INSTITUTO TECNOLÓGICO AUTÓNOMO DE MÉXICO

CENTRO DE INVESTIGACIÓN ECONÓMICA

WORKING PAPER SERIES

Liquidity Contractions and Prepayment Risk on Collateralized Asset Markets

Miguel Angel Iraola
Instituto Tecnológico Autónomo de México

Juan Pablo Torres-Martínez
Universidad de Chile

August 2012
Working Paper12-04

Av. Camino Santa Teresa #930
México D.F., 10700
Tel. +52(55)5628-4197

LIQUIDITY CONTRACTIONS AND PREPAYMENT RISK ON COLLATERALIZED ASSET MARKETS

MIGUEL A. IRAOLA AND JUAN PABLO TORRES-MARTÍNEZ

Abstract. This paper presents a dynamic general equilibrium model with default and collateral requirements. In contrast with previous literature, our model allows for liquidity contractions and general prepayment specifications. We show that liquidity substantially affects credit and prepayment risks, and that different borrowers may follow differentiated payment strategies: whereas some pay, others prepay or default. The lack of liquidity increases debtors' willingness to continue paying, even thought prepayment cost could be higher than the collateral value. This mechanism rationalizes underwater mortgages. We prove existence of equilibrium, and provide a numerical example illustrating the main determinants of optimal payment strategies.

Keywords. Collateralized Asset Markets - Liquidity Constraints - Prepayment Risk
JEL Classification. D50, D52.

1. Introduction

The recent financial crisis of 2007-2009 has clearly revealed the central role played by financial markets liquidity and its interaction with credit and prepayment risks (cf., Geanakoplos et al. (2012)). These should be essential ingredients of modern macroeconomic theories. However, there is a lack of general equilibrium models identifying the major determinants of financial risks. This paper is an attempt to close this gap. We propose a dynamic general equilibrium model with default, collateral requirements, and liquidity contractions.

We extend Geanakoplos and Zame (1997, 2002, 2007) two-period general equilibrium model with default and collateral requirements to a three-period setting with long-lived securities. In this direction, Araujo, Páscoa and Torres-Martínez (2011) show equilibrium existence in an infinite-horizon economy with long-lived securities and sequential trading without imposing exogenous debt limits or transversality conditions (cf., Magill and Quinzii (1994, 1996), Hernandez and Santos (1996), Levine and Zame (1996), Araujo Páscoa and Torres-Martínez (2002), Kubler and Schmedders (2003)). A major divergence of our model with respect to previous literature is the incorporation of liquidity contractions modeled as a shrinkage of the set of available securities. Indeed, Araujo,

Date: August, 2012.
Miguel A. Iraola acknowledges financial support from CONACYT (Mexican Research Council) through project 81825. J.P.Torres-Martínez acknowledges financial support from CONICYT (Chilean Research Council) through Fondecyt project 1120294.

Páscoa and Torres-Martínez (2011) require the set of securities to be constant over time.[1] This modification drastically changes optimal agents' behavior.

We consider an economy with a variable set of credit contracts collateralized by durable goods which are seized in case of default.[2] Each credit contract is characterized by its emission node, coupon payment, prepayment rule, and collateral requirements. After the emission of a credit line, borrowers have the possibility to pay the coupon or close short positions by either delivering the collateral or prepaying. If the set of credit lines were constant, debtors would deliver the minimum between the market value of debt (i.e., prepayment cost) and the constituted collateral. However, liquidity contractions allow for differentiated optimal payment schemes across agents.

We assume that credits are financed by securitization of debts. Thus, each debt contract variety is securitized into a pass-through security delivering aggregate debtors payments. This financial structure allows to consider diverse Mortgage Backed Securities (MBS). As it is well known, the implicit yield-to-maturity of a MBS could be affected by credit and prepayment risks (cf., Becketti (1989)). Although some forward-looking specifications of prepayment rules are able to eliminate prepayment risks —for instance, rules defined as a high enough present value of future commitments—, our model is compatible with a great variety of prepayment specifications. For example, backward-looking prepayment rules, defined as the unpaid portion of a debt, are compatible with our approach.

The fourth quarter of 2011, over 22% of all residential properties with a mortgage (around 11.1 million) were underwater.[3] A mortgage is considered underwater if the borrower continues paying the coupons, even though the remaining value of the loan (prepayment cost) is higher than the underlying collateral. In a model without credit tightening, as in Araujo, Páscoa and Torres-Martínez (2011), underwater mortgages are not possible in equilibrium because borrowers optimally decide to default. However, in our model the lack of liquidity makes debtors more willing to pay coupons which could rationalize underwater mortgages. This result is in the spirit of the additional enforcement literature. However, we do not face the same problems in showing existence of equilibrium (see Ferreira and Torres-Martínez (2010)). In our model, collateral guarantees and commodity markets

[1]Although the extension of our model to an infinite horizon does not entail substantial technical difficulties, this three-period version of the model allows a neater presentation of our main results.

[2]Different additional enforcement mechanisms have been considered in the literature of general equilibrium with credit risk. The effect of utility penalties on payment behavior has been analyzed by Dubey, Geanakoplos and Shubik (1989, 2005), and Zame (1993); participation constraints have been considered by Kehoe and Levine (1993), Kocherlakota (1996), and Alvarez and Jermann (2000); bankruptcy mechanisms have been considered by Araujo and Páscoa (2002), Sabarwal (2003), and Poblete-Cazenave and Torres-Martínez (2012).

[3]According to CoreLogic data: http://www.corelogic.com/about-us/news/corelogic-reports-negative-equity-increase-in-q4-2011.aspx

clearing conditions induce endogenous upper bounds on debt positions. Thus, existence of equilibrium is proved without imposing additional debt constraints (cf., Hart (1975), Geanakoplos and Zame (1997, 2002, 2007), and Araujo Páscoa and Torres-Martínez (2002, 2011)).

We provide a numerical example illustrating all possible payment strategies in our model: payment, prepayment, and default. We show that different agents may adopt differentiated optimal payment decisions and discuss the effect of financial markets liquidity on these decisions. In particular, it is shown that underwater mortgages are a possible equilibrium outcome as a consequence of liquidity contractions. Moreover, the analysis of optimal payment decisions reveals that, independently of the existence of alternative credit opportunities, if the cost associated with closing a debt position is lower than the present value of commitments, then agents close this position by prepaying or defaulting. Additionally, agents whose optimal decisions are not affected by collateral constraints maintain short positions if the closing cost is higher than the present value of future commitments. However, these results are substantially modified in the presence of more favorable credit contracts. In fact, the existence of alternative credit opportunities may trigger agents' decision to close debts although the cost of this action could be higher than the present value of commitments.

The rest of the paper proceeds as follows: Section 2 sets out the model, notation and equilibrium definition. Section 3 provides a numerical example, Section 4 establishes existence of equilibrium, Section 5 characterizes optimal payment strategies, and finally we provide some concluding remarks.

2. Model

Information structure. We consider a dynamic economy $\mathcal{E}$ with three periods. There is uncertainty about the state of nature that will be realized, which belongs to a finite set S. The common and symmetric information available at period $t \in \{0, 1, 2\}$ is given by a partition of S, denoted by $\mathbb{F}_t$. We assume that there is no information at $t = 0$, i.e., $\mathbb{F}_0 = \{S\}$. Available information increases through time and economic agents are perfectly informed in the last period. That is, (i) $\mathbb{F}_{t+1}$ is as fine as $\mathbb{F}_t$, where $t \in \{0, 1\}$; and (ii) $\mathbb{F}_2 = \{\{s\} : s \in S\}$.

A node is a pair (t, σ), where $t \in \{0, 1, 2\}$ and $\sigma \in \mathbb{F}_t$. Let D be the set of nodes in the economy —or event-tree— and ξ_0 be the unique initial node. We denote by $t(\xi)$ the date associated with a node ξ, and by D_t the set of nodes dated t. We refer to $\mu = (t(\mu), \sigma_\mu)$ as a successor of $\xi = (t(\xi), \sigma_\xi)$, denoted by $\mu > \xi$, when both $t(\mu) > t(\xi)$ and $\sigma_\mu \subseteq \sigma_\xi$. Let ξ^+ be the set of immediate successor nodes of $\xi \in D$.

Physical markets. At each node in D there is a finite and ordered set of commodities, L, which are traded in spot markets and may suffer transformations through time. A bundle of commodities $v \in \mathbb{R}^L_+$ consumed at $\xi \in D$ is transformed into a bundle $Y_\mu v$ at each node $\mu \in \xi^+$, where Y_μ is a

$(L \times L)$-matrix with non-negative entries. Let $p_\xi = (p_{\xi,l}\,;\, l \in L) \in \mathbb{R}_+^L$ be the vector of spot prices at $\xi \in D$ and $p = (p_\xi\,;\, \xi \in D)$ be the process of commodity prices.

Financial instruments. At each $\xi \in D \setminus D_2$ a finite and ordered set $J(\xi)$ of collateralized credit contracts can be issued. Promises associated with $j \in J(\xi)$ are pooled into a pass-through security identified with the same subindex j (i.e., security j distributes payments made by borrowers of credit contract j). We denote by $q_\xi = (q_{\xi,j}\,;\, j \in J(\xi)) \in \mathbb{R}_+^{J(\xi)}$ the vector of unitary prices of credit contracts issued at $\xi \in D \setminus D_2$, and by $q = (q_\xi\,;\, \xi \in D \setminus D_2)$ the process of unitary prices. Without loss of generality, at each $\xi \in D \setminus D_2$, we identify the unitary price of a security $j \in J(\xi)$ with the unitary price of the associated credit contract.

Securities issued in the first period can be renegotiated. Hence, let $\pi_\mu = (\pi_{\mu,j}\,;\, j \in J(\xi_0)) \in \mathbb{R}_+^{J(\xi_0)}$ be the unitary resale price of securities at $\mu \in D_1$, and denote by $\pi = (\pi_\mu\,;\, \mu \in D_1)$ the process of pass-through resell prices. Let $\mathcal{P} := \mathbb{R}_+^{D\times L} \times \prod_{\xi \in D\setminus D_2} \mathbb{R}_+^{J(\xi)} \times \mathbb{R}_+^{D_1 \times J(\xi_0)}$ be the space of unitary commodity and financial prices (p, q, π).

Financial trading rules. The seller of one unit of credit contract $j \in J(\xi)$ receives at ξ an amount of resources $q_{\xi,j}$, is burdened to constitute a collateral bundle of commodities $C_{\xi,j} \in \mathbb{R}_+^L \setminus \{0\}$, and promises to pay a coupon $A_{\mu,j}(p, q, \pi)$ at each node $\mu > \xi$, with $A_{\mu,j} : \mathcal{P} \to \mathbb{R}_+$. It is assumed that borrowers hold and consume collateral guarantees.

At terminal nodes, since the only enforcement in case of default is the seizure of collateral, borrowers follow strategic default. That is, borrowers of one unit of a credit contract $j \in J(\xi_0)$ pay at $\mu \in D_2$ the minimum between the original promise $A_{\mu,j}(p, q, \pi)$ and the market value of the collateral guarantee $p_\mu C_{\mu,j}$, where $C_{\mu,j} := Y_\mu Y_{\mu^-} C_{\xi_0,j}$ and μ^- is the immediate predecessor node of μ. Analogously, given $\xi \in D_1$, borrowers of one unit of $j \in J(\xi)$ pay at each node $\mu \in \xi^+$ the minimum between $A_{\mu,j}(p, q, \pi)$ and $p_\mu C_{\mu,j}$, where $C_{\mu,j} := Y_\mu C_{\xi,j}$. Thus, at terminal nodes, lenders can perfectly foresight borrowers' payments. To shorten notation, given $\mu \in D_2$, let $R_{\mu,j}(p, q, \pi) := \min\{A_{\mu,j}(p, q, \pi), p_\mu C_{\mu,j}\}$ be the unitary payment of security $j \in J(\mu^-) \cup J(\xi_0)$ at node μ.

At intermediate nodes, heterogeneous payments across agents could be observed as a consequence of liquidity shrinkages. That is, at each $\xi \in D_1$, different borrowers of a credit contract $j \in J(\xi_0)$ may adopt different decisions: some of them pay, while others prepay or default on their promises. It is assumed that each credit line incorporates a prepayment rule. This rule specifies payments in order to reduce the amount of debt before terminal nodes. More precisely, borrowers of $j \in J(\xi_0)$ can reduce at $\xi \in D_1$ their short-positions in one unit by paying an amount of resources $B_{\xi,j}(p, q, \pi)$. The prepayment rule $B_{\xi,j} : \mathcal{P} \to \mathbb{R}_+$ satisfies $B_{\xi,j}(p, q, \pi) \geq A_{\xi,j}(p, q, \pi),\ \forall (p, q, \pi) \in \mathcal{P}$. We discuss the generality of our approach to prepayment rules at the end of this section.

Given $\xi \in D \setminus D_2$, buyers of one unit of pass-through security $j \in J(\xi)$ pay $q_{\xi,j}$, which entitles them to obtain a payment $N_{\mu,j}$ at each $\mu > \xi$. Unitary payments are endogenously determined and are such that, node by node, resources distributed to lenders of security j match borrowers' deliveries. Let $D^+ = \{(\mu, j) : \exists \xi \in D, (\mu > \xi) \wedge (j \in J(\xi))\}$ be the set of pairs (μ, j) such that μ is a node where security j could yield payments.

Households. There is a finite set of agents, denoted by H. Each agent $h \in H$ is characterized by a utility function $U^h : \mathbb{R}_+^{D\times L} \to \mathbb{R}$ and a commodity endowment process $w^h = (w^h_\xi\,;\, \xi \in D) \in \mathbb{R}_+^{D\times L}$. At $\xi \in D$, each agent $h \in H$ chooses autonomous consumption bundles $x^h_\xi \in \mathbb{R}^L_+$, i.e., a consumption in excess of the required collateral. Also, each agent h selects at $\xi \in D \setminus D_2$ a debt portfolio $\varphi^h_\xi = (\varphi^h_{\xi,j}; j \in J(\xi)) \in \mathbb{R}_+^{J(\xi)}$. For each intermediate node $\xi \in D_1$, $\varphi^{\alpha,h}_{\xi,j} \in [0, \varphi^h_{\xi_0,j}]$ denotes the position on debt contract $j \in J(\xi_0)$ that h honors and maintains open. Analogously, $\varphi^{\beta,h}_{\xi,j} \in [0, \varphi^h_{\xi_0,j}]$ is the part of agent h debt that is prepaid at $\xi \in D_1$. Thus, agent h defaults on $(\varphi^h_{\xi_0,j} - \varphi^{\alpha,h}_{\xi,j} - \varphi^{\beta,h}_{\xi,j})$ units of contract $j \in J(\xi_0)$ at $\xi \in D_1$.

Since borrowers consume collateral bundles, the total consumption at a node $\xi \in D$ is given by

$$c_\xi(x^h, \varphi^h, \varphi^{\alpha,h}) := \begin{cases} x^h_\xi + \sum\limits_{j\in J(\xi)} C_{\xi,j}\varphi^h_{\xi,j}, & \text{when } \xi = \xi_0; \\ x^h_\xi + \sum\limits_{j\in J(\xi)} C_{\xi,j}\varphi^h_{\xi,j} + \sum\limits_{j\in J(\xi_0)} C_{\xi,j}\varphi^{\alpha,h}_{\xi,j}, & \text{when } \xi \in D_1; \\ x^h_\xi, & \text{when } \xi \in D_2. \end{cases}$$

The vector $\theta^h_\xi := (\theta^h_{\xi,j}; j \in J(\xi_0) \cup J(\xi)) \in \mathbb{R}_+^{J(\xi_0)\cup J(\xi)}$ denotes the portfolio of passthrough securities of agent $h \in H$ at node $\xi \in D \setminus D_2$.

Given prices $(p, q, \pi) \in \mathcal{P}$ and unitary security payments $N := (N_{\xi,j}; (\xi, j) \in D^+) \in \mathcal{N} := \mathbb{R}_+^{D^+}$, the objective of each household $h \in H$ is to maximize utility by choosing a plan

$$(x^h, \theta^h, \varphi^h, \varphi^{\alpha,h}, \varphi^{\beta,h}) \in \mathcal{X} := \mathbb{R}_+^{D\times L} \times \prod_{\xi\in D\setminus D_2} \mathbb{R}_+^{J(\xi_0)\cup J(\xi)} \times \prod_{\xi\in D\setminus D_2} \mathbb{R}_+^{J(\xi)} \times \mathbb{R}_+^{D_1\times J(\xi_0)} \times \mathbb{R}_+^{D_1\times J(\xi_0)},$$

which satisfies the following constraints:

$$(\mathrm{B}_{\xi_0}) \qquad p_{\xi_0} c_{\xi_0}(x^h, \varphi^h, \varphi^{\alpha,h}) + \sum_{j\in J(\xi_0)} q_{\xi_0,j}\theta^h_{\xi_0,j} \le p_{\xi_0} w^h_{\xi_0} + \sum_{j\in J(\xi_0)} q_{\xi_0,j}\varphi^h_{\xi_0,j};$$

for all $\xi \in D_1$,

$$\begin{aligned} (\mathrm{B}_\xi) \qquad & p_\xi c_\xi(x^h, \varphi^h, \varphi^{\alpha,h}) + \sum_{j\in J(\xi)} q_{\xi,j}\theta^h_{\xi,j} + \sum_{j\in J(\xi_0)} \pi_{\xi,j}\theta^h_{\xi,j} \\ & \le p_\xi \left(w^h_\xi + Y_\xi c_{\xi_0}(x^h, \varphi^h, \varphi^{\alpha,h})\right) + \sum_{j\in J(\xi)} q_{\xi,j}\varphi^h_{\xi,j} + \sum_{j\in J(\xi_0)} \left(\pi_{\xi,j} + N_{\xi,j}\right)\theta^h_{\xi_0,j} \\ & \quad - \sum_{j\in J(\xi_0)} \left(A_{\xi,j}(p,q,\pi)\varphi^{\alpha,h}_{\xi,j} + B_{\xi,j}(p,q,\pi)\varphi^{\beta,h}_{\xi,j} + p_\xi C_{\xi,j}\left(\varphi^h_{\xi_0,j} - \varphi^{\alpha,h}_{\xi,j} - \varphi^{\beta,h}_{\xi,j}\right)\right); \end{aligned}$$

$$(\mathrm{S}_\xi) \qquad \varphi^{\alpha,h}_{\xi,j} + \varphi^{\beta,h}_{\xi,j} \le \varphi^h_{\xi_0,j}, \quad \forall j \in J(\xi_0);$$

and, for all $\xi \in D_2$,

$$(\mathrm{B}_\xi) \qquad p_\xi x^h_\xi \leq p_\xi \left(w^h_\xi + Y_\xi c_{\xi^-}(x^h, \varphi^h, \varphi^{\alpha,h})\right) + \sum_{j \in J(\xi_0) \cup J(\xi^-)} N_{\xi,j}\theta^h_{\xi^-,j}$$
$$- \left(\sum_{j \in J(\xi^-)} R_{\xi,j}(p,q,\pi)\varphi^h_{\xi^-,j} + \sum_{j \in J(\xi_0)} R_{\xi,j}(p,q,\pi)\varphi^{\alpha,h}_{\xi^-,j} \right).$$

Given $(p,q,\pi,N) \in \mathcal{P} \times \mathcal{N}$, the choice set of $h \in H$ —denoted by $\Gamma^h(p,q,\pi,N)$— is the collection of plans in $\mathcal{X}$ that satisfy budget constraints $(\mathrm{B}_\xi)_{\xi \in D}$ and portfolio restrictions $(\mathrm{S}_\xi)_{\xi \in D_1}$.

Definition. *An equilibrium for $\mathcal{E}$ is given by prices and unitary payments $(\overline{p}, \overline{q}, \overline{\pi}, \overline{N}) \in \mathcal{P} \times \mathcal{N}$ jointly with allocations $(\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h, \overline{\varphi}^{\alpha,h}, \overline{\varphi}^{\beta,h})_{h \in H} \in \mathcal{X}^H$ such that,*

(i) For each $h \in H$, $(\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h, \overline{\varphi}^{\alpha,h}, \overline{\varphi}^{\beta,h}) \in argmax\,\{U^h(z),\, z \in \Gamma^h(\overline{p}, \overline{q}, \overline{\pi}, \overline{N})\}$.
(ii) Asset markets are cleared,

$$\begin{aligned} \sum_{h \in H} \left(\overline{\theta}^h_{\xi,j} - \overline{\varphi}^h_{\xi,j}\right) &= 0, \quad \forall \xi \in D \setminus D_2,\, \forall j \in J(\xi). \\ \sum_{h \in H} \left(\overline{\theta}^h_{\mu,j} - \overline{\theta}^h_{\xi_0,j}\right) &= 0, \quad \forall \mu \in D_1,\, \forall j \in J(\xi_0). \end{aligned}$$

(iii) Physical markets are cleared,

$$\begin{aligned} \sum_{h \in H} c_{\xi_0}(\overline{x}^h, \overline{\varphi}^h, \overline{\varphi}^{\alpha,h}) &= \sum_{h \in H} w^h_{\xi_0}, \\ \sum_{h \in H} c_\xi(\overline{x}^h, \overline{\varphi}^h, \overline{\varphi}^{\alpha,h}) &= \sum_{h \in H} \left(w^h_\xi + Y_\xi c_{\xi^-}(\overline{x}^h, \overline{\varphi}^h, \overline{\varphi}^{\alpha,h})\right), \quad \forall \xi > \xi_0. \end{aligned}$$

(iv) Security payments are compatible with deliveries,

$$\overline{N}_{\xi,j} \sum_{h \in H} \overline{\theta}^h_{\xi_0,j} = \sum_{h \in H} \left(A_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi})\overline{\varphi}^{\alpha,h}_{\xi,j} + B_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi})\overline{\varphi}^{\beta,h}_{\xi,j} + \overline{p}_\xi C_{\xi,j}\overline{\varphi}^{\gamma,h}_{\xi,j} \right),\ \forall \xi \in D_1,\, \forall j \in J(\xi_0);$$

$$\overline{N}_{\mu,j} \sum_{h \in H} \overline{\theta}^h_{\xi_0,j} = R_{\mu,j}(\overline{p}, \overline{q}, \overline{\pi}) \sum_{h \in H} \overline{\varphi}^{h,\alpha}_{\mu^-,j}, \quad \forall \mu \in D_2,\, \forall j \in J(\xi_0);$$

$$\overline{N}_{\mu,j} = R_{\mu,j}(\overline{p}, \overline{q}, \overline{\pi}), \quad \forall \mu \in D_2,\, \forall j \in J(\mu^-);$$

where, for each $(h, \xi, j) \in H \times D_1 \times J(\xi_0)$, $\overline{\varphi}^{\gamma,h}_{\xi,j} := \overline{\varphi}^h_{\xi_0,j} - \overline{\varphi}^{\alpha,h}_{\xi,j} - \overline{\varphi}^{\beta,h}_{\xi,j}$ denotes default by agent h on debt contract j at node ξ.

Notice that, equilibrium existence could easily be proved if security prices and payments were zero at each node. Indeed, any pure spot commodity market equilibrium is an equilibrium for our financial economy. However, when credit lines involve non-zero promises and collateral do not fully depreciate through time, it is natural to expect positive deliveries for traded contracts (cf., Steinert and Torres-Martínez (2007)). Hence, in Section 4 below we prove the existence of a *non-trivial*

equilibrium, i.e., an equilibrium such that, for some $\xi \in D \setminus D_2$, there exists $j \in J(\xi)$ such that $(N_{\mu,j})_{\mu>\xi} \neq 0$.

Security payments as the mean value of borrowers' deliveries.

Given an equilibrium $\left((\overline{p}, \overline{q}, \overline{\pi}, \overline{N}); (\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h, \overline{\varphi}^{\alpha,h}, \overline{\varphi}^{\beta,h})_{h\in H}\right) \in \mathcal{P} \times \mathcal{N} \times \mathcal{X}^H$, assume that credit contract $j \in J(\xi_0)$ is traded. Since at each $\xi \in D_1$, payment, prepayment and default rates are given by

$$\tau^p_{\xi,j} := \frac{\sum\limits_{h\in H} \overline{\varphi}^{\alpha,h}_{\xi,j}}{\sum\limits_{h\in H} \overline{\varphi}^{h}_{\xi_0,j}}; \qquad \tau^{pp}_{\xi,j} := \frac{\sum\limits_{h\in H} \overline{\varphi}^{\beta,h}_{\xi,j}}{\sum\limits_{h\in H} \overline{\varphi}^{h}_{\xi_0,j}}; \qquad \tau^d_{\xi,j} := \frac{\sum\limits_{h\in H} \left(\overline{\varphi}^{h}_{\xi_0,j} - \overline{\varphi}^{\alpha,h}_{\xi,j} - \overline{\varphi}^{\beta,h}_{\xi,j}\right)}{\sum\limits_{h\in H} \overline{\varphi}^{h}_{\xi_0,j}},$$

unitary security payments can be rewritten as a weighted mean of borrowers' deliveries. That is, $\overline{N}_{\xi,j} = \tau^p_{\xi,j} A_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi}) + \tau^{pp}_{\xi,j} B_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi}) + \tau^d_{\xi,j} \overline{p}_\xi C_{\xi,j}$. Additionally, at each terminal node $\mu \in \xi^+$, we have $\overline{N}_{\mu,j} = (1 - \tau^{pp}_{\xi,j} - \tau^d_{\xi,j}) R_{\mu,j}(\overline{p}, \overline{q}, \overline{\pi})$, implying that three forces could make security payments lower than coupon values: previous and current rates of default, jointly with prepayment risk.

Positive margins between collateral and credit values.

Let $\left((\overline{p}, \overline{q}, \overline{\pi}, \overline{N}), (\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h, \overline{\varphi}^{\alpha,h}, \overline{\varphi}^{\beta,h})_{h\in H}\right) \in \mathcal{P} \times \mathcal{N} \times \mathcal{X}^H$ be a non-trivial equilibrium. As in Geanakoplos and Zame (1997, 2002, 2007), under strict monotonicity of preferences, the following non-arbitrage condition holds: for each $\xi \in D \setminus D_2$ and each debt contract $j \in J(\xi)$, the collateral value is greater than the amount of credit, i.e., $\overline{p}_\xi C_{\xi,j} - \overline{q}_{\xi,j} > 0$. Indeed, if this condition is not satisfied, agents could take advantage of an unlimited arbitrage opportunity. They may increase their utility by increasing the short position on contract j issued at ξ, buying the associated collateral bundle with the borrowed resources, and defaulting at successor nodes $\mu \in \xi^+$. The existence of this arbitrage opportunity contradicts the optimality of individual plans.

Examples of debt contracts and prepayment rules.

Our model is compatible with a great variety of coupon and prepayment specifications. Indeed, to ensure the existence of non-trivial equilibria we only need the continuity of these functions (see Section 4). Thus, given prices $(p, q, \pi) \in \mathcal{P}$ such that $p \gg 0$, our framework includes the following characterizations of a debt contract $j \in J(\xi_0)$:

(a) *Promises and prepayments in real terms.* For each $\mu > \xi_0$, define $A_{\mu,j}(p, q, \pi) := p_\mu a_{\mu,j}$ such that $a_{\mu,j} \in \mathbb{R}^L_+$. Then, the coupon value coincides with the market value of a given commodity bundle. Analogously, for each $\mu \in D_1$, prepayment rules can be specified in real terms as $B_{\mu,j}(p, q, \pi) = p_\mu b_{\mu,j}$, where $b_{\mu,j} \geq a_{\mu,j}$.

(b) *Fixed interest rates and backward-looking prepayment.* Suppose that,

$$A_{\mu,j}(p, q, \pi) = \frac{q_{\xi_0,j}}{(d + d^2)} \frac{p_\mu a}{p_{\xi_0} a}, \quad \forall \mu > \xi_0,$$

where $d = \frac{1}{(1+i)}$, $i > 0$, and the bundle $a \in \mathbb{R}^L_{++}$ is used to compute a price index, i.e., $\frac{p_\mu a}{p_\xi a}$ intended to measure the purchase power variation between $\xi \in D$ and $\mu \in \xi^+$. In this case, borrowers who honor their commitments pay a real interest i per period. At each $\xi \in D_1$, the prepayment rule is defined as

$$B_{\xi,j}(p,q,\pi) := (1+i) q_{\xi_0,j} \frac{p_\mu a}{p_{\xi_0} a},$$

which satisfies the condition $B_{\xi,j}(p,q,\pi) > A_{\xi,j}(p,q,\pi)$. Essentially, $(B_{\xi,j}; \xi \in D_1)$ is a backward-looking prepayment rule and, therefore, after the payment of the coupon at $\xi \in D_1$, the cost of prepaying a debt $B_{\xi,j}(p,q,\pi) - A_{\xi,j}(p,q,\pi)$ is equal to the unpaid portion of the loan's face value $\frac{(1+i)}{(2+i)} q_{\xi_0,j}$ adjusted by the price index.[4]

Finally, a simple modification of this example allows to consider nominal debt contracts,

$$(A_{\mu,j}, B_{\xi,j})(p,q,\pi) = \left(\frac{q_{\xi_0,j}}{(d+d^2)}, \frac{q_{\xi_0,j}}{d} \right), \quad \forall \mu > \xi_0, \forall \xi \in D_1.$$

(c) *Forward-looking prepayment rules.* Some financial instruments avoid prepayment risk specifying a forward-looking prepayment cost. Thus, borrowers who want to close debts before terminal nodes are required to deliver the present value of promises. That is, strictly positive discount factors $(\rho(\mu); \mu \in D_2)$ are specified such that, the prepayment cost $B_{\xi,j}(p,q,\pi)$ at node $\xi \in D_1$ is given by either $A_{\xi,j}(p,q,\pi) + \sum_{\mu \in \xi^+} \rho(\mu) A_{\mu,j}(p,q,\pi)$ or $A_{\xi,j}(p,q,\pi) + \sum_{\mu \in \xi^+} \rho(\mu) R_{\mu,j}(p,q,\pi)$. In the former case, the prepayment rule does not take into account that borrowers may default at terminal nodes and, therefore, induces relatively more costly prepayment values compared with the latter.

Discount factors could be exogenously determined to ensure a lower bound for investment returns, even when all borrowers prepay. Alternatively, future payments could be discounted considering idiosyncratic characteristics of potential borrowers, with the aim of limiting prepayment risk. To this end, it is sufficient to ensure that agents are more impatient than the implicit inter-temporal discount induced by the financial contract (see Section 5 for a description of borrowers' behavior in terms of their idiosyncratic discount factors).

---

[4]Functions $(A_{\mu,j})_{\mu > \xi_0}$ and $(B_{\xi,j})_{\xi \in D_1}$ are not well defined for $p_{\xi_0} = 0$. However, by means of a lower bound for first period commodity prices, and after mild modifications, we make the specifications of coupon and prepayment rules above compatible with our equilibrium existence result (see Section 4).

Equilibrium prices can be normalized to satisfy $\|p_{\xi_0}\|_\Sigma + \|q_{\xi_0}\|_\Sigma = 1$ and, under strict monotonicity of preferences, the following non-arbitrage conditions hold: $p_{\xi_0} C_{\xi_0,k} > q_{\xi_0,k}$, $\forall k \in J(\xi_0)$. Therefore, adding the latter non-arbitrage inequality across assets we get $\|p_{\xi_0}\|_\Sigma > \Upsilon_{\xi_0} := \left(1 + \left(\max_{l \in L} \sum_{j \in J(\xi_0)} C_{\xi_0,j,l}\right)\right)^{-1}$.

Hence, in equilibrium, the following continuous functions coincide with $(A_{\mu,j})_{\mu > \xi_0}$ and $(B_{\xi,j})_{\xi \in D_1}$,

$$\left(\widehat{A}_{\mu,j}, \widehat{B}_{\xi,j}\right)(p,q,\pi) := q_{\xi_0,j} \left(\frac{1}{(d+d^2)} \Theta(\mu), \frac{1}{d} \Theta(\xi) \right), \quad \forall \mu > \xi_0, \forall \xi \in D_1,$$

where for each $\nu \in D$, $\Theta(\nu) := \min \left\{ \frac{p_\nu a}{p_{\xi_0} a}, \frac{p_\nu a}{(\min_{l \in L} a_l)\, \Upsilon_{\xi_0}} \right\}$.

3. Heterogeneous Behavior under Liquidity Constraints

In our model, different borrowers of identical credit contracts may exhibit heterogenous optimal payment strategies. The objective of this section is to illustrate this possibility. Thus, by means of an example, we show that in equilibrium we can observe: (i) prepayment of debt in presence of cheaper credit options; (ii) prepayment without alternative access to credit; (iii) payment of promises as a consequence of the absence of liquidity; (iv) default on the original promises.

Before presenting our numerical example, assuming strict monotonicity of preferences, we would like to highlight some simple situations where debtor's optimal payment strategy is uniform across agents.

Terminal nodes. All borrowers of a credit contract choose the same optimal decisions in the final period. They honor their commitments only if promises are lower than the collateral value.

Low collateral value at intermediate nodes. Given a node $\xi \in D_1$ and a security $j \in J(\xi_0)$, suppose that the collateral value is lower than the coupon value, $p_\xi C_{\xi,j} < A_{\xi,j}(p,q,\pi)$. Then, the optimal strategy is to default because the collateral bundle could be consumed at a lower cost by defaulting and buying back the collateral bundle.

Notice that, given prices $(p,q,\pi) \in \mathcal{P}$, prepayment and default on a contract $j \in J(\xi_0)$ coexist at a node $\xi \in D_1$ only when these strategies cost the same, $B_{\xi,j}(p,q,\pi) = p_\xi C_{\xi,j}$. Indeed, both decisions finalize the contractual commitment and, thus, borrowers who want to close the contract before terminal nodes will always choose the cheapest strategy. Moreover, if $B_{\xi,j}(p,q,\pi) \neq p_\xi C_{\xi,j}$, some agents could pay while others close the position at ξ only when

$$A_{\xi,j}(p,q,\pi) < B_{\xi,j}(p,q,\pi) < p_\xi C_{\xi,j} \quad \text{or} \quad A_{\xi,j}(p,q,\pi) < p_\xi C_{\xi,j} < B_{\xi,j}(p,q,\pi).$$

In the first case, some borrowers of j may pay the coupon while others prepay. In the second case, an underwater mortgage, some borrowers may default while others honor the promise maintaining the short position. We illustrate these possibilities in the following example.

Example. Assume that there is uncertainty only between $t = 0$ and $t = 1$. In $t = 1$ there are three states of nature $\{u, m, d\}$. Thus, let $D = \{0, u, m, d, u^+, m^+, d^+\}$ be the event-tree. There is only one commodity in the economy, which appreciates 50% between periods $t = 0$ and $t = 1$ if nodes $\{u, m\}$ are reached, depreciates a 13/22 when node d occurs, and it is perfectly durable between periods $t = 1$ and $t = 2$. At each node, the commodity price is normalized to one.

Credit contracts are issued at nodes $\{0, m\}$ and are securitized into pass-through securities. One unit of credit contract j_0 issued at $\xi = 0$ delivers q_{0,j_0} to the borrower, which is burdened to constitute a collateral of $C_{0,j_0} = 11/4$ and has the commitment to pay coupons $A_{\xi,j_0} = 1$ at nodes

$\xi \neq \{0, m^+\}$ and $A_{\xi,j_0} = 2$ at node $\xi = m^+$. The constituted collateral must be maintained through the duration of the contract. Borrowers may prepay their debt at nodes $\xi \in \{u, m, d\}$ delivering an amount of commodity B_{ξ,j_0}, where $(B_{u,j_0}, B_{m,j_0}, B_{d,j_0}) = (5/4,\ 3/2,\ 5/4)$. On the other hand, one unit of debt contract j_m issued at node $\xi = m$ delivers q_{m,j_m} to borrowers, who must constitute a collateral $C_{m,j_m} = 33/8$ and commit to pay a coupon $A_{\xi,j_m} = 1$ at node $\xi = m^+$.

Agents can invest on securities associated with the pooling of credit contracts. The security associated with credit contract j_0 is negotiated at all nodes in periods $t \in \{0, 1\}$ and distributes payments made by borrowers. Unitary payments of security j_0 at node $\xi > 0$ is denoted by N_{ξ,j_0}. The security associated with credit contract j_m is negotiated only at node m and delivers N_{m^+,j_m} at node $\xi = m^+$.

Agents $h \in \{A, B, C\}$ are characterized by the following utility functions and endowments,

$$\begin{aligned}
U^A(x_0, x_u, x_m, x_d, x_{u^+}, x_{m^+}, x_{d^+}) &= x_0 + \frac{3}{24}x_u + \frac{3}{24}x_m + \frac{12}{24}x_d + \frac{12}{96}x_{u^+} + \frac{12}{96}x_{m^+} + \frac{48}{96}x_{d^+};\\
(w_0^A, w_u^A, w_m^A, w_d^A, w_{u^+}^A, w_{m^+}^A, w_{d^+}^A) &= (3/2, 0, 0, 0, 0, 0, 0);\\
\\
U^B(x_0, x_u, x_m, x_d, x_{u^+}, x_{m^+}, x_{d^+}) &= x_0 + \frac{2}{24}x_u + \frac{2}{24}x_m + \frac{8}{24}x_d + \frac{1}{96}x_{u^+} + \frac{1}{96}x_{m^+} + \frac{4}{96}x_{d^+};\\
(w_0^B, w_u^B, w_m^B, w_d^B, w_{u^+}^B, w_{m^+}^B, w_{d^+}^B) &= (2, 1, 1, 1, 0, 0, 1);\\
\\
U^C(x_0, x_u, x_m, x_d, x_{u^+}, x_{m^+}, x_{d^+}) &= x_0 + \frac{1}{24}x_u + \frac{1}{24}x_m + \frac{4}{24}x_d + \frac{4}{96}x_{u^+} + \frac{4}{96}x_{m^+} + \frac{16}{96}x_{d^+};\\
(w_0^C, w_u^C, w_m^C, w_d^C, w_{u^+}^C, w_{m^+}^C, w_{d^+}^C) &= (2, 1, 1, 1, 0, 0, 1).
\end{aligned}$$

Agents choose allocations of consumption and financial positions to maximize utility subject to budget constraints and portfolio restrictions defined in the previous section. After normalizing commodity prices to one at each node, an equilibrium for this economy is given by prices and payments

$$[(q_{0,j_0}, q_{m,j_m}); (\pi_{u,j_0}, \pi_{m,j_0}, \pi_{d,j_0})] = [(3/4,\ 1/2); (1/4, 0,\ 1/4)]\,;$$

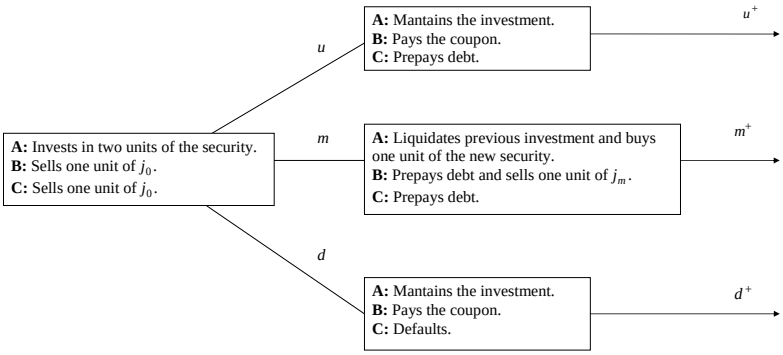
$$\left[(N_{u,j_0}, N_{m,j_0}, N_{d,j_0}, N_{u^+,j_0}, N_{m^+,j_0}, N_{d^+,j_0}); N_{m^+,j_m}\right] = [(9/8,\ 3/2,\ 17/16,\ 1/2,\ 0,\ 1/2); 1]\,.$$

jointly with consumption allocations

$$\begin{aligned}
(x_0^A, x_u^A, x_m^A, x_d^A, x_{u^+}^A, x_{m^+}^A, x_{d^+}^A) &= (0, 9/4,\ 10/4,\ 17/8,\ 13/4,\ 7/2,\ 25/8);\\
(x_0^B, x_u^B, x_m^B, x_d^B, x_{u^+}^B, x_{m^+}^B, x_{d^+}^B) &= (11/4,\ 33/8,\ 33/8,\ 9/8,\ 25/8,\ 25/8,\ 9/8);\\
(x_0^C, x_u^C, x_m^C, x_d^C, x_{u^+}^C, x_{m^+}^C, x_{d^+}^C) &= (11/4,\ 31/8,\ 29/8, 1,\ 31/8,\ 29/8, 2);
\end{aligned}$$

and financial positions described in the figure below.[5]

[5]The individual optimality of these allocations has been verified through a simplex algorithm.



As utility functions are linear in consumption, marginal rates of substitution between two immediate successor nodes are measures of individual impatience. In this sense, agent A is relatively patient. Moreover, as agent $A's$ endowment is concentrated at $t = 0$, A decides to invest in the first period. Agent B, who is more impatient than A between $t = 0$ and $t = 1$, and the most impatient consumer between periods $t = 1$ and $t = 2$, prefers to borrow at $t = 0$. Agent C, who is as patient as A between the last two periods, is the most impatient agent between periods $t = 0$ and $t = 1$ and, therefore, borrows resources at $t = 0$ to anticipate consumption.

However, between periods $t = 1$ and $t = 2$, B is more impatient than C. Therefore, at node u, where both borrowers could prepay their debts, B decides to pay the coupon and C prepays. Hence, even though agents have enough resources to prepay their debts, this decision depends on preferences and endowments. Furthermore, if there are more convenient borrowing options, the most impatient agents may prepay their debts and make use of these alternative credit instruments. For instance, at node m, agent B prepays and issues the new credit contract.

We would like to highlight that agents do not necessarily default on their debt when the collateral value is lower than the prepayment value (underwater mortgage). This decision depends on financial markets' liquidity and debtors' wealth. For instance, since agent B is impatient, prefers to pay coupons at d and d^+ rather than to close the contract by delivering the collateral guarantee at d. □

It could seem that the results of the example above crucially depend on the absence of rental markets for the consumption good. However, this is not the case. For instance, assume the existence

of a rental contract at node d, which specifies a rental price 65/73, and assume that rented goods' depreciation is 58/73 (we allow higher depreciation of the durable good when it is not consumed by its owner). In this case, our original result is not altered. First, from the lender's point of view, the rate of return on renting the consumption good $\left(\frac{15/73}{1-65/73} = 15/8\right)$ is lower than the rate of return provided by the existing security $\left(\frac{1/2}{1/4} = 2\right)$. Second, from the borrower's point of view, the renting price is too high. Therefore, it is preferable to maintain the original financial positions.

4. Equilibrium Existence

Although our model allows for incomplete financial markets and real assets, it is possible to show the existence of a non-trivial equilibrium. As in the seminal model of Geanakoplos and Zame (1997, 2002, 2007), commodity markets feasibility induce endogenous upper bounds on debt positions. Therefore, market feasible financial positions are bounded and the economy can be compactified to find an equilibrium allocation as a Cournot-Nash equilibrium of a generalized game.

Theorem. *An economy $\mathcal{E}$ that satisfies the following assumptions has a non-trivial equilibrium.*

(A1) *For each $h \in H$, U^h is continuous, strictly increasing, and strictly quasi-concave.*

(A2) *For each $h \in H$, $(W^h_\xi : \xi \in D) \in \mathbb{R}^{D\times L}_{++}$, with $W^h_{\xi_0} := w^h_{\xi_0}$ and $W^h_\xi := w^h_\xi + Y_\xi W^h_{\xi^-}$, $\forall \xi > \xi_0$.*

(A3) *Given $(\xi, j) \in D^+$, $A_{\xi,j} : \mathcal{P} \to \mathbb{R}_+$ is continuous.*

(A4) *Given $(\xi, j) \in D_1 \times J(\xi_0)$, $B_{\xi,j}$ is continuous and satisfies $B_{\xi,j}(\cdot) \geq A_{\xi,j}(\cdot)$.*

(A5) *There exist $\xi \in D$ and $j \in J(\xi)$ such that, for each commodity price $p \in \mathbb{R}^{D\times L}_{++}$ there is a successor node $\mu \in \xi^+$ for which $\min\{A_{\mu,j}(p,\cdot), \|Y_\mu C_{\mu,j}\|_\Sigma\} > 0$.*

Proof. We construct a non-trivial equilibrium for our economy as a Nash equilibrium of a generalized game $\mathcal{G}$ where abstract players choose prices and security payments, and agents maximize objective functions in truncated budget sets.

Spaces of strategies. In the generalized game, feasible commodity and asset prices are restricted to $\Delta := \prod_{\xi\in D} \Delta_\xi$, where for each node $\xi \in D \setminus D_2$, $\Delta_\xi := \left\{v \in \mathbb{R}^L_+ \times \mathbb{R}^{J(\xi)\cup J(\xi_0)}_+ : \|v\|_\Sigma = 1\right\}$, and for each terminal node $\xi \in D_2$, $\Delta_\xi := \left\{v \in \mathbb{R}^L_+ : \|v\|_\Sigma = 1\right\}$. Additionally, our equilibrium definition guarantees that there exists $\Omega > 0$ such that, each $(x^h, \theta^h, \varphi^h, \varphi^{\alpha,h}, \varphi^{\beta,h})_{h\in H} \in \mathcal{X}^H$ satisfying $(\mathrm{S}_\xi)_{\xi\in D_1}$ and market clearing conditions (ii)-(iii) is bounded from above by $\Omega(1,\ldots,1) \in \mathcal{X}^H$. Let $\mathcal{X}(\Omega)$ be the collection of allocations in $\mathcal{X}$ lower than or equal to 2Ω. Finally, since Δ is compact, Assumptions (A3)-(A4) and condition (iv) in the equilibrium definition guarantee that unitary security payments associated with traded debt contracts are bounded. Thus, there exists $\Phi > 0$ such that, for each traded debt contract j we have $N_{\mu,j} < \Phi, \forall \mu \in D : (\mu, j) \in D^+$.

Hence, for each security $j \in J(\xi)$, given $\mu > \xi$ it is assumed that $N_{\mu,j} \in \mathcal{N}_{\mu,j}(\Phi) := [0, \Phi]$. Let $\mathcal{N}(\Phi) := \prod_{(\mu,j)\in D^+} \mathcal{N}_{\mu,j}(\Phi)$.

Players characterization. The game $\mathcal{G}$ has a finite number of players whose objectives are:

(i) Given $((p, q, \pi), N) \in \Delta \times \mathcal{N}(\Phi)$, each agent $h \in H$ maximizes U^h in $\Gamma^h(p, q, \pi, N) \cap \mathcal{X}(\Omega)$.

(ii) Given $(x^h, \theta^h, \varphi^h, \varphi^{\alpha,h}, \varphi^{\beta,h})_{h\in H} \in \mathcal{X}(\Omega)^H$,

- A player chooses $(p_{\xi_0}, q_{\xi_0}) \in \Delta_{\xi_0}$ to maximize

$$p_{\xi_0} \sum_{h\in H} \left(c_{\xi_0}(x^h, \varphi^h, \varphi^{\alpha,h}) - w^h_{\xi_0}\right) + q_{\xi_0} \sum_{h\in H} \left(\theta^h_{\xi_0} - \varphi^h_{\xi_0}\right).$$

- For each $\xi \in D_1$, a player chooses $(p_\xi, q_\xi, \pi_\xi) \in \Delta_\xi$ to maximize

$$\sum_{h\in H} \left(p_\xi \left(c_\xi(x^h, \varphi^h, \varphi^{\alpha,h}) - w^h_\xi - Y_\xi c_{\xi_0}(x^h, \varphi^h, \varphi^{\alpha,h})\right) + q_\xi \left(\theta^h_\xi - \varphi^h_\xi\right) + \pi_\xi \left(\theta^h_\xi - \theta^h_{\xi_0}\right)\right).$$

- For each $\xi \in D_2$, a player chooses $p_\xi \in \Delta_\xi$ to maximize

$$p_\xi \sum_{h\in H} \left(c_\xi(x^h, \varphi^h, \varphi^{\alpha,h}) - w^h_\xi - Y_\xi c_{\xi^-}(x^h, \varphi^h, \varphi^{\alpha,h})\right).$$

(iii) Given $\left((p, q, \pi), (x^h, \theta^h, \varphi^h, \varphi^{\alpha,h}, \varphi^{\beta,h})_{h\in H}\right) \in \Delta \times \mathcal{X}(\Omega)^H$,

- For each $(\mu, j) \in D_1 \times J(\xi_0)$, a player chooses $N_{\mu,j} \in [R_{\mu,j}(p, q, \pi), \Phi]$ to maximize

$$-\left(N_{\mu,j} \sum_{h\in H} \varphi^h_{\xi_0,j} - \sum_{h\in H} \left(A_{\mu,j}(p, q, \pi)\varphi^{\alpha,h}_{\mu,j} + B_{\mu,j}(p, q, \pi)\varphi^{\beta,h}_{\mu,j} + p_\mu C_{\mu,j}\varphi^{\gamma,h}_{\xi,j})\right)\right)^2.$$

- For each $(\mu, j) \in D_2 \times J(\xi_0)$, a player chooses $N_{\mu,j} \in \mathcal{N}_{\mu,j}(\Phi)$ to maximize

$$-\left(N_{\mu,j} \sum_{h\in H} \varphi^h_{\xi_0,j} - R_{\mu,j}(p, q, \pi) \sum_{h\in H} \varphi^{h,\alpha}_{\mu^-,j}\right)^2.$$

- For each $(\mu, j) \in D_2 \times J(\xi)$, a player chooses $N_{\mu,j} \in \mathcal{N}_{\mu,j}(\Phi)$ to maximize $-\left(N_{\mu,j} - R_{\mu,j}(p, q, \pi)\right)^2$.

A vector $\left((\overline{p}, \overline{q}, \overline{\pi}, \overline{N}), (\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h, \overline{\varphi}^{\alpha,h}, \overline{\varphi}^{\beta,h})_{h\in H}\right) \in \Delta \times \mathcal{N}(\Phi) \times \mathcal{X}(\Omega)^H$ is a *Cournot-Nash equilibrium* for the generalized game $\mathcal{G}$ if it solves all the problems above.

Existence of Cournot-Nash equilibria. Under Assumptions (A1)-(A4), each player in the generalized game $\mathcal{G}$ has a continuous correspondence of admissible strategies, with non-empty, compact, and convex values. Also, players' objective functions are continuous and quasi-concave on their own strategy. Since $\Delta \times \mathcal{N}(\Phi) \times \mathcal{X}(\Omega)^H$ is non-empty, convex, and compact, Berge's Maximum Theorem guarantees that best-reply correspondences are upper hemicontinuous and have non-empty, compact and convex values. Applying Kakutani Fixed Point Theorem to the set-value mapping which associates to each $z \in \Delta \times \mathcal{N}(\Phi) \times \mathcal{X}(\Omega)^H$ the cartesian product of players' best-reply strategies to z, we obtain an equilibrium for $\mathcal{G}$.

From Cournot-Nash to non-trivial equilibria. Let $\left((\overline{p},\overline{q},\overline{\pi},\overline{N}),(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h},\overline{\varphi}^{\beta,h})_{h\in H}\right)$ be a Cournot-Nash equilibrium for $\mathcal{G}$. Then, for each $h\in H$, the allocation $(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h},\overline{\varphi}^{\beta,h})$ belongs to $\Gamma^h(\overline{p},\overline{q},\overline{\pi},\overline{N})\cap\mathcal{X}(\Omega)$ and, therefore, it satisfies inequalities $(\mathrm{B}_\xi)_{\xi\in D}$ and $(\mathrm{S}_\xi)_{\xi\in D_1}$.

Adding restrictions (B_{ξ_0}) across agents we conclude that, the objective function of the player who chooses $(\overline{p}_{\xi_0},\overline{q}_{\xi_0})$ has an optimal value less than or equal to zero. Since $(\overline{p}_{\xi_0},\overline{q}_{\xi_0})\in\Delta_{\xi_0}$, this implies that $\sum_{h\in H}\left(c_{\xi_0}(\overline{x}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h})-w^h_{\xi_0}\right)\leq 0$ and $\sum_{h\in H}\left(\overline{\theta}^h_{\xi_0}-\overline{\varphi}^h_{\xi_0}\right)\leq 0$. Hence, for each agent h, $(\overline{x}^h_{\xi_0},\overline{\theta}^h_{\xi_0},\overline{\varphi}^h_{\xi_0})\leq\Omega(1,\ldots,1)$. That is, upper bounds on individual allocations chosen at ξ_0 are non-binding. For this reason, monotonicity of preferences implies that $\overline{p}_{\xi_0}\gg 0$ and that budget constraints at ξ_0 are binding. We conclude that the equilibrium value of the objective function of the player that chooses $(\overline{p}_{\xi_0},\overline{q}_{\xi_0})$ is zero, which in turn implies that commodity markets feasibility condition holds at ξ_0 and, for each $j\in J(\xi_0)$, $\sum_{h\in H}\overline{\theta}^h_{\xi_0,j}\leq\sum_{h\in H}\overline{\varphi}^h_{\xi_0,j}$, $\overline{q}_{\xi_0,j}\sum_{h\in H}\left(\overline{\theta}^h_{\xi_0,j}-\overline{\varphi}^h_{\xi_0,j}\right)=0$.

Fix an intermediate node $\xi\in D_1$. The definition of Φ guarantees that, for each $j\in J(\xi_0)$,

$$\overline{N}_{\xi,j}\sum_{h\in H}\overline{\varphi}^h_{\xi_0,j}=\sum_{h\in H}\left(A_{\xi,j}(\overline{p},\overline{q},\overline{\pi})\overline{\varphi}^{\alpha,h}_{\xi,j}+B_{\xi,j}(\overline{p},\overline{q},\overline{\pi})\overline{\varphi}^{\beta,h}_{\xi,j}+\overline{p}_\xi C_{\xi,j}(\overline{\varphi}^h_{\xi_0,j}-\overline{\varphi}^{\alpha,h}_{\xi,j}-\overline{\varphi}^{\beta,h}_{\xi,j})\right).$$

From this identity, adding (B_ξ) across agents and given that $\sum_{h\in H}\left(\overline{\theta}^h_{\xi_0}-\overline{\varphi}^h_{\xi_0}\right)\leq 0$, we get

$$\sum_{h\in H}\left(\overline{p}_\xi\left(c_\xi(\overline{x}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h})-w^h_\xi-Y_\xi c_{\xi_0}(\overline{x}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h})\right)+\overline{q}_\xi\left(\overline{\theta}^h_\xi-\overline{\varphi}^h_\xi\right)+\overline{\pi}_\xi\left(\overline{\theta}^h_\xi-\overline{\theta}^h_{\xi_0}\right)\right)\leq 0.$$

Therefore, as it was the case at ξ_0, upper bounds on individual allocations chosen at ξ are non-binding. For this reason, monotonicity of preferences ensures that $\overline{p}_\xi\gg 0$ and that budget constraints at ξ are satisfied with equality. Thus, at node ξ commodity markets feasibility conditions hold and

$$\sum_{h\in H}\left(\overline{\theta}^h_{\xi,j}-\overline{\varphi}^h_{\xi,j}\right)\leq 0,\quad \overline{q}_{\xi,j}\sum_{h\in H}\left(\overline{\theta}^h_{\xi,j}-\overline{\varphi}^h_{\xi,j}\right)=0,\quad \forall j\in J(\xi),$$

$$\sum_{h\in H}\left(\overline{\theta}^h_{\xi,j}-\overline{\theta}^h_{\xi_0,j}\right)\leq 0,\quad \overline{\pi}_{\xi,j}\sum_{h\in H}\left(\overline{\theta}^h_{\xi,j}-\overline{\theta}^h_{\xi_0,j}\right)=0,\quad \forall j\in J(\xi_0).$$

Fix a terminal node $\xi\in D_2$. Analogous arguments to those made above guarantee that, for each $j\in J(\xi^-)$, $\overline{N}_{\xi,j}=R_{\xi,j}(\overline{p},\overline{q},\overline{\pi})$. Also, for each $j\in J(\xi_0)$,

$$\overline{N}_{\xi,j}\sum_{h\in H}\overline{\varphi}^h_{\xi_0,j}=R_{\xi,j}(\overline{p},\overline{q},\overline{\pi})\sum_{h\in H}\overline{\varphi}^{\alpha,h}_{\xi^-,j}.$$

These properties —jointly with the inequalities $\sum_{h\in H}\overline{\theta}^h_{\xi^-,k}\leq\sum_{h\in H}\overline{\theta}^h_{\xi_0,k}\leq\sum_{h\in H}\overline{\varphi}^h_{\xi_0,k}$ and $\sum_{h\in H}\overline{\theta}^h_{\xi^-,j}\leq\sum_{h\in H}\overline{\varphi}^h_{\xi^-,j}$, which hold for any $(k,j)\in J(\xi_0)\times J(\xi^-)$— guarantee that after adding (B_ξ) across agents we get that,

$$\overline{p}_\xi\sum_{h\in H}\left(c_\xi(\overline{x}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h})-w^h_\xi-Y_\xi c_{\xi_0}(\overline{x}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h})\right)\leq 0.$$

Since $\overline{p}_\xi \in \Delta_\xi$, there is no excess of demand in commodity markets at ξ and, hence, upper bounds on individual allocations chosen at ξ are non-binding. We conclude that commodity markets feasibility condition holds at ξ and that $\overline{p}_\xi \gg 0$. Therefore, $\overline{p} \gg 0$ and commodity market clearing conditions hold at D.

Given $\xi \in D \setminus D_2$ and $j \in J(\xi)$, $\overline{N}_{\mu,j} \geq R_{\mu,j}(\overline{p}, \overline{q}, \overline{\pi})$, $\forall \mu \in \xi^+$. Since $\overline{p} \gg 0$, Assumption (A5) guarantees that there exists at least one security with non-trivial payments. Moreover, for each security with non-trivial payments, the market clearing condition holds at the emission node. Otherwise $\overline{q}_{\xi,j} = 0$, a contradiction with the strict monotonicity of preferences and the fact that upper bounds on optimal individual allocations are non-binding. Therefore, for each $\xi \in D \setminus D_2$ and $j \in J(\xi)$ such that $(N_{\mu,j})_{\mu>\xi} \neq 0$, we obtain $\sum_{h\in H} \left(\overline{\theta}^h_\xi - \overline{\varphi}^h_\xi\right) = 0$.

If for some $\xi \in D$ and $j \in J(\xi)$, $\sum_{h\in H} \left(\overline{\theta}^h_{\xi,j} - \overline{\varphi}^h_{\xi,j}\right) < 0$, then $\overline{q}_{\xi,j} = 0$ and $(\overline{N}_{\mu,j})_{\mu>\xi} = 0$. Therefore, maintaining optimality, for each $h \in H$ we can substitute $\overline{\theta}^h_{\xi,j}$ with $\widehat{\theta}^h_{\xi,j} := \overline{\varphi}^h_{\xi,j}$. Also, if there exist $(\mu, j) \in D_1 \times J(\xi_0)$ for which $\sum_{h\in H} \left(\overline{\theta}^h_{\mu,j} - \overline{\theta}^h_{\xi_0}\right) < 0$, then $\overline{\pi}_{\mu,j} = 0$ and $(\overline{N}_{\xi,j})_{\xi\in D_2} = 0$.[6] Therefore, we can substitute $\overline{\theta}^h_{\mu,j}$ with $\widehat{\theta}^h_{\mu,j} := \overline{\theta}^h_{\xi_0,j}$ maintaining optimality. After these modifications, financial market clearing conditions hold.

Furthermore, these substitutions guarantee that, for each $\xi \in D_1$ and $j \in J(\xi_0)$,

$$\overline{N}_{\xi,j} \sum_{h\in H} \overline{\theta}^h_{\xi_0,j} = \sum_{h\in H} \left(A_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi})\overline{\varphi}^{\alpha,h}_{\xi,j} + B_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi})\overline{\varphi}^{\beta,h}_{\xi,j} + \overline{p}_\xi C_{\xi,j}(\overline{\varphi}^h_{\xi_0,j} - \overline{\varphi}^{\alpha,h}_{\xi,j} - \overline{\varphi}^{\beta,h}_{\xi,j}) \right).$$

Also, for each $\xi \in D_2$ and $j \in J(\xi_0)$, we have

$$\overline{N}_{\xi,j} \sum_{h\in H} \overline{\theta}^h_{\xi_0,j} - R_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi}) \sum_{h\in H} \overline{\varphi}^{\alpha,h}_{\xi^-,j} = 0.$$

It follows that $\left((\overline{p}, \overline{q}, \overline{\pi}, \overline{N}), (\overline{x}^h, \widehat{\theta}^h, \overline{\varphi}^h, \overline{\varphi}^{\alpha,h}, \overline{\varphi}^{\beta,h})_{h\in H}\right)$ satisfies conditions (ii)-(iv) in our equilibrium definition, with at least one security with non-trivial payments.

Therefore, to ensure that $\left((\overline{p}, \overline{q}, \overline{\pi}, \overline{N}), (\overline{x}^h, \widehat{\theta}^h, \overline{\varphi}^h, \overline{\varphi}^{\alpha,h}, \overline{\varphi}^{\beta,h})_{h\in H}\right)$ is a non-trivial equilibrium for $\mathcal{E}$ it is sufficient to show that, for each $h \in H$ the allocation $\overline{z}^h := (\overline{x}^h, \widehat{\theta}^h, \overline{\varphi}^h, \overline{\varphi}^{\alpha,h}, \overline{\varphi}^{\beta,h})$ is an optimal choice in $\Gamma^h(\overline{p}, \overline{q}, \overline{\pi}, \overline{N})$. Suppose by contradiction that for some $h \in H$ there exists another allocation $\widetilde{z}^h := (\widetilde{x}^h, \widetilde{\theta}^h, \widetilde{\varphi}^h, \widetilde{\varphi}^{\alpha,h}, \widetilde{\varphi}^{\beta,h}) \in \Gamma^h(\overline{p}, \overline{q}, \overline{\pi}, \overline{N})$ such that,

$$U^h\left(\left(c_\xi(\widetilde{x}^h, \widetilde{\varphi}^h, \widetilde{\varphi}^{\alpha,h})\right)_{\xi\in D}\right) > U^h\left(\left(c_\xi(\overline{x}^h, \overline{\varphi}^h, \overline{\varphi}^{\alpha,h})\right)_{\xi\in D}\right).$$

Since $\overline{z}^h$ is in the interior (relative to $\mathcal{X}$) of $\Gamma^h(\overline{p}, \overline{q}, \overline{\pi}, \overline{N}) \cap \mathcal{X}(\Omega)$ and U^h is strictly quasi-concave, there exists $\lambda \in (0, 1)$ such that, $(x^h_\lambda, \theta^h_\lambda, \varphi^h_\lambda, \varphi^{\alpha,h}_\lambda, \varphi^{\beta,h}_\lambda) := \lambda\overline{z}^h + (1-\lambda)\widetilde{z}^h \in \Gamma^h(\overline{p}, \overline{q}, \overline{\pi}, \overline{N}) \cap \mathcal{X}(\Omega)$ and $U^h\left(\left(c_\xi(x^h_\lambda, \varphi^h_\lambda, \varphi^{\alpha,h}_\lambda)\right)_{\xi\in D}\right) > U^h\left(\left(c_\xi(\overline{x}^h, \overline{\varphi}^h, \overline{\varphi}^{\alpha,h})\right)_{\xi\in D}\right)$, a contradiction.

Therefore $\left((\overline{p}, \overline{q}, \overline{\pi}, \overline{N}), (\overline{x}^h, \widehat{\theta}^h, \overline{\varphi}^h, \overline{\varphi}^{\alpha,h}, \overline{\varphi}^{\beta,h})_{h\in H}\right)$ is a non-trivial equilibrium for $\mathcal{E}$. □

[6] This may be a consequence of debt prepayment.

5. Characterizing Prepayment and Credit Risks

In this section we provide necessary and sufficient conditions that induce borrowers to close short positions before terminal nodes, either prepaying or defaulting. These conditions depend on observable market variables and contractual characteristics.

We begin with results that characterize optimal payment strategies independently of the existence of alternative credit opportunities. More precisely, at each intermediate node $\xi \in D_1$, if the cost associated with closing a debt position on $j \in J(\xi_0)$ is lower than the present value of commitments, then agents prepay or default on their j-debt. In addition, if the cost of closing a debt is higher than the present value of future commitments, then borrowers whose optimal actions are not affected by collateral constraints pay the coupons and maintain the short position open.

Proposition 1. *Under Assumptions (A1)-(A2), assume that for all agents $h \in H$, $U^h : \mathbb{R}^{D\times L} \to \mathbb{R}$ is continuously differentiable. Let $\left((\overline{p}, \overline{q}, \overline{\pi}, \overline{N}), (\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h, \overline{\varphi}^{\alpha,h}, \overline{\varphi}^{\beta,h})_{h\in H}\right)$ be an equilibrium. For each $h \in H$, let $(\lambda^h(\eta))_{\eta\in D}$ be agent h's Kuhn-Tucker multipliers associated with budget constraints.*

Fix $(h, \xi, j) \in H \times D_1 \times J(\xi_0)$ such that $\overline{\varphi}^h_{\xi_0,j} > 0$ and define

$$\Phi^h_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi}) = \min\{B_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi}), \overline{p}_\xi C_{\xi,j}\} - \left(A_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi}) + \sum_{\mu\in\xi^+} \frac{\lambda^h(\mu)}{\lambda^h(\xi)} R_{\mu,j}(\overline{p}, \overline{q}, \overline{\pi})\right).$$

If $\Phi^h_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi}) < 0$, then agent h closes short positions on j at ξ.
If $\Phi^h_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi}) > 0$ then agent h reduces short-positions on j at ξ only when collateral constraints associated with credit contract j are active at ξ.

Proof. From Arrow and Enthoven (1961), the usual Kuhn-Tucker conditions are necessary for optimality. From the partial derivatives of agent h's Lagrangian function with respect to $\overline{x}^h_\xi$ and $\overline{\varphi}^{\alpha,h}_{\xi,j}$ we obtain,

$$\overline{p}_\xi C_{\xi,j} = A_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi}) + \sum_{\mu\in\xi^+} \frac{\lambda^h(\mu)}{\lambda^h(\xi)} R_{\mu,j}(\overline{p}, \overline{q}, \overline{\pi}) + \frac{\kappa^h_{\xi,j} + \nu^h_\xi C_{\xi,j} - \eta^{\alpha,h}_{\xi,j}}{\lambda^h(\xi)},$$

where $\kappa^h_{\xi,j} \geq 0$ is the Kuhn-Tucker multiplier of constraint $\overline{\varphi}^{\alpha,h}_{\xi,j} + \overline{\varphi}^{\beta,h}_{\xi,j} \leq \overline{\varphi}^h_{\xi_0,j}$, $\nu^h_\xi \in \mathbb{R}^L_+$ is the vector of multipliers associated with $\overline{x}^h_\xi \geq 0$, and $\eta^{\alpha,h}_{\xi,j} \geq 0$ is the multiplier of the non-negativity constraint of $\overline{\varphi}^{\alpha,h}_{\xi,j}$. From this condition, and using the partial derivative of agent h's Lagrangian function with respect to $\overline{\varphi}^{\beta,h}_{\xi,j}$ we have,

$$B_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi}) = A_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi}) + \sum_{\mu\in\xi^+} \frac{\lambda^h(\mu)}{\lambda^h(\xi)} R_{\mu,j}(\overline{p}, \overline{q}, \overline{\pi}) + \frac{\eta^{\beta,h}_{\xi,j} + \nu^h_\xi C_{\xi,j} - \eta^{\alpha,h}_{\xi,j}}{\lambda^h(\xi)},$$

where $\eta^{\beta,h}_{\xi,j} \geq 0$ is the multiplier of the non-negativity constraint of $\overline{\varphi}^{\beta,h}_{\xi,j}$. Thus, we obtain that,

$$\Phi^h_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) = \frac{\nu^h_\xi C_{\xi,j} - \eta^{\alpha,h}_{\xi,j}}{\lambda^h(\xi)} + \frac{\min\{\eta^{\beta,h}_{\xi,j}, \kappa^h_{\xi,j}\}}{\lambda^h(\xi)}.$$

Therefore, $\Phi^h_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) < 0$ implies that $\eta^{\alpha,h}_{\xi,j} > 0$. Thus, when $\Phi^h_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) < 0$ agent h closes short positions on j at ξ. On the other hand, suppose that $\Phi^h_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) > 0$ and that agen h's collateral constraints for credit contract j are not active at ξ, i.e., $\nu^h_\xi = 0$. Then, $\min\{\eta^{\beta,h}_{\xi,j}, \kappa^h_{\xi,j}\} > 0$. Hence, both $\overline{\varphi}^{\beta,h}_{\xi,j} = 0$ and $\overline{\varphi}^{\alpha,h}_{\xi,j} + \overline{\varphi}^{\beta,h}_{\xi,j} = \overline{\varphi}^h_{\xi_0,j}$, implying $\overline{\varphi}^{\alpha,h}_{\xi,j} = \overline{\varphi}^h_{\xi_0,j}$. □

The previous proposition shows that agents close debts when either prepayment or default cost is low. Furthermore, independently of the existence of alternative credit opportunities, when collateral constraints do not affect optimal behavior, underwater loans are possible in equilibrium. Indeed, suppose that there exists a node $\xi \in D_1$ such that, for some $j \in J(\xi_0)$ and $h \in H$ we have $\overline{\varphi}^h_{\xi_0,j} > 0$ and

$$B_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) > \overline{p}_\xi C_{\xi,j} > A_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) + \sum_{\mu\in\xi^+} \frac{\lambda^h(\mu)}{\lambda^h(\xi)} R_{\mu,j}(\overline{p},\overline{q},\overline{\pi}).$$

In this situation, if agent h demands autonomous consumption of all commodities used as collateral by credit contract j, then the short position on this asset is maintained at ξ.

The existence of alternative credit opportunities may expand borrowers' options to close debts, a possibility that is particularly relevant when the cost of this action is higher than the present value of commitments. The following result shows that, if alternative credit opportunities are available, borrowers determine optimal payment strategies by comparing collateral margins and expected commitments across debt contracts.

Proposition 2. *Under Assumptions (A1)-(A2), assume that for all agents $h \in H$, $U^h : \mathbb{R}^{D\times L} \to \mathbb{R}$ is continuously differentiable. Let $\left((\overline{p},\overline{q},\overline{\pi},\overline{N}), (\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h, \overline{\varphi}^{\alpha,h}, \overline{\varphi}^{\beta,h})_{h\in H}\right)$ be an equilibrium. Fix $(\xi, j) \in D_1 \times J(\xi_0)$ such that $\Psi_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) := \min\{B_{\xi,j}(\overline{p},\overline{q},\overline{\pi}), \overline{p}_\xi C_{\xi,j}\} - A_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) > 0$. For each $h \in H$, let $(\lambda^h(\eta))_{\eta\in D}$ be agent h's Kuhn-Tucker multipliers associated with budget constraints.*

Then, an agent h closes short positions on j when there is a credit line $k \in J(\xi)$ for which

$$\frac{C_{\xi,k}}{\overline{q}_{\xi,k}} \leq \frac{C_{\xi,j}}{\Psi_{\xi,j}(\overline{p},\overline{q},\overline{\pi})}, \quad \text{and} \quad \sum_{\mu\in\xi^+} \lambda^h(\mu)\frac{R_{\mu,k}(\overline{p},\overline{q},\overline{\pi})}{\overline{q}_{\xi,k}} < \sum_{\mu\in\xi^+} \lambda^h(\mu)\frac{R_{\mu,j}(\overline{p},\overline{q},\overline{\pi})}{\Psi_{\xi,j}(\overline{p},\overline{q},\overline{\pi})}.$$

In this situation, agent h prepays debt j if and only if $B_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) \leq \overline{p}_\xi C_{\xi,j}$.

Proof. Assume that for some debt contract $k \in J(\xi)$ conditions above hold. Suppose that, after issuing $\overline{\varphi}^h_{\xi_0,j}$ units of j at ξ_0, j-borrower h maintains a position $\overline{\varphi}^{\alpha,h}_{\xi,j} \in (0, \overline{\varphi}^h_{\xi_0,j}]$ at node ξ. Therefore, h should pay $A_{\xi,j}(\overline{p},\overline{q},\overline{\pi})\overline{\varphi}^{\alpha,h}_{\xi,j}$, consume the collateral bundle $C_{\xi,j}\overline{\varphi}^{\alpha,h}_{\xi,j}$, and deliver a payment $R_{\mu,j}(\overline{p},\overline{q},\overline{\pi})\overline{\varphi}^{\alpha,h}_{\xi,j}$ at each terminal node $\mu \in \xi^+$. It can be shown that this strategy is not optimal.

Indeed, consider the following alternative: agent h closes the short position $\overline{\varphi}^{\alpha,h}_{\xi,j}$ and trades $\widetilde{\varphi}_k\overline{\varphi}^{\alpha,h}_{\xi,j}$ units of debt contract k, where $\widetilde{\varphi}_k = \frac{\Psi_{\xi,j}(\overline{p},\overline{q},\overline{\pi})}{\overline{q}_{\xi,k}}$. There is no additional cost at ξ, i.e., $\Psi_{\xi,j}(\overline{p},\overline{q},\overline{\pi})\overline{\varphi}^{\alpha,h}_{\xi,j} - \overline{q}_{\xi,k}\widetilde{\varphi}_k\overline{\varphi}^{\alpha,h}_{\xi,j} = 0$. Since $\frac{C_{\xi,k}}{\overline{q}_{\xi,k}} \leq \frac{C_{\xi,j}}{\Psi_{\xi,j}(\overline{p},\overline{q},\overline{\pi})}$, the original consumption bundle at ξ satisfies agent h's new collateral requirements. Finally, the new payments at terminal nodes imply that the Lagrangian function increases, as $\sum_{\mu\in\xi^+} \lambda^h(\mu)\left(R_{\mu,j}(\overline{p},\overline{q},\overline{\pi}) - R_{\mu,k}(\overline{p},\overline{q},\overline{\pi})\widetilde{\varphi}_k\right)\overline{\varphi}^{\alpha,h}_{\xi,j} > 0$.

Hence, any strategy that maintains open a short position on j at ξ is not optimal. □

Notice that, Propositions 1 and 2 show that the availability of alternative credit opportunities could be incompatible with the existence of individuals who partially finance their consumption through long term credit, i.e., the existence of $h \in H$ such that $\overline{x}^h_\xi \gg \sum_{j\in J(\xi_0)} C_{\xi,j}\overline{\varphi}^{h,\alpha}_{\xi,j} > 0$ at some $\xi \in D_1$.[7]

It follows from the result above that the presence of atractive alternative credit opportunities avoids underwater loans in equilibrium. In this direction, and under the appropriate specification of credit contracts, the model of Araujo, Páscoa, and Torres-Martínez (2011) is a particular case of our framework. Indeed, given $(\xi, j) \in D_1 \times J(\xi_0)$ suppose that there exists a debt contract $k \in J(\xi)$ with the same collateral requirements and future promises as j, i.e., $C_{\xi,k} = C_{\xi,j}$ and $R_{\mu,k}(p,q,q) = R_{\mu,j}(p,q,\pi)$, $\forall(p,q,\pi) \in \mathcal{P}$, $\forall\mu \in \xi^+$. Then, j-borrowers close their debt at ξ when $\Psi_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) < \overline{q}_{\xi,k}$ and they would be indifferent if both magnitudes were equal. Since Araujo, Páscoa and Torres-Martínez (2011) do not allow for liquidity contractions, the prepayment cost is implicitly given by $A_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) + \overline{q}_{\xi,k}$. Therefore, all borrowers will optimally close debts at intermediate nodes, defaulting if the collateral value $\overline{p}_\xi C_{\xi,j}$ is lower than the prepayment cost $A_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) + \overline{q}_{\xi,k}$. Thus, this model does not capture underwater mortgages.

Our previous propositions are based on individual income shadow values. However, under some circumstances optimal payment strategies can be specified in terms of observable variables.

Corollary. *Under Assumptions (A1)-(A2), assume that for all agents $h \in H$, $U^h : \mathbb{R}^{D\times L} \to \mathbb{R}$ is continuously differentiable. Let $\left((\overline{p},\overline{q},\overline{\pi},\overline{N}),(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h},\overline{\varphi}^{\beta,h})_{h\in H}\right)$ be an equilibrium. Fix $(\xi,j) \in D_1 \times J(\xi_0)$ for which $\Psi_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) > 0$. If the following conditions are satisfied,*

$$\frac{C_{\xi,k}}{\overline{q}_{\xi,k}} \leq \frac{C_{\xi,j}}{\Psi_{\xi,j}(\overline{p},\overline{q},\overline{\pi})}, \quad \textit{and} \quad \left(\frac{R_{\mu,k}(\overline{p},\overline{q},\overline{\pi})}{\overline{q}_{\xi,k}}\right)_{\mu\in\xi^+} < \left(\frac{R_{\mu,j}(\overline{p},\overline{q},\overline{\pi})}{\Psi_{\xi,j}(\overline{p},\overline{q},\overline{\pi})}\right)_{\mu\in\xi^+},$$

then all agents close their j-debts at ξ.

[7] In fact, under this condition, Proposition 1 shows that agent h maintains short positions on credit contracts on $J(\xi_0)$ at node ξ. However, Proposition 2 implies that this behavior is not optimal if there exist better credit options.

6. Concluding Remarks

We propose a model of long-term loans backed by physical collateral, in which borrowers may prepay their debts before terminal nodes. This model extends Geanakoplos and Zame (1997, 2002, 2007) theoretical framework to allow for long-term loans and liquidity contractions. Under mild conditions, we prove existence of equilibrium and provide a theoretical characterization of optimal payment strategies.

We show that in equilibrium, agents decide to close their debts before terminal dates —either prepaying or defaulting— if closing a short position is less costly than the expected present value of commitments. However, this condition is not homogeneous across agents and, hence, optimal payment strategies depend on individual characteristics. Moreover, this decision also depends on financial markets liquidity. The absence of better credit opportunities makes some individuals more prone to honor original commitments in order to maintain the consumption of collateralized durable goods. That is, borrowers can react to liquidity shrinkages by paying coupons of debt instead of closing short positions. Therefore, the lack of liquidity could make it optimal for borrowers to honor their commitments even though the collateral value were lower than the prepayment value (underwater mortgage).

We provide a numerical example illustrating that optimal payment strategies —payment, prepayment, and default— depend on individual characteristics and financial markets liquidity.

It is well known that, without liquidity contractions, collateral avoids Ponzi schemes and equilibrium exists in infinite horizon collateralized asset markets. Furthermore, the absence of asset pricing bubbles on durable commodity prices avoids bubbles on collateralized securities (see Araujo, Páscoa, and Torres-Martínez (2002, 2011)). As a matter of future research, we plan to extend these results to our model with liquidity contractions and prepayment rules.

References

[1] Alvarez, F., and U. J. Jermann (2000): "Efficiency, Equilibrium, and Asset Pricing with Risk of Default," *Econometrica,* 68, 4, 775-797.

[2] Araujo, A., Páscoa, M.R. (2002): "Bankruptcy in a Model of Unsecured Claims," *Economic Theory,* 20, 455-481.

[3] Araujo, A., M.R. Páscoa, and J.P. Torres-Martínez (2002): "Collateral Avoids Ponzi Schemes in Incomplete Markets," *Econometrica,* 70, 1613-1638.

[4] Araujo, A., M.R. Páscoa, and J.P. Torres-Martínez (2011): "Long-lived Collateralized Assets and Bubbles," *Journal of Mathematical Economics*, 47, 3, 260-271.

[5] Arrow, K., and A. Enthoven (1961): "Quasi-Concave Programming," *Econometrica*, 29, 779-800.

[6] Becketti, S. (1989): "The Prepayment Risk of Mortgage-Backed Securities," *Economic Review*, Federal Reserve Bank of Kansas City, 74, 2, 406-420.

[7] Dubey, P., J. Geanakoplos, and M. Shubik (1989): "Default and Efficiency in a General Equilibrium Model with Incomplete Markets," Cowles Foundation Discussion Paper No. 879R.

[8] Dubey, P., J. Geanakoplos, and M. Shubik (2005): "Default and Punishment in General Equilibrium," *Econometrica*, 73, 1-37.

[9] Ferreira, T.R., and J.P. Torres-Martínez (2010): "The Impossibility of Effective Enforcement Mechanisms in Collateralized Credit Markets," *Journal of Mathematical Economics*, 46, 3, 332-342.

[10] Geanakoplos, J., R. Axtell, D.J. Farmer, P. Howitt, B. Conlee, J. Goldstein, M. Hendrey, N.M. Palmer, and C. Yang (2012): "Getting at Systemic Risk via an Agent-Based Model of the Housing Market," *American Economic Review: Papers and Proceedings*, 102, 3, 53-58.

[11] Geanakoplos, J., and W.R. Zame (1997): "Collateral, Default and Market Crashes," Cowles Foundation Discussion Paper.

[12] Geanakoplos, J., and W.R. Zame (2002): "Collateral and the Enforcement of Intertemporal Contracts," Cowles Foundation Discussion Paper.

[13] Geanakoplos, J., and W.R. Zame (2007): "Collateralized Asset Markets," UCLA Working Paper.

[14] Hart, O. (1975): "On the Optimality of Equilibrium when the Market Structure is Incomplete", *Journal of Economic Theory*, 11, 418-443.

[15] Hernandez, A., and M. Santos (1996): "Competitive Equilibria for Infinite-Horizon Economies with Incomplete Markets," *Journal of Economic Theory*, 71, 102-130.

[16] Kehoe, T., and D.K. Levine (1993): "Debt Constrained Asset Markets," *Review of Economic Studies*, 60, 865-888.

[17] Kocherlakota, N. (1996): "Implications of Efficient Risk Sharing without Committment," *Review of Economic Studies*, 63, 4, 595-609.

[18] Kubler, F., and K. Schmedders (2003): "Stationary equilibria in asset pricing models with incomplete markets," *Econometrica* 71, 1767-1793.

[19] Levine, D.K., and W.R. Zame (1996): "Debt Constraints and Equilibrium in Infinite Horizon Economies with Incomplete Markets," *Journal of Mathematical Economics*, 26, 103-131.

[20] Magill, M., and M. Quinzii (1994): "Infinite Horizon Incomplete Markets," *Econometrica*, vol. 62, 4, 853-880.

[21] Magill, M., and M. Quinzii (1996): "Incomplete Markets over an Infinite Horizon: Long-lived Securities and Speculative Bubbles," *Journal of Mathematical Economics*, 26, 133-170.

[22] Poblete-Cazenave, R. and Torres-Martínez, J.P. (2012): "Equilibrium with Limited-Recourse Collateralized Loans," *Economic Theory,* DOI:10.1007/s00199-011-0685-8.

[23] Sabarwal, T. (2003): "Competitive Equilibria with Incomplete Markets and Endogenous Bankruptcy," *Contributions to Theoretical Economics,* 3, 1.

[24] Steinert, M., and J.P. Torres-Martinez (2007): "General equilibrium in CLO markets," *Journal of Mathematical Economics*, 43, 6, 709-734.

[25] Zame, W. (1993): "Efficiency and the Role of Default when Security Markets are Incomplete," *American Economic Review*, 83, 1142-1164.

Center for Economic Research, CIE-ITAM
Av. Camino de Santa Teresa 930, Colonia Héroes de Padierna
Delegación Magdalena Contreras, C.P. 10700, México.
E-mail address: miguel.iraola@itam.mx

Department of Economics, University of Chile
Diagonal Paraguay 257, Santiago, Chile.
E-mail address: juan.torres@fen.uchile.cl