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M E X I C O

MESSAGE SPACES FOR PERFECT CORRELATED EQUILIBRIA

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Abstract. We show that a perfect correlated equilibrium distribution of an N-person game, as defined by Dhillon and Mertens (1996) can be achieved using a finite number of copies of the strategy space as the message space.

1. Introduction

Dhillon and Mertens (1996; henceforth, DM) introduced the concept of perfect correlated equilibria (PCE) as a refinement of correlated equilibria (Aumann, 1974)—see the Introduction to DM for a rationale for this refinement. DM define a PCE as a perfect equilibrium (Selten, 1975) of an extended game obtained by using a correlation device. A perfect correlated equilibrium distribution (PCED) is then a correlated equilibrium distribution that is achievable as a perfect equilibrium of some extended game. DM also provided a characterization of PCEDs of two-player games using a canonical message space. In this paper, we provide an extension of their result to the N-player case.

Using a simple two-player game, DM show that the direction revelation principle fails to hold for PCEDs, i.e., certain PCEDs are not obtainable by using the strategy space as the message space. The reason for this failure is that players need more "coordinates" in their messages to encode information about how they are to tremble. For the two player case, DM show that the two-fold product of the strategy space suffices as the message space. Here, we analyze the N-person case and show (Theorem 3.1) that every PCED can be obtained using a finite number of copies of the strategy space as the message space. We also provide a heuristic argument that strongly suggests the possibility that using just two copies will not be enough in general.

The messages that support a PCE in our construct are L-tuples (for some positive integer L) of pure strategies for each player that actually represent a hierarchy of beliefs of his opponents about what he would play when he gets a message. Unfortunately, the number L depends on the particular PCED under consideration and hence we have not been able to get a uniform bound for it. We hope that future work will be able to provide a tight bound for the dimension of the message spaces derived here.

2. Definitions

Let Γ be a a finite normal form game with player set $\mathcal{N} = \{1, \ldots, N\}$. For each player n, let S_n be his finite set of pure strategies; and let $S = \prod_{n \in \mathcal{N}} S_n$. For any finite set X, denote by $\Sigma(X)$ the set of all probability distributions over X. For simplicity, we will write Σ_n to denote the set $\Sigma(S_n)$ of mixed strategies of player n.

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A correlation device is a an $(N+1)$-tuple $d = \big((M_n)_{n\in\mathcal{N}}, P\big)$ where for each n, M_n is player n's finite message space, and P is a probability distribution on $M = \prod_n M_n$. Given a correlation device, one defines an extended game Γ_d as follows: first nature chooses a message $m \in M$ according to P and informs each player of his coordinate of m; then, the players play Γ. Thus, a pure strategy for player n in Γ_d is a function $\tau_n : M_n \to S_n$, and payoffs are defined by taking expectations w.r.t. P.

Definition 2.1. A perfect correlated equilibrium (PCE) of Γ is a pair (d, τ) where d is a correlation device and τ is a perfect equilibrium of the game Γ_d. The distribution over S induced by τ is called a perfect correlated equilibrium distribution (PCED).

In defining PCEDs, it is irrelevant whether we consider the normal form or the extensive form perfect equilibria of Γ_d. Indeed, in the extensive form of the the game Γ_d, no information set of a player succeeds another one of his. Hence, the normal form and the extensive form perfect equilibria of Γ_d coincide. Conceptually, it is simpler to view the perfect equilibria of Γ_d as being in behavioural strategies, which are functions from the message spaces of players to their sets of mixed strategies in Γ. Throughout this paper, therefore, we will be working with behavioural strategies in the extended game.

3. Canonical Message Spaces

The main result of our paper is the following theorem showing that every PCED can be achieved using a finite number of copies of the strategy space as the message space.

Theorem 3.1. *$\mu \in \Sigma(S)$ is a PCED if and only if there exists a positive integer L and a device $d = (S^L, P)$ such that:*

(α) *The marginal distribution of P on the first factor is μ, i.e., $\mu(s^1) = \sum_{s^2,\dots,s^L} P(s^1, s^2, \dots, s^L)$ for all $s^1 \in S$.*

(β) *The pure strategy profile ρ given by $\rho_n(s_n^1, \dots, s_n^L) = s_n^1$ is a perfect equilibrium of Γ_d. More precisely, there exists a sequence $\big(\varepsilon^0(k), \dots, \varepsilon^{L-1}(k)\big)$ in $(0,1)^L$ converging to zero such that ρ is a best reply against every element of the following sequence of completely mixed strategies: given the message $(s_n^1, \dots, s_n^L)$, player n plays the uniform strategy in Σ_n with probability $\varepsilon^0(k)$ and with probability $(1-\varepsilon^0(k))$ plays the strategy that is the nested combination $(\varepsilon^1(k), \dots, \varepsilon^{L-1}(k))\Box(s_n^1, \dots, s_n^L)$ given by*

$$\big(1-\varepsilon^1(k)\big)\, s_n^1 + \varepsilon^1(k)\left[\big(1-\varepsilon^2(k)\big)\, s_n^2 + \varepsilon^2(k)\left[\cdots + \varepsilon^{L-2}(k)\left[\big(1-\varepsilon^{L-1}(k)\big)\, s_n^{L-1} + \varepsilon^{L-1}(k) s_n^L\right]\cdots\right]\right].$$

The proof of the Theorem relies crucially on the following Lemma, which is a minor variant of Proposition 2 in Blume, Brandenberger, and Dekel (1991). The only difference is that they consider a sequence of probability distributions while we consider a sequence of *vectors* of probability distributions. It is easily verified that their proof applies to our case as well; therefore, we omit the proof of the Lemma.

Lemma 3.2. *Let $X_1, \dots, X_I$ be a finite collection of finite sets. Let $\mu(k)$ be a sequence in $\prod_{i=1}^I \Sigma(X_i)$. Then there exist (i) a positive integer $L \leqslant \sum_i |X_i|$; (ii) for each $l = 1, \dots, L$, a vector μ^l in $\prod_{i=1}^I \Sigma(X_i)$; and (iii) a sequence $(\varepsilon^1(k), \dots, \varepsilon^{L-1}(k))$ in $(0,1)^{L-1}$ converging to zero such that a subsequence of $\mu(k)$ is expressible as the nested combination $(\varepsilon^1(k), \dots, \varepsilon^{L-1}(k))\Box(\mu^1, \dots, \mu^L)$.*

Proof of Theorem 3.1. The sufficiency of our condition is obvious. We will therefore prove its necessity. Let (d, τ) be a PCE that induces μ, where $d = \big((M_n)_{n\in\mathcal{N}}, P\big)$ is the correlation device. There exists a sequence $\tau(k)$ of completely mixed behavioural strategies in Γ_d converging to τ such that τ is a best reply against every element of the sequence. Express $\tau(k)$ as $(1-\varepsilon^0(k))\tilde{\tau}(k) + \varepsilon^0(k)\tau^*$ where τ^* is the uniform strategy profile and $\varepsilon^0(k)$ is a sequence of positive numbers converging to zero. We then have that the limit of the sequence of $\tilde{\tau}(k)$ is τ. Viewing $\tilde{\tau}(k)$ as a sequence in $\prod_n (\Sigma_n)^{M_n}$ we can now apply Lemma 3.2 to obtain: (i) an integer L; (ii) for each $l = 1\ldots, L$, $n \in \mathcal{N}$ and $m_n \in M_n$ a mixed strategy $\tau^l_{m_n}$ in Σ_n; and (iii) a sequence $\varepsilon(k)$ in $(0,1)^{L-1}$ converging to zero, such that (by replacing the sequence of $\tilde{\tau}(k)$'s with an appropriate subsequence) we have that for each n, m_n, $\tilde{\tau}_{m_n}(k) = (\varepsilon^1(k), \ldots, \varepsilon^{L-1}(k))\Box(\tau^1_{m_n}, \ldots, \tau^L_{m_n})$. Obviously, $\tau^1_{m_n}$ is the limit of the sequence $\tilde{\tau}_{m_n}(k)$, which by construction is τ_{m_n}.

Construct a correlation device $\tilde{d}$ as follows. The message space for each player n is $\tilde{M}_n = \big(M_n \times S^L_n\big)$; and the lottery, call it $\tilde{P}$, on $\tilde{M} = \prod_n \tilde{M}_n$ is obtained by first choosing m according to P and then, given m, choosing for each player n, and each $l = 1, \ldots, L$, a pure strategy s^l_n according to $\tau^l_{m_n}$. Consider the following pure strategy ρ in $\Gamma_{\tilde{d}}$: given the message $(m_n, s^1_n, \ldots, s^L_n)$, player n plays s^1_n. By construction, the distribution on Σ that is induced by $(\tilde{d}, \rho)$ is μ. We claim now that ρ is a perfect equilibrium of $\Gamma_{\tilde{d}}$. Indeed, it follows again from the construction above that ρ is a best reply against the following sequence $\rho(k)$ of completely mixed strategy profiles: given a message $(m_n, s^1_n, \ldots, s^L_n)$, player n plays the uniform mixture over S_n with probability $\varepsilon^0(k)$ and with probability $(1-\varepsilon^0(k))$ plays the the nested combination $(\varepsilon^1(k), \ldots, \varepsilon^{L-1}(k))\Box(s^1_n, \ldots, s^L_n)$.

Observe that both the pure strategy ρ and the the sequence $\rho(k)$ are implementable even if we do not inform the players of the original messages m_n. Moreover, ρ would, a fortiori, be a best reply against the sequence—players have fewer strategies available now. Thus, deleting the original messages from $\tilde{M}$ yields the result. $\Box$

Remark. In the two-player case, the integer L in the statement of Theorem 3.1 can be taken to be 2. Indeed, in the Proof above, every element of the sequence of $\tau(k)$'s can be chosen as a convex combination of τ and a fixed, completely mixed strategy. Therefore, the sequence $\tilde{\tau}(k)$ can be taken to be of the form $(1-\epsilon^1(k))\tau + \epsilon^1(k)\tau'$ for some mixed strategy τ' and some sequence of $\epsilon^1(k)$'s converging to zero. Thus, $\tilde{\tau}(k)$ is the nested combination $\epsilon^1(k)\Box(\tau, \tau')$. In other words, L can be be taken to be 2.[1]

4. Discussion

Theorem 3.1 not only provides a canonical message space for achieving a PCED, but also provides a description of a sequence of perturbed strategies that renders the correlated equilibrium perfect. Given the message $(s^1_n, \ldots, s^L_n)$ player n plays the pure strategy s^1_n. All the other players, on the other hand, believe that player n might play any of the coordinates with positive probability but that the probability of s^l_n is infinitely smaller than that of s^{l-1}_n for each $l > 1$. Thus, modulo the fact that each player plays the uniform strategy with a small probability—and this, only for the technical reason that the strategies in the game Γ_d have to be completely mixed—the vector $(s^1_n, \ldots, s^L_n)$ represents a hierarchy of beliefs of the other players, in the sense of Blume, Brandenberger, and Dekel (1991).

[1]This argument is essentially the same as the one used by DM in their characterization of PCEDS for two player games.

In terms of its interpretation, the DM characterization of the PCEDs of two-player games is similar. However, their result is stronger in the sense that the message space can be taken to be $S \times S$. The reason for this difference is the equivalence between perfection and admissibility in the two-player case: it suffices for either player to have only one alternative theory about his opponent's behaviour in the extended game. The nonlinearities involved in the N-person case means that such a result is not possible. To see why this is the case, consider an N-player game and a finite collection of perfect equilibria in all of which, one of the players, say player 1, plays a pure strategy, say T. If we are to implement a convex combination of these equilibria using only two copies of S, then player 1's message space would be of the same dimension as his strategy space. Suppose now that this player has two other pure strategies, M and B. Furthermore, suppose that each of the perfect equilibria requires that the relative magnitudes of the trembles, ε_M and ε_B, resp., for M and B, are different across equilibria. Specifically, one equilibrium requires that $\varepsilon_M = \varepsilon_B$, another that $\varepsilon_M^2 = \varepsilon_B$, etc. (Such conditions are possible in the N-person case, since it is possible that for one player's equilibrium strategy to be a best reply, the probability that player 1 deviates to B is of the same order as the probability that he deviates to M *and* a certain subset of the others also deviate to a non-equilibrium strategy.) In order to implement a convex combination of these perfect equilibria as a PCE, Player 1 would need more than just three messages.

As remarked in the Introduction, an important unresolved issue is that of providing a uniform bound on L. One way to answer this question is to provide a bound on the dimension of the message space required to implement a PCED, i.e., to show that there exists a number k (that depends only only the cardinalities of the strategy sets of the players) such that every PCED can be implemented using a message space with at most k messages.

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