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Vote Revelation: Empirical Characterization of Scoring Rules

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Vote revelation: empirical characterization of scoring rules

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Abstract

In this paper I consider choice correspondences defined on an extended domain: the decisions are assumed to be taken not by individuals, but by committees and, in addition to the budget sets, committee composition is observable and variable. For the case of varying committees choosing over a fixed set of two alternatives I provide a full characterization of committee choice structures that may be rationalized with sincere scoring. For the general case of multiple alternatives a necessary implication of choice by sincere scoring is provided.

1 Introduction

Consider an observer trying to make sense of the goings on in a secretive committee, such as the old Soviet Politburo. Such an observer would not have any direct evidence about preferences of individual committee members, nor would he be likely to observe the rules the committee uses to make

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its decisions. Nevertheless, our Kremlinologist does have some information to work with. For one, he may have a reasonably good idea of the options the committee members are facing. He would also be able to observe the committee decision: perhaps, it would come out in the Pravda. Finally, the committee membership is public knowledge (he could determine it by observing the figures standing on the observation deck of Lenin's Mausoleum during the Revolution Day parade). What sort of deductions would it be possible to make about the unobservable preferences and preference aggregation rules within the committee from this information?

Alternatively, we may be observing the decisions of a relatively transparent committee, which has formal decision rules that are explicitly set in a published law. Still, many things may be unknown about what happens inside the doors and inside the minds of the committee members. Do they vote strategically or sincerely? Do they take into account preferences of and/or information possessed by their fellow committee members? If only committee decisions are made public, with votes and deliberations remaining secret, could we still test theories about the goings on inside the committee?

In fact, not much could be said from a single observation of the committee decision alone. However, as I try to establish in this paper, it turns out that, if a number of observations of decisions taken by a committee with variable membership is available, one can use the available data to test certain hypotheses about the committee functioning.

The approach I use here is, in fact, quite standard, being based on the ideas of revealed preference and rationalizability, that have long been standard foundations of economic analysis. Ever since Houthakker (1950) it has been known that a simple consistency condition on choices (the Strong Axiom of Revealed Preference, SARP) is a necessary and sufficient condition for being able to explain individual choices with rational preference maximization. Of course, this approach has long been a basis for the formal decision theory used by political scientists, as well as economists. The standard textbook treatments of the political theory, such as Austin-Smith and Banks (1999), in fact, start by formally presenting it. Once, however, it is used to support the preference theory, most of the time, the primary attention is paid to preference aggregation into decisions. In contrast, the fact that the group decisions may be used to uncover both individual preferences and group decision rules is frequently ignored.

It is not to say, that this has never been suggested. Thus, for instance, when Blair *et al.* (1976) characterized such restrictions on choice structures

as would derive from maximizing preferences that are merely acyclic, rather than transitive, this could, of course, be interpreted as characterizing choices made by committees of rational members with some of those members exercising veto power. In a different context, Peters and Wakker (1991) have discussed empirical consequences of bargaining solutions, as do, for instance, Chambers and Echenique in a recent paper (2011). However, though well-established, the tradition of revealed preference approach to group decisions has not been much developed recently. In particular, I am aware of no studies establishing "signatures" imposed on collective decisions by most commonly used voting rules. It is precisely this that I attempt to do in this paper.

In fact, when in recent years concepts of choice and revealed preference have received substantial renewed attention in economics, it was mostly in the context of individual decision-making. This attention has been derived from the new focus on "boundedly rational" decision-making procedures different from the usual rational preference maximization. In this context one might mention, among many others, Manzini and Mariotti (2007) work on "sequential rationalizability" or Masatlioglu and Ok (2005) study of choice with status-quo bias, both of which attempt to establish restrictions imposed on choices by distinct decision-making procedures. Other recent studies, such as Caplin and Dean (2009) and Caplin, Dean and Martin (2011) attempt to explore the restrictions that various "boundedly rational" procedures would impose on records that are somewhat more detailed than the usual choice data, though still plausibly observable: the insight that is also of key importance to the present research.

In the situations described at the beginning of this introduction, the group decision data is, in fact, richer than usual: in addition to the record of choices from a given set of alternatives, we have the committee membership at each decision point to consider. Thus, if we want to test a given theory of how the committee works, we have more information to base our testing on. Even on incomplete data (i.e., when not all possible observations might be there), we may observe enough to do this.

In this study I concentrate on a particular class of theories about the internal committee workings. I will generally assume that each committee consists of rational members who decide using some scoring rule: those voting rules, such as the simple "first past the post" plurality or the Borda Count, in which individuals are asked to provide each alternative with a numeric score (reflecting their preferences), the individual scores are added up and the alternative with the highest aggregate score is chosen. These rules have

long been characterized by social choice theorists (see Smith 1973, Young 1975 or Myerson 1995). My objective is to formulate the natural restrictions on observations implied by these rules. Even when the particular scoring rule is unknown, such restrictions turn out to be non-trivial.

The objective of this work is clearly related to the study of empirical content of sincere (vs. strategic) voting by Degan and Merlo (2009). In fact, as suggested at the start of this introduction, if the formal decision rule is known, this work may be reinterpreted precisely as the test of voter sincerity: if I know how the votes are counted, violations of the conditions established here could only be interpreted as indications that the scores do not directly reflect rational individual preference. Thus, to the extent one maintains the assumption that voters are rational, sincere voting would be falsified in this case. Likewise, this paper is related to Kalandrakis (2010) work on rationalizing individual voting decisions. This paper crucially differs from both Degan and Merlo (2009) and Kalandrakis (2010), however, in that I do not assume observability of individual votes (nor do I impose anything in addition to rationality on individual preferences).

Rather, individual votes are "revealed" here from the observations of the group choices. One hopes to characterize the conditions under which these revealed scores are consistent. For the simple case of varying committees choosing over a fixed pair of alternatives, in fact, such characterization turns out to coincide with the conditions for the existence of additive probability measures over a finite state space, representing a given binary relation "at least as likely as", established by Kraft et al. (1959). The necessary and sufficient condition for such a representation has a clear "SARP-like" acyclicity form. For the more general case of multiple alternatives natural necessary conditions of the "SARP"-type emerge, though the complete characterization is, so far, unknown.

The rest of this paper is organized as follows. In section two I provide the basic model set-up. In section three I consider the simple case of two alternatives and provide a characterization of the restrictions on the committee choice structures that make them consistent with choice by scoring In section four I extend the analysis to the case of three or more alternatives. Section five considers the case of plurality rule. Section six concludes.

2 Basic Set-up

In this section I closely follow my earlier note (Gomberg 2011). Consider a finite set $N = \{1, 2, ...n\}$ of agents and a finite set $X = \{x_1, x_2...x_m\}$ of alternatives. A set of alternatives to be considered by a *committee* $S \in 2^N \setminus \{\varnothing\}$ is $B \in 2^X \setminus \{\varnothing\}$; following the standard terminology of individual choice theory, I shall call B the *budget set*. If a committee S is offered a choice from the budget set B the committee choice is recorded as $\varnothing \neq C(B, S) \subset B$. The *committee choice structure* is defined as a pair $(\mathcal{E}, C(.,.))$ where $\mathcal{E} \subset 2^X \setminus \{\varnothing\} \times 2^N \setminus \{\varnothing\}$ is the record of which budget sets where considered by which committees and $C : \mathcal{E} \to X$, such that $C(B, S) \subset B$ is the non-empty-valued choice correspondence, recording committee choices.

In order to explain observed committee choice structures I shall, in general, assume that each agent $i \in N$ has rational (complete and transitive) preferences $\succsim_i$defined over X. The committee choice structure provides a record of observed committee choices, which may be used by an observer to deduce the preference profiles and the preference aggregation rules the committee uses. In this paper I concentrate on a particular class of such rules: the scoring rules, a class that includes such distinct procedures as the plurality vote (in which the winner is an alternative that is chosen by the largest number of voters), the Borda Count (in which alternatives get assigned the most points for being someone's top choice, a point less for being a second choice, etc., the scores get summed up over all the voters and the alternative with the largest score wins), or the Approval Voting (in which an individual is allowed to mark alternatives as acceptable or unacceptable, and the alternative which has been marked as acceptable by the largest number of voters gets chosen). Overall, I shall assume that agents are non-strategic, in that they ignore who else is in the committee (as noted above, the conditions I am deriving here might, if the formal rule is observable, be viewed as empirical implications of sincere voting itself). However, I shall allow the votes to depend on the budget sets under consideration (as would be the case in a sincere Borda Count). Thus, if the set of alternatives B, a vote of agent $i \in S$ is a function $v_i^B : B \to \mathbb{R}$.

Given a vote from each of its members a committee S chooses an alternative that gets the highest score

$$C^{scoring}(B, S) = \arg\max_{x \in B} \sum_{i \in S} v_i^B(x)$$

where $\sum_{i \in S} v_i^B(x)$ is called the *score* received by an alternative $x \in B$ in voting by committee S. Such a choice structure is said to be generated by the scoring rule.

Following Myerson (1995) I shall allow agents to submit votes that are distinct from reporting their preference orderings. In fact, for the purposes of defining a scoring rule one does not need to assume that the votes themselves derive from rational preferences. However, as the scoring rules require agents to report a ranking of alternatives in B by means of their votes $v_i \in \mathbb{R}^k$. Though in general such a ranking may not necessarily represent a rational preference (and thus, for instance, could be inconsistent over the different budget sets B), I shall concentrate on voting that, indeed, can be viewed as a sincere representation of individual preferences. Formally, given a rational preference profile $\succsim = (\succsim_1, \succsim_2, ..., \succsim_n)$ I shall say that a committee vote v_i^B is *strictly consistent with preferences* if $x \succsim_i y$ if and only if $v_i^B(x) \geq v_i^B(y)$. I shall say that a committee vote v_i^B is *weakly consistent with preferences* if $x \succsim_i y$ implies $v_i^B(x) \geq v_i^B(y)$.

If a committee choice structure is such that for any $(B, S) \in \mathcal{E}$

$$C(B, S) = C^{scoring}(B, S)$$

where the votes are consistent with preferences for some rational preference profile $\succsim$. I shall say that $\succsim$ rationalizes $(\mathcal{E}, C(.,.))$ via a scoring rule.

It should be noted, that unless the choice structure is extended by allowing observing variations in committee membership, scoring rules would, at first glance, appear particularly unpromising from the standpoint of this research: it would seem that nearly every possible committee decision could be explained by some sort of scoring applied to an unobserved preference profile of a fixed committee. Thus, if one defines, in the spirit of Salant and Rubinstein (2008) work on the choice with frames, the choice correspondence as

$$C_c(B) = \{x : x \in C(B, S) \text{ for some committee } S\}$$

little, if anything appears to be imposed on $C_c(.)$ (some restrictions may be derived from the relative cardinalities of B and N, if the latter is observed, but that appears to be it). However, it turns out that more can be said if committee membership and its variations are observed.

3 Revealed Scoring: the case of two alternatives

I shall first consider the simple case, in which the number of alternatives is equal to 2. In this case the only interesting budget set is $B = X = \{x_1, x_2\}$, so that the entire variation comes off the committee membership. The choice C here is a mapping from a subset of the set of observed committees $\mathcal{E} \subset 2^N \setminus \{\varnothing\}$, that may take one of only three values: $\{x_1\}$, $\{x_2\}$ or $\{x_1, x_2\}$. Of course, if committee members may change their votes arbitrarily based on either committee membership or the set of alternatives involved, not much could be done here. For this reason, I shall assume that voters are restricted to be strictly consistent with sincere voting.

It is clear that not every such committee choice structure would be rationalizable with sincere scoring. Crucially, the notion of sincere scoring studied here implies that each individual's votes are independent of the committee composition. Hence, if we ever observe that for two disjoint committees $S \cap T = \varnothing$ we have $C(S) = C(T) = x_i$ it must, indeed, follow that $C(S \cup T) = x_i$. This, property, introduced, for instance, in characterizations of scoring rules by Smith (1973) and Young (1975) is usually known as the *reinforcement* axiom. Clearly, reinforcement must be a necessary condition for the rationalizability here desired. But the scoring has an even stronger implication for the actual scores that committees assign to alternatives: the score difference between the alternatives must be added up if two disjoint committees are joined.

In fact, if sincere scoring is the rule used, the difference w between the scores assigned to x_1 and to x_2 by the committee S

$$w(S) = \sum_{i \in S} v_i^B(x_1) - \sum_{i \in S} v_i^B(x_2)$$

will define a (signed) measure on the finite measurable space $\left(N, 2^N\right)$, as long as one naturally sets $w(\varnothing) = 0$, since $w(S \cup T) = w(S) + w(T) - w(S \cup T)$ for any two committees $S, T \in 2^N$.

Unfortunately, we do not observe the actual scores or their differences, but only choices, which correspond to the sign of w. Defining $\mathcal{E}^* = \mathcal{E} \cup \varnothing$ it may be convenient to summarize our observations with a function $f : \mathcal{E}^* \to \{-1, 0, 1\}$ defined by the

$$f(S) = sign(w(S)) = \begin{cases} -1, \text{ if } C(S) = \{x_2\} \\ 0, \text{if } C(S) = \{x_1, x_2\} \text{ or } S = \varnothing \\ 1, \text{if } C(S) = \{x_1\} \end{cases}$$

This function f is, of course, non-additive. If, however, we can consistently with it assign individual vote differences $w\{j\}$ to each individual so that the committee differences defined as

$$sign(w(S)) = sign\left(\sum_{j \in S} w(\{j\})\right) = f(S)$$

, we shall obtain a scoring-based theory that would explain how the observed choice structure arose!

Fortunately, it turns out that this problem is closely related to well-established problems in utility theory. In fact, a very similar mathematical problem emerges if one considers the question of when could a binary relation "at least as likely as" over a finite states space be represented by a probability measure, which has been posed and solved by Kraft et al. (1959). Indeed, the following example they construct implies that the reinforcement alone, though necessary, is not sufficient for such a theory to be possible.

Example 1 *Suppose* $N = \{1, 2, 3, 4, 5\}$ *and* $f(\{4\}) = f(\{2, 3\}) = f(\{1, 5\}) = f(\{1, 3, 4\}) = 1$ *whereas* $f(\{1, 3\}) = f(\{1, 4\}) = f(\{3, 4\}) = f(\{2, 5\}) = -1$. *It can be checked that it is possible to complete this set of observed choices in a way that would not violate reinforcement. However, it is not hard to see that this set of choices is not consistent with sincere scoring, as it would imply that* $2w(\{1\}) + w(\{2\}) + 2w(\{3\}) + 2w(\{4\}) + w(\{5\})$ *is simultaneously positive and negative!*

Consequently, a stronger condition, which I shall call *strong reinforcement,* is required, which is analogous to strong additivity of Kraft et al. (1959). Following Fishburn (1986) it can be presented as follows. Consider two collections (of equal cardinality) of committees $\mathbf{S} = (S_1, S_2, ..., S_m)$ and $\mathbf{T} = (T_1, T_2, ..., T_m)$. Note, that an empty set is taken here as a possible committee and that a committee might be repeated several times within a collection. Denote as $n_j(\mathbf{S})$ the number of committees in the collection $\mathbf{S}$ that individual j is included in. We say that $\mathbf{S} \approxeq \mathbf{T}$ if for each individual $j \in N$ $n_j(\mathbf{S}) = n_j(\mathbf{T})$.

- The choice correspondence C satisfies *strongly reinforcement* if for each pair of committee collections $\mathbf{S}, \mathbf{T}$ such that $\mathbf{S} \approxeq \mathbf{T}$ if $f(S_i) > f(T_i)$ or $f(S_i) = f(T_i) = 0$ for $i = 1, 2, ..., m-1$ then not $f(S_m) > f(T_m)$.

It should be noted that strong additivity is indeed a strong property, which implies a number of desirable conditions of the choice structures. Thus, it can be easily seen to imply the reinforcement property itself. It turns out that, in fact, strong additivity characterizes choice structures that can be explained with sincere scoring.

Theorem 1 *A committee choice structure* $(\mathcal{E}, C(.,.))$ *may be generated by a scoring rule strictly consistent with rational preferences if and only if the choice structure satisfies strong reinforcement.*

Proof. The necessity part is straightforward, since if it were not the case, there would exist a pair of committee collections $\mathbf{S} \approxeq \mathbf{T}$ such that $f(S_i) > f(T_i)$.or $f(S_i) = f(T_i) = 0$ for all $i = 1, 2, ..., m-1$ and $f(S_m) > f(T_m)$ However, as $f(S_i) = sign(w(S_i)) = sign\left(\sum_{j \in S_i} w(\{j\})\right)$ it follows that $\sum_{j \in S_i} w(\{j\}) > \sum_{j \in T_i} w(\{j\})$ or $\sum_{j \in S_i} w(\{j\}) = \sum_{j \in T_i} w(\{j\}) = 0$ for $i = 1, 2, ...m-1$ and $\sum_{j \in S_m} w(\{j\}) > \sum_{j \in T_m} w(\{j\})$, which, if we some across the committees in each collection, in turn would imply that $\sum_{j \in N} n_j(\mathbf{S}) w(\{j\}) > \sum_{j \in N} n_j(\mathbf{T}) w(\{j\})$.

The proof of sufficiency closely follows that of Theorem 4.1 in Fishburn (1970). If all committees make the same choice, the theorem is trivially true, therefore, I shall henceforth assume that there exists at least one pair of committees $(S, T) \in \mathcal{E}^* \times \mathcal{E}^*$ such that $f(S) > f(T)$ Let $K \in \mathbb{N}$ be equal to the number of distinct committee pairs $S, T \in \mathcal{E}$ $(S \neq T)$ such that $f(S) > f(T)$ and $M \in \mathbb{Z}_+$ be equal to one half of the number of committee pairs $(S, T) \in \mathcal{E}^* \times \mathcal{E}^*$ such that $f(S) = f(T) = 0$. Note that the latter includes all committee pairs of the form $(S, \varnothing)$ and $(\varnothing, T)$. Clearly, $K + M \leq 2^n < \infty$.

For each committee S let the indicator function

$$1_S(j) = \begin{cases} 1 \text{ if } j \in S \\ 0 \text{ if } j \notin S \end{cases}$$

Clearly, if for each of the first $k = 1, 2..., K$ committee pairs S^k, T^k defined above we may write

$$\sum_{j=1}^{n} w(\{j\}) a_j^k > 0$$

and for each of the following $k = K+1, K+2, ...K+M$ committee pairs S^k, T^k we may write

$$\sum_{j=1}^{n} w(\{j\}) a_j^k = 0$$

where $a_j^k = (1_{S^k}(j) - 1_{T^k}(j)) \in \{-1, 0, 1\}$, the weights $\sum_{j=1}^{n} w(\{j\})$ may be interpreted as the individual vote difference assignment consistent with the observed choice structure (note, in particular, that this would imply that $\sum_{j=1}^{n} w(\{j\}) 1_S(j) = 0$ for every S such that $f(S) = 0$)

Suppose this is impossible. Then by Theorem 4.2 in Fishburn (1970), know as the Theorem of the Alternative, there must exist a collection of number $r_{k,}$ $k = 1, 2, ..., M+K$, such that the first K of these are non-negative and not all zero so that for every $j = 1, 2, ..., n$

$$\sum_{k=1}^{K+M} r_k a_j^k = 0$$

In fact, since all a_j^k are rational by construction, all r_k may be chosen to be integers. If for some $k > K$ there is an $r_k < 0$ one may replace a_j^k with $-a_j^k$ to make it positive (this is possible since if $f\left(S^k\right) = f\left(T^k\right)$ one may interchange S^k and T^k). Consider now two committee collections $\mathbf{S}$ and $\mathbf{T}$ such that each committee S^k is repeated r^k times in $\mathbf{S}$ and each committee T^k is repeated r^k times in $\mathbf{T}$. By construction the cardinality of each committee collection is equal to $\sum_{k=1}^{K+M} r_k$ and from the preceding equation it follows that the number of times each individual is included in committees in each collection is

$$n_j(\mathbf{S}) = \sum_{k=1}^{K+M} r_k 1_{S^k}(j) = \sum_{k=1}^{K+M} r_k 1_{T^k}(j) = n_j(\mathbf{T})$$

and, hence $\mathbf{S} \approxeq \mathbf{T}$. But by construction we have $f\left(S^k\right) \geq f\left(T^k\right)$ for all $k = 1, 2, ...K+M$, with the first K inequalities strict. Hence, the strong additivity of the committee choice structure is violated. QED

4 Three or more alternatives

The case of three or more alternatives does not allow for a reduction of the problem to an existence of a single measure on the committee space. Still, supposing that committees are making their decisions using scoring rules implies that each committee produces a ranking, represented by the score in question. Maintaining the strict consistency with sincere voting assumption of the previous section, I shall try to "reveal" as much as possible about individual votes. As in the set-up section, some of these results and concepts have been introduced in an earlier note of mine (Gomberg 2011).

I shall start by defining the direct preference revelation

- *Direct revelation.* For each $(B, S) \in \mathcal{E}$ a pair of nested binary relations $P^*_{B,S} \subset R^*_{B,S}$ on B is defined by

  (i) let $x \in C(B, S)$ then $xR^*_{B,S}y$ for any $y \in B$

  (ii) let $x \in C(B, S)$ and $y \notin C(B, S)$ for some $y \in B$ then $xP^*_{B,S}y$

This constitutes a record of direct preference revelation: if an alternative is chosen, it implies it received at least as high a score as any other feasible alternative and a strictly higher score than any feasible alternative not chosen..

The reinforcement axiom of Smith (1973) and Young (1975) provides us with a way of extending these revealed scoring relations, often even when a particular pair (B, S) is not in $\mathcal{E}$. This axiom states that if each of the two disjoint committees makes the same choice, the union of those two committees has to follow it. It is easy to see that every scoring rule would satisfy it: thus, for instance, if $C(\{a, o\}, \{1, 2\}) = C(\{a, o\}, \{3, 4\}) = \{a\}$ we may not have $C(\{a, o\}, \{1, 2, 3, 4\}) = \{o\}$. Furthermore, individual preference revelation is sometimes possible in this framework as well: if one ever observes an individual choosing an alternative when alone, this reveals his/her preference that would be unchanged even when the budget set changes. This motivates the following extension of the score revelation

- *Reinforcement.* The binary relations $P_{B,S} \subset R_{B,S}$ on B are defined by

  (i) xP^*y implies xPy, xR^*y implies xRy,

  (ii) For any $B \in 2^X \setminus \{\varnothing\}$ and any $S, T \in 2^N \setminus \{\varnothing\}$ such that $S \cap T = \varnothing$, $xR_{B,S}y$ and $xR_{B,T}y$ imply that $xR_{B,S\cup T}y$

(iii) For any $B \in 2^X \setminus \{\varnothing\}$ and any $S, T \in 2^N \setminus \{\varnothing\}$ such that $S \cap T = \varnothing$, $xP_{B,S}y$ and $xR_{B,T}y$ imply that $xP_{B,S\cup T}y$

(iv) For any $B \in 2^X \setminus \{\varnothing\}$ and any $S, T \in 2^N \setminus \{\varnothing\}$ such that $S \subset T (T \setminus S \neq \varnothing)$, $xP_{B,S}y$ and $yR_{B,T}x$ imply that $yP_{B,T\setminus S}x$

(v) For any $B \in 2^X \setminus \{\varnothing\}$ and any $S, T \in 2^N \setminus \{\varnothing\}$ such that $S \subset T (T \setminus S \neq \varnothing)$, $xR_{B,S}y$ and $yP_{B,T}x$ imply that $yP_{B,T\setminus S}x$

(vi) For any $B \in 2^X \setminus \{\varnothing\}$ and any $i \in N$, $xP_{B,\{i\}}y$ implies $xR_{D,\{i\}}y$ for all $D \in 2^X \setminus \{\varnothing\}$

The statements $xP_{B,S}y$ (respectively, $xR_{B,S}y$) may be understood as "x is revealed (directly or indirectly) to have obtained a higher (respectively, at least as high) score than y in a vote by a committee S over the budget set B". Of course, no matter how obtained, scoring revelation cannot be self-contradicting. Thus, for instance, if $C(\{a, o\}, \{1, 2\}) = C(\{a, o\}, \{3, 4\}) = \{a\}$ one may not have $C(\{a, o\}, \{1, 2, 3, 4\}) = \{o\}$. In fact, since the binary relations $R_{B,S}$ and $P_{B,S}$ refer to the number of votes, the relation should be transitive (if more people vote for x than for y and more people vote for y than for z more people should be voting for x than for z).

Of course, as noted above, one may be able to make inferences about individual preferences, for instance, from direct or indirect observations of singleton coalitions, by defining an individual revealed preference relation P_i as follows:

- *Individual preference revelation.*

  (i) If $xR_{B,\{i\}}y$ for some $B \in 2^X \setminus \{\varnothing\}$ then $xR_{D,\{i\}}y$ for any D s.t. $x, y \in D$

  (ii) if $xP_{B,\{i\}}y$ for some $B \in 2^X \setminus \{\varnothing\}$ then $xP_{D,\{i\}}y$ for any D s.t. $x, y \in D$

Note, that we may want to define the "usual" revealed preference relations xR_iy and xP_iy in this case.

Once the binary relations $R_{B,S}$ and $P_{B,S}$ are thus extended, we may formulate the following simple axiom:

Axiom 1 (Committee Axiom of Revealed Preference (CARP)) [1]
For any $B \in 2^X \setminus \{\varnothing\}$, any $S \in 2^N \setminus \{\varnothing\}$ and any $x_1, x_2, ...x_n \in B$, $x_1R_{B,S}x_2, x_2R_{B,S}x_3...x_{n-1}R_{B,S}x_n$ implies $\neg(x_nP_{B,S}x_1)$

[1] The naming suggestion for this axiom belongs to Norman Schofield

Example 2 *Consider the budget set* $B = \{a, b, c\}$ *and the four disjoint committees* S_1, S_2, S_3 *and* T. *Let* $C(B, S_1) = a$, $C(B, S_2) = b, C(B, S_3) = c$, $C(B, S_1 \cup T) = b, C(B, S_2 \cup T) = c, C(B, S_3 \cup T) = a$. *It is not hard to see that this implies that* $bP_{B,T}cP_{B,T}aP_{B,T}b$ *which, of course, contradicts Axiom 1: committee* T *should be giving alternative* b *a higher score than alternative* c*, alternative* c *a higher score than alternative* a*, and alternative* a *the higher score than alternative* b*, which is impossible.*

It should be noted that, taking $D = X$ we may observe that, with the individual revelation taken into account, CARP implies the usual Strong Axiom of Revealed Preference (SARP) for the individual preference revelation.

It is clear that CARP would have to hold if a committee of rational individuals is deciding by sincere votes using a scoring rule, since otherwise we'd have to accept either cycles in individual preferences or in group scores (as in the example above). Hence, the next result follows immediately from the construction.

Proposition 1 *A committee choice structure* $(\mathcal{E}, C(.,.))$ *may be generated by a scoring rule strictly consistent with rational preferences only if the implied* $R_{B,S}$ *and* $P_{B,S}$ *satisfy CARP for each* (B, S).

It should be stressed that this result provides only a necessary and not a sufficient condition for rationalizability with scoring. In fact, counterexamples to the converse are not hard to generate, as possibilities for indirect score revelation are by no means exhausted with application of reinforcement.

Example 3 *Consider a budget set* $B = \{a, b\}$ *and four committees* S_1, S_2,T_1 *and* T_2 *such that* $S_i \cap T_j = \varnothing$. *Suppose* $\{a\} = C(B, T_1) = C(B, T_2) = C(B, S_1 \cup T_1) = C(B, S_2 \cup T_2)$, *while* $\{b\} = C(B, S_1 \cup T_2) = C(B, S_2 \cup T_1)$. *Of course, if these decisions were arrived to by scoring, this would imply (by reinforcement) that both* S_i *would have to be choosing* b *as well. The votes of* S_1 *are sufficient to overturn the preference of* T_2*, but not of* T_1 *for* a*, so we can conclude that the advantage in votes that* T_1 *gives to* a *is strictly bigger than that given by* T_2. *However, the votes of* S_2 *overturn the choice of* T_1*, but not the choice of* T_2*, so the vote advantage of* a *in* T_2 *is strictly bigger than that in* T_1. *Clearly, this is impossible, unless the scores assigned are not independent of committee membership.*

The establishment of the exact conditions for rationalizability with scoring when there are more than 2 alternatives is, thus, for the moment, an open question.[2]

5 Revealed plurality with three or more alternatives

One rule of particular interest may be plurality voting (frequently known as the First Past the Post, FPTP): a scoring rule in which voters give a score of 1 to their top choice and 0 to everything else. Note, however, that due to the cardinality of the range, plurality may only be *weakly* consistent with preferences as long as there are three or more alternatives. In what follows, I shall say that plurality voting is "sincere", if agents assign their non-zero score to their top alternatives. Furthermore, for simplicity, I shall disallow individual indifference, as doing otherwise would raise issues with tie-breaking rules, which I would rather postpone for the moment. In other words, I shall attempt to construct rationalizations only with strict preferences.

Assuming, thus, sincerity and a rational preference profile preference profile $\succ = (\succ_1, \succ_2, ..., \succ_n)$ the plurality rule generates the following committee

[2] The above example might be addressed directly by forcing a further extension of binary relations $R_{B,S}$ and $P_{B,S}$. For instance, one might require the following property to hold:

- *Inbetweenness*

  (i) For any $B \in 2^X \setminus \{\varnothing\}$ and any $S, T \in 2^N \setminus \{\varnothing\}$ such that $S \cap T = \varnothing$, $xR_{B,S}yR_{B,S}zR_{B,S}w$, $zR_{B,T}wR_{B,T}xR_{B,T}y$ and $yR_{B,S\cup T}z$ then $xR_{B,S\cup T}w$

  (ii) Furthermore, if, in addition $xP_{B,S}y$ or $zP_{B,S}w$, or $zP_{B,T}w$ or $xP_{B,T}y$ then $xP_{B,S\cup T}w$

Even that, however, is not enough, as further counterexamples might be generated unless we require something along the lines of

- *Monotonicity*

  (i) For any $B \in 2^X \setminus \{\varnothing\}$ and any $S_1, S_2, T_1, T_2 \in 2^N \setminus \{\varnothing\}$ such that $S_i \cap T_j = \varnothing$, if $xR_{B,T_i}y, i = 1, 2; xR_{B,S_1\cup T_1}y, yP_{B,S_2\cup T_1}x$ and $yR_{B,S_1\cup T_2}x$ then $yP_{B,S_2\cup T_2}x$

  (ii) For any $B \in 2^X \setminus \{\varnothing\}$ and any $S_1, S_2, T_1, T_2 \in 2^N \setminus \{\varnothing\}$ such that $S_i \cap T_j = \varnothing$, if $xR_{B,T_i}y, i = 1, 2; xP_{B,S_1\cup T_1}y, yR_{B,S_2\cup T_1}x$ and $yR_{B,S_1\cup T_2}x$ then $yP_{B,S_2\cup T_2}x$

Whether further extension of revealed scores is required here remains to be established.

choice structure formally becomes

$$C^{pl}(B,S;\succsim) = \arg\max_{x\in B}\{i \in S : x \succsim_i y \forall y \in B\}$$

If a committee choice structure is such that for any $(B,S) \in \mathcal{E}$

$$C(B,S) = C^{pl}(B,S;\succ)$$

for some rational preference profile $\succ$I shall say that $\succ$ rationalizes $(\mathcal{E}, C(.,.))$ via plurality voting.

Overall, I am going to take a parallel approach to that taken in the previous section by defining two binary relation$P^{pl}_{B,S} \subset R^{pl}_{B,S}$, The statement $xP^{pl}_{B,S}y$ (respectively $xR^{pl}_{B,S}y$) may be interpreted as a record of revelation (whether direct, or indirect) of strictly more (respectively, as many as) members of the committee S having x as their top choice from a budget set B containing y in comparison with the number of committee members having y as their top choice. Informally, $xR^{pl}_{B,S}y$ may be read as "at least as many members of S vote for x as for y from the budget set B"

Since plurality is a scoring rule, there is a substantial commonality with the previous characterization. Thus, as above we shall have

- *Direct revelation*

  (i) let $x \in C(B,S)$ then $xR^{*pl}_{B,S}y$ for any $y \in B$

  (ii) let $x \in C(B,S)$ and $y \notin C(B,S)$ for some $y \in B$ then $xP^{*pl}_{B,S}y$

As the plurality rule is also a scoring rule, the reinforcement condition is essentially unchanged

- *Reinforcement.* The binary relations $P^{pl}_{B,S} \subset R^{pl}_{B,S}$ on B are defined by

  (i) $xP^{*pl}y$ implies $xP^{pl}y, xR^{*pl}y$ implies $xR^{pl}y$,

  (ii) For any $B \in 2^X \setminus \{\varnothing\}$ and any $S,T \in 2^N \setminus \{\varnothing\}$ such that $S \cap T = \varnothing$, $xR^{pl}_{B,S}y$ and $xR^{pl}_{B,T}y$ imply that $xR^{pl}_{B,S\cup T}y$

- +(iii) For any $B \in 2^X \setminus \{\varnothing\}$ and any $S,T \in 2^N \setminus \{\varnothing\}$ such that $S \cap T = \varnothing$, $xP^{pl}_{B,S}y$ and $xR^{pl}_{B,T}y$ imply that $xP^{pl}_{B,S\cup T}y$

  (iv) For any $B \in 2^X \setminus \{\varnothing\}$ and any $S,T \in 2^N \setminus \{\varnothing\}$ such that $S \subset T (T \setminus S \neq \varnothing)$, $xP^{pl}_{B,S}y$ and $yR^{pl}_{B,T}x$ imply that $yP^{pl}_{B,T\setminus S}x$

(v) For any $B \in 2^X \setminus \{\varnothing\}$ and any $S, T \in 2^N \setminus \{\varnothing\}$ such that $S \subset T (T \setminus S \neq \varnothing)$, $xR^{pl}_{B,S}y$ and $yP^{pl}_{B,T}x$ imply that $yP^{pl}_{B,T\setminus S}x$

Other conditions may be formulated for scoring rules, such as the in-betweenness and monotonicity mentioned above as necessary to deal with the counterexample in the previous section.

There are, however, a number of additional situations in which we could make inferences, assuming decisions are taken by plurality. Thus, for instance, if $X = \{a, o, b\}$ and $N = \{1, 2, 3, 4\}$ then the observation $C(\{a, o\}, \{1, 2, 3\}) = \{a\}$ implies that $C(\{a, o\}, \{1, 2, 3, 4\}) = \{b\}$ is impossible: adding (or, for that matter, subtracting) a single committee member may not fully overturn the strict plurality. At best, it would have to be $C(\{a, o\}, \{1, 2, 3, 4\}) = \{a, b\}$. This, in fact, is a key property of plurality vote as compared to other scoring rules. I shall therefore incorporate such deduction into my revealed plurality:

- *Continuity*

  (i) For any $B \in 2^X \setminus \{\varnothing\}$, any $S \in 2^N \setminus \{\varnothing\}$ and any $i \in N$, $xP^{pl}_{B,S}y$ implies $xR^{pl}_{B,S\cup\{i\}}y$

  (ii) For any $B \in 2^X \setminus \{\varnothing\}$, any $S \in 2^N \setminus \{\varnothing\}$ and any $i \in S$, $xP^{pl}_{B,S}y$ implies $xR^{pl}_{B,S\setminus\{i\}}y$

Another property of the relations follows from the fact that individual has only one positive vote, which, if he is sincere goes to his top alternative (which I have assumed to be unique):

- *Single individual vote*

  if $xP^{pl}_{B,\{i\}}y$ for some $y \in B$ then $xP^{pl}_{B,\{i\}}z$ for any $z \in B$, $z \neq x$

As before, information on individual choices, if available, is of particular interest: it provides information that is going to extend to all budget sets. However, there is substantial difference here with the *strictly consistent* scoring rule characterization in the previous section, as plurality vote is necessarily only weakly consistent with preferences: it fails to differentiate between the non-top choices. In particular, not much may be inferred from "indifference" in individual scores. Thus, I shall define individual preference revelation relations P_i on X as follows:

- *Individual preference revelation*

  If $xP^{pl}_{B,\{i\}}y$ for some $B \in 2^X \setminus \{\varnothing\}$ then $xP^{pl}_i y$

Information about individual preferences might be useful to further enrich the revealed plurality relations as follows:

- *Top choice*

  let $xP^{pl}_i y$ for every $y \in B \subset X$ and $x \in B$ then $xP_{B,\{i\}}y$ for every $y \in B$

  On the other hand, if it is clear that an alternative is not a top choice for an individual, including this individual may not improve its ranking with respect to any other outcomes.

- *Independence of dominated alternatives* (see Ching 1996 characterization of plurality)

  Let $yP^{pl}_i x$ for some $x, y \in B$ and some $i \in N$. Then for any $z \in B$ if $zR^{pl}_{B,S}x$ then $zR^{pl}_{B,S\cup\{i\}}x$ and if $zP^{pl}_{B,S}x$ then $zP^{pl}_{B,S\cup\{i\}}x$

As is the case with the characterization in the previous section, plurality vote should result in a transitive vote ranking for each B, S. Hence, as before, we have a "SARP"-like requirement for each possible pair B, S

Axiom 2 (Plurality Axiom of Revealed Preference (PARP)) *For any* $B \in 2^X \setminus \{\varnothing\}$, *any* $S \in 2^N \setminus \{\varnothing\}$ *and any* $x_1, x_2, ...x_n \in B, x_1R^{pl}_{B,S}x_2, x_2R^{pl}_{B,S}x_3...x_{n-1}R^{pl}_{B,S}x_n$ *implies* $\neg\left(x_nP^{pl}_{B,S}x_1\right)$

Furthermore, individual preference revelation should satisfy the actual SARP (as noted above, in the characterization of the scoring rules above this was implied by the CARP axiom itself). This is no longer the case here, as the individual vote over any set of alternatives only indicates the most preferred of these. Hence, the SARP will have to be assumed directly

Axiom 3 (Strong Axiom of Revealed Preference (SARP)) *For any* $i \in N$ $x_1P^{pl}_i x_2....P^{pl}_i x_n$ *implies* $\neg x_nP^{pl}_i x_1$

It is straightforward to see that

Proposition 2 *Consider a choice correspondence $C^{pl}(B, S; \succsim)$ generated by the plurality rule from some rational preference profile $\succsim$ then it satisfies PARP for each (B, S) and SARP for each $i \in N$.*

Unfortunately, as the following example shows, this condition is not sufficient for rationalizability with plurality

Example 4 *Let $X = \{a, b, c, d\}$ and $N = \{1, 2, 3\}$. Let $C(\{a, b\}, N) = \{a\}, C(\{a, b, c\}, N) = \{b\}$ and $C(\{a, b, c, d\}, N) = \{a\}$. Since there is no variation in the committee membership, this committee choice structure trivially satisfies Axioms 1 and 2. However, it can be easily seen that it cannot be rationalized via plurality. Indeed, for a to be the unique plurality choice from $\{a, b, c, d\}$ it has to have, at least, 2 affirmative votes. This, however, implies that it would have at least as many votes in the choice from $\{a, b, c\}$, so that the actual choice b must have, at least, 3 affirmative votes, which it would have preserved in the choice from $\{a, b\}$. Which, of course, implies that a should get, at least, 4 votes in this case, which are simply unavailable.*

6 Conclusions and further research

This paper introduces the notion of a committee choice structure and establishes a necessary and sufficient condition for such a choice structure to be rationalizable via scoring rules when the committees decide over the fixed set of two alternatives. For three or more alternatives, so. far it has been possible to establish a set of properties of committee choice structures that are necessary consequences of sincere scoring-based committee decisions It remains to see if this could be strengthened to a concise sufficient condition for rationalizability with scoring. An interesting further extension of the model would be to consider the consequences of particular scoring rules, such as plurality, approval or the Borda Count.

In terms of practical application, this paper provides conditions on the choice structures that would have to be violated for models more complicated then "sincere scoring" being possible to test. It should be noted that "sincerity", as defined here, is simply a statement that voters always maintain the same ranking of alternatives and do not change their scores based on the identity of other people in the committee they are a part of. Of course, if voters behave in this way, it is impossible to distinguish a possible case of "strategic" voting from simply following a fixed preference relation.

However, in many environments following one's preferences would imply a violation of sincerity as here defined. Thus, for instance, if voters have interdependent preferences with other committee members, or noisy signals about a common value of different alternatives, then they may change their behavior based on the identities of other committee members. Thus, if the committee, consisting of such voters, even if it uses a scoring rule, its choices would likely violate conditions derived here. This suggests, that the approach in this paper may be used to develop tests for presence of preference interdependence or common value in voting settings when only committee decisions and memberships are observed.

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