots N$, and let $\widetilde{\beta}$ and $\widehat{\beta}$ denote respectively the smallest and largest of these N values. If the perfect foresight root $\bar{\beta}_2$ satisfies $\tilde{\beta} < \bar{\beta}_2 < \widehat{\beta}$, one can solve for $\pi_{ij} \neq 0$, $i, j = 1, \cdots, N$, such that (11) is satisfied.

The usual argument for the existence FSSE in the linear model is as follows: Given Π, an N state FSSE is a tuple $(x_1, \cdots, x_N)$ such that

$$k x_s = \textstyle\sum_{j=1,\cdots,N} \pi_{sj} x_j, \qquad s = 1, \cdots, N,$$

which can also be written as

$$[k \cdot I_N - \Pi] \cdot x = 0_N, \tag{12}$$

where I_N is the N-dimensional identity matrix. It is known that solutions exist with non-degenerate probabilities and with $x_i \neq x_j$ for some $i \neq j$ if and only if $|k| < 1$.[14]

By comparing (11) and (12) it should be clear that the conditions for existence of the two kinds of equilibria are very different. As noted earlier, FSE generate FSSE as special cases when $h(x_{t-1}; s) = x_s$.

4. Concluding Remarks

In this paper we introduced FSE, an equilibrium concept that subsumes FSSE, and demonstrated conditions under which they exist. We noted the extreme degree of flexibility that one has in constructing FSE since h need not be continuous and the set Z, the null space of Π, can be large. This shows once again that the requirement of self-fulfilling beliefs in itself is very weak and far from letting one pin down the equilibrium behaviour of economic systems.[15] In our analysis, the multiplicity is driven by the fact that the state variable is endogenous and is not tied down sufficiently by the fact that it is described by a function relating its value across successive periods.

It remains to investigate the welfare properties of FSE under a conditional notion of optimality and the precise connection between FSE and the general class

[14] When $N = 2$ this is very easy to check.

[15] Cass and Shell (1983) made the same point in their seminal work on sunspots.

of sunspot equilibria studied by Woodford (1986), Chiappori and Guesnerie (1993) and Woodford (1994). Given the large degree of multiplicity of FSE, it would also be interesting to investigate whether any of these might be robust in the sense of Goenka and Shell (1997), and which might be stable under adaptive learning rules.

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