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Convergence Results for Unanimous Voting

César Martinelli
Instituto Tecnológico Autónomo de México

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Av. Camino a Santa Teresa # 930
Col. Héroes de Padierna
México, D.F. 10700
M E X I C O

Convergence Results for Unanimous Voting*

César Martinelli
Departamento de Economía
and Centro de Investigación Económica
Instituto Tecnológico Autónomo de México
México, D.F. 10700

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Abstract

This paper derives a necessary condition for unanimous voting to converge to the perfect information outcome when voters are only imperfectly informed about the alternatives. Under some continuity assumptions, the condition turns out to be necessary and sufficient for the existence of a sequence of equilibria that exhibits convergence to the perfect information outcome. The requirement is equivalent to that found by Milgrom (1979) to be necessary and sufficient for convergence of the price to the true value of an object in a single-prize auction. An example illustrates that convergence to the "right" decision may be reasonably fast for small electorates. However, if voters have common preferences, unanimity is not the optimal voting rule. Unanimity rule makes sense in the example only as a way to make sure that the viewpoint of a minority is respected.

*This paper is a follow up on previous joint work with John Duggan, who also contributed with comments to this paper. Nicola Persico suggested exploiting the analogy between unanimous voting and single-object auctions. I am grateful to both of them.

1 Introduction

A group of individuals must decide between two alternatives, A or C. Individuals might have different preferences over A and C. A possible decision rule is to select C if everybody agrees that C is a better alternative than A, and to select A otherwise. This decision rule makes sense in several real life situations. C may be an unusually risky alternative, or one carrying extraordinary moral consequences. An example is a jury decision about the death penalty in a criminal trial, in countries where this punishment is (still) possible. Another possibility is that A represents the status quo, so that C should be adopted only if it is Pareto improving. Individuals in the group might be representatives of a larger society, as in a parliamentary committee or a board of directors. Requiring unanimous consent in the group may be a way to ensure that the interests of minorities are taken into account if a significant departure from the status quo is being considered.

The appropriateness of unanimous consent as a decision rule is less clear if members of the group have common values with respect to some characteristic of the alternatives and are privately informed about this characteristic. In the jury example, jurors may have different information about the extent of the responsibility of the defendant in a criminal act due to differences in their perspectives and ability to judge the evidence available. In the committee example, committee members may have private information about the costs and benefits of departing from the status quo. In both cases, what is in question is the ability of unanimous voting to aggregate information about the quality of the alternatives, so that C is adopted if it is really Pareto superior to A. From this perspective, two types of mistakes are possible: To choose A even though everybody would favor C if all the private information were common knowledge, and to choose C if at least some member of the group would favor A if all the private information were common knowledge.

The literature on information aggregation in elections has shown that, under different non-unanimous voting rules, increasing the size of a committee typically leads to the decision of the committee to converge in probability to what it would have been were the true characteristic of the alternatives known to voters. This paper investigates whether this result applies in the case of unanimous voting.

We start our investigation in section 2 by setting up a model with a countable number of states or values of the unknown characteristic of the alternatives, with very little restrictions on what the preferences of different voters might be, and in which voters' private information takes the form of a signal generated by a countably-additive probability measure. This framework is much more general than those adopted by Feddersen and Pesendorfer (1998) and Duggan and Martinelli (1998), who study unanimous voting in the context of two states, common preferences among voters, and either two possible signals or a continuum of signals. The extra generality of the current paper is desirable because, as argued later on, common preferences is not the right assumption to judge unanimous voting. We also want to abstract from irrelevant details of the signal-generating process.

In section 3, a necessary condition for convergence is derived. It states that it must be possible for a voter to be arbitrarily sure that the characteristic of the alternatives favors voting for A, if it is actually the case that someone would like to vote for A were the true characteristic known to voters. Note that the requirement is only that it should be *possible* for a voter to be arbitrarily sure; it is not required that a voter is perfectly informed with any positive probability. This condition is also sufficient for the probability of choosing C by mistake to go to zero along *every* sequence of Nash equilibria. Interestingly, the condition is equivalent to that found by Milgrom (1979) to be necessary and sufficient for convergence of the winning bid to the true value of an object in a single-prize auction.

In section 4, it is shown that, under some continuity assumptions, in a two-state model the requirement described in the previous paragraph is necessary and sufficient for the existence of a sequence of equilibria along which the probabilities of both types of mistakes go to zero. This result is robust in the sense that, if the condition is nearly satisfied, then there is a sequence of Nash equilibria along which the probabilities of both mistakes get close to zero. A numerical example in section 5 illustrates that convergence to the "right" decision may be reasonably fast for small electorates. However, if voters have common preferences, unanimity is not the optimal voting rule.

Some of these results contrast sharply with those obtained by Feddersen and Pesendorfer (1998). We leave for the last section a discussion of the relation of this paper with previous literature on information aggregation in elections and auctions.

2 The model

There is a countable infinity of voters ($i = 1, 2, \dots$). A von Neumann-Morgenstern utility function $u_i(\cdot, \cdot)$ describes the preferences of voter i. The first argument is a point $z \in \mathcal{Z} = \{z_1, z_2, \dots\}$. The countable set $\mathcal{Z}$ represents the possible circumstances or states. The second argument is the social decision $d \in \{C, A\}$. Define

$$\mathcal{Z}_c = \{z \in \mathcal{Z} : u_i(z, A) - u_i(z, C) < 0 \text{ for all } i\}$$

and

$$\mathcal{Z}_a = \{z \in \mathcal{Z} : u_i(z, A) - u_i(z, C) \geq 0 \text{ for some voter } i\}.$$

The sets $\mathcal{Z}_c, \mathcal{Z}_a$ are non-empty. There are two positive real numbers r and q such that, if $z' \in \mathcal{Z}_a$, then there is an infinite number of voters such that $u_i(z', A) - u_i(z', C) > r$ and such that $u_i(z, A) - u_i(z, C) > -q$ for every $z \in \mathcal{Z}_c$. Note that there is no explicit randomness in the selection of the sequence of voters. If we prefer to think of voters' preferences as being drawn according to some probability measure, then the results in the following sections apply to every realization of the sequence of voters that satisfies the (very weak) restrictions on preferences described above.

Each voter receives a signal, which is private knowledge. Let $\mathcal{S}$ be the set of possible signals. Define

$$\Omega = \mathcal{Z} \times \mathcal{S} \times \mathcal{S} \times \cdots .$$

A typical point $\omega \in \Omega$ is $(z, s_1, s_2, \dots)$, where $z \in \mathcal{Z}$ and s_i is the signal received by voter i . Let $\mathcal{A}$ be a σ-field of subsets of $\mathcal{S}$, and let

$$\mathcal{F} = 2^{\mathcal{Z}} \times \mathcal{A} \times \mathcal{A} \times \cdots .$$

Each voter is assumed to have a probability measure P on $\mathcal{F}$. The random variable Z is defined by $Z(z, s_1, s_2, \dots) = z$, and represents the unknown true circumstances. It is assumed that $P\{Z = z\} > 0$ for every $z \in \mathcal{Z}$. Conditional on Z and under P, the signals are independent and identically distributed random variables. The definitions of σ-field, probability measure, and random variable are those of, say, Billingsley (1986).

The social decision d is determined in an election in which the first n voters participate. After receiving their signals, voters simultaneously cast a vote in favor of A or in favor of C. If all n voters vote for C, the social decision is C; otherwise the social decision is A. That is, unanimity is required in order to adopt C. In terms of the jury problem, which we use to illustrate the results, C represents "conviction" and A represents "acquittal."

Given n, a strategy for a voter is an $\mathcal{A}$-measurable mapping $p_{ni} : \mathcal{S} \to \{0, 1\}$, where $p_{ni}(s_i)$ is the probability that voter i votes in favor of C. We consider only pure strategies; mixed strategies are easily incorporated by including in the signal space a dimension unrelated to Z and used for randomization.

The above formulation defines a Bayesian game for every n. Given a sequence of Nash equilibria of the election game, indexed by n, we treat d_n as a sequence of random variables. We are interested in finding conditions under which

$$\lim_{n\to\infty} P\{d_n = C | Z \in \mathcal{Z}_c\} = 1 \quad \text{and} \quad \lim_{n\to\infty} P\{d_n = C | Z \in \mathcal{Z}_a\} = 0.$$

That is, with probability approaching one, A is adopted whenever an (arbitrarily small) fraction of voters would favor A if the state of the world were known, and C is adopted otherwise.

It turns out that the problem is similar to that of finding conditions such that the winning bid converges to the true value of the object at an auction, a problem studied by Wilson (1977) and Milgrom (1979). The following condition is similar to that given by Milgrom (1979). We will say that P provides *clear evidence in favor of acquittal* if, for every $z \in \mathcal{Z}_a$,

$$\inf_{S \in \mathcal{A}} \frac{P\{s_i \in S | Z \in \mathcal{Z}_c\}}{P\{s_i \in S | Z = z\}} = 0.$$

3 General results

Theorem 1 *If there is any sequence of Nash equilibria such that*

$$\lim_{n\to\infty} P\{d_n = C|Z \in \mathcal{Z}_c\} = 1 \quad \textit{and} \quad \lim_{n\to\infty} P\{d_n = C|Z \in \mathcal{Z}_a\} = 0$$

then P provides clear evidence in favor of acquittal. Moreover, if P provides clear evidence in favor of acquittal, every sequence of Nash equilibria satisfies

$$\lim_{n\to\infty} P\{d_n = C|Z \in \mathcal{Z}_a\} = 0.$$

□

Proof Suppose first that

$$\lim_{n\to\infty} P\{d_n = C|Z \in \mathcal{Z}_c\} = 1 \text{ and } \lim_{n\to\infty} P\{d_n = C|Z \in \mathcal{Z}_a\} = 0.$$

From $\lim_{n\to\infty} P\{d_n = C|Z \in \mathcal{Z}_c\} = 1$ and

$$\begin{aligned} P\{d_n = C|Z \in \mathcal{Z}_c\} &= \prod_{i=1}^{n} (1 - P\{p_{ni}(s_i) = 0|Z \in \mathcal{Z}_c\}) \\ &\leq \exp\left(-\sum_{i=1}^{n} P\{p_{ni}(s_i) = 0|Z \in \mathcal{Z}_c\}\right), \end{aligned}$$

we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} P\{p_{ni}(s_i) = 0|Z \in \mathcal{Z}_c\} = 0.$$

From $\lim_{n\to\infty} P\{d_n = C|Z \in \mathcal{Z}_a\} = 0$ and $P\{Z = z\} > 0$ for every $z \in \mathcal{Z}$, we get $\lim_{n\to\infty} P\{d_n = C|Z = z\} = 0$ for every $z \in \mathcal{Z}_a$. Since

$$\begin{aligned} P\{d_n = C|Z = z\} &= \prod_{i=1}^{n} (1 - P\{p_{ni}(s_i) = 0|Z = z\}) \\ &\geq 1 - \sum_{i=1}^{n} P\{p_{ni}(s_i) = 0|Z = z\}, \end{aligned}$$

we have that, for every $z \in \mathcal{Z}_a$,

$$\liminf_{n\to\infty} \sum_{i=1}^{n} P\{p_{ni}(s_i) = 0|Z = z\} \geq 1.$$

Thus, for every $z \in \mathcal{Z}_a$,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n P\{p_{ni}(s_i) = 0 | Z \in \mathcal{Z}_c\}}{\sum_{i=1}^n P\{p_{ni}(s_i) = 0 | Z = z\}} \quad = \quad 0.$$

Then, for every $z \in \mathcal{Z}_a$ there must exist positive integers $j = j(n, z) \leq n$ such that

$$\lim_{n\to\infty} \frac{P\{p_{nj}(s_j) = 0 | Z \in \mathcal{Z}_c\}}{P\{p_{nj}(s_j) = 0 | Z = z\}} \quad = \quad 0.$$

(see, for instance, the Lemma in Milgrom (1979)). Let $S^{n,z} = p_{nj}^{-1}(0)$. Thus, for every $z \in \mathcal{Z}_a$,

$$\lim_{n\to\infty} \frac{P\{s_j \in S^{n,z} | Z \in \mathcal{Z}_c\}}{P\{s_j \in S^{n,z} | Z = z\}} \quad = \quad 0.$$

Since the signals are independent and identically distributed,

$$\lim_{n\to\infty} \frac{P\{s_i \in S^{n,z} | Z \in \mathcal{Z}_c\}}{P\{s_i \in S^{n,z} | Z = z\}} \quad = \quad 0.$$

This proves the first part of the theorem. With respect to the second part, suppose there is a sequence of Nash equilibrium n-tuples $(p_{n1}, \ldots, p_{nn})$ such that

$$\liminf_{n\to\infty} P\{d_n = C | Z = z'\} \geq \alpha$$

for some $\alpha > 0$ for some $z' \in \mathcal{Z}_a$, but P provides clear evidence in favor of A.

From the assumption of clear evidence, there is some $S \in \mathcal{A}$ such that

$$\text{(1)} \qquad \frac{P\{s_i \in S | Z \in \mathcal{Z}_c\}}{P\{s_i \in S | Z = z'\}} \quad < \quad \frac{r\alpha P\{Z = z'\}}{qP\{Z \in \mathcal{Z}_c\}}.$$

Let $I(z')$ be the set of individuals such that $u_i(z', A) - u_i(z', C) > 1/b$. We claim that there is no S satisfying (1) such that $P\{p_{ni}(s_i) = 0 | s_i \in S\} = 1$ for every $i \in I(z')$ for n large enough. Suppose there is such S. Then

$$\sum_{i=1}^n P\{p_{ni}(s_i) = 0 | Z = z'\} \geq \#\{i \in I(z'), i \leq n\} P\{s_i \in S | Z = z'\}.$$

Since there is an infinite number of individuals in $I(z')$, the left side of this inequality goes to infinity with n. But then, since

$$\begin{aligned} P\{d_n = C|Z = z'\} &= \prod_{i=1}^{n} (1 - P\{p_{ni}(s_i) = 0|Z = z'\}) \\ &\leq \exp\left(-\sum_{i=1}^{n} P\{p_{ni}(s_i) = 0|Z = z'\}\right), \end{aligned}$$

we obtain $\lim_{n\to\infty} P\{d_n = C|Z = z'\} = 0$, a contradiction.

We claim now that there is some S satisfying equation (1) such that for some $i \in I(z')$ and some $n \geq i$, $P\{p_{ni}(s_i) = 0|s_i \in S\} = 0$. If this were not the case, we would have that for every $i \in I(z')$, every $n \geq i$, and every S satisfying equation (1), neither $S \bigcap p_{ni}^{-1}(0)$ nor any (measurable) subset of it would satisfy equation (1). Thus,

$$S \bigcap \left(\bigcup_{i \in I(z'),\, i \leq m \leq n} p_{mi}^{-1}(0) \right)$$

would not satisfy equation (1) for arbitrary n. But then, for arbitrarily large n,

$$S \bigcap \left(\bigcap_{i \in I(z'),\, i \leq m \leq n} p_{mi}^{-1}(1) \right)$$

would satisfy equation (1). Let i' be the first voter in $I(z')$. Thus, either

$$S \bigcap \left(\bigcap_{n \geq i'} \bigcap_{i \in I(z'),\, i \leq n} p_{ni}^{-1}(1) \right)$$

satisfies equation (1), which contradicts the previous claim, or it has zero probability conditional on $Z = z'$. But the last cannot be since S satisfies equation (1).

Now consider such S, i, and n as described in the previous paragraph. Define the strategy

$$p_{ni}^{*}(s) = \begin{cases} 0 & \text{if } s \in S, \\ p_{ni}(s) & \text{otherwise.} \end{cases}$$

Let e_{ni} be the expected value of u_i if every voter behaves according to the corresponding Nash equilibrium strategy, and let e^*_{ni} be the expected value of u_i if voter i adopts the strategy p^*_{ni} and every other voter behaves according to the corresponding Nash equilibrium strategy. Then

$$e^*_{ni} - e_{ni} \geq r\alpha P\{s_i \in S|Z = z'\}P\{Z = z'\} - qP\{s_i \in S|Z \in \mathcal{Z}_c\}P\{Z \in \mathcal{Z}_c\}$$

$$= P\{s_i \in S|Z = z'\} \times \left(r\alpha P\{Z = z'\} - qP\{Z \in \mathcal{Z}_c\}\frac{P\{s_i \in S|Z \in \mathcal{Z}_c\}}{P\{s_i \in S|Z = z'\}}\right)$$

$$> 0.$$

But this is a contradiction to the assumption that p_{ni} is an equilibrium strategy. ■

4 Continuous signals

In this section we consider a version of the general model with some additional assumptions. First, we assume that there are only two states, $z_c \in \mathcal{Z}_c$ and $z_a \in \mathcal{Z}_a$.

Second, the set of possible signals is assumed to be $(\underline{S}, \overline{S}) \subseteq (-\infty, \infty)$. $\mathcal{A}$ is taken to be the σ-field of Borel subsets of $(\underline{S}, \overline{S})$. Let

$$F(x|z) = P\{s \in (\underline{S}, x]|Z = z\}$$

for $z = z_a, z_c$. These distribution functions are assumed to be absolutely continuous with respect to Lebesgue measure, with continuous densities $f(s|z_a)$, $f(s|z_c)$ such that $f(s|z_a), f(s|z_c) > 0$ for $s \in (\underline{S}, \overline{S})$. The likelihood ratio, $f(s|z_c)/f(s|z_a)$, is strictly increasing on $(\underline{S}, \overline{S})$. Note that the definition of clear evidence in the context of this section specializes to

$$\lim_{s \downarrow \underline{S}} \frac{f(s|z_c)}{f(s|z_a)} = 0.$$

Third, we assume that each voter belongs to one of a finite number of types, $k = 0, 1, \ldots, K$. If i is a type-0 voter, $u_i(A, z) - u_i(C, z) < 0$ for

$z = z_c, z_a$. That is, type-0 voters prefer to convict regardless of the state. If i is a type-k voter, for $k = 1, \ldots, K$,

$$u_i(A, z_c) - u_i(C, z_c) = -q_k \text{ and } u_i(A, z_a) - u_i(C, z_a) = r_k$$

for some $q_k, r_k > 0$. There is an infinite number of voters of each type, except possibly type 0. We have

Theorem 2 *In the model with two states, continuous signals, and finite number of types, there is a sequence of Nash equilibria such that*

$$\lim_{n\to\infty} P\{d_n = C|Z = z_c\} = 1 \text{ and } \lim_{n\to\infty} P\{d_n = C|Z = z_a\} = 0$$

if and only if P provides clear evidence in favor of acquittal. □

Proof Necessity follows from Theorem 1. Also, from Theorem 1,

$$\lim_{n\to\infty} P\{d_n = C|Z = z_a\} = 0$$

along any sequence of Nash equilibria. It remains to be shown that if $\lim_{s\downarrow\underline{S}} f(s|z_c)/f(s|z_a) = 0$ there exists in fact a sequence of Nash equilibria such that along that sequence

$$\lim_{n\to\infty} P\{d_n = C|Z = z_c\} = 1.$$

We will use throughout the proof the fact that, under assumptions of this section, if $\lim_{s\downarrow\underline{S}} f(s|z_c)/f(s|z_a) = 0$,

(*i*) $f(s|z_c)/f(s|z_a)$ is continuous and

$$0 < F(s|z_c)/F(s|z_a) < f(s|z_c)/f(s|z_a) \text{ for } s \in (\underline{S}, \overline{S}),$$

(*ii*) $(1 - F(s|z_c))/(1 - F(s|z_a))$ is continuous and strictly increasing on $s \in (\underline{S}, \overline{S})$, with

$$\frac{1 - F(s|z_c)}{1 - F(s|z_a)} > 1 \text{ for } s \in (\underline{S}, \overline{S}) \quad \text{and} \quad \lim_{s\downarrow\underline{S}} \frac{1 - F(s|z_c)}{1 - F(s|z_a)} = 1,$$

(*iii*) For any $\alpha > 0$,

$$\left(\frac{1-F(s|z_c)}{1-F(s|z_a)}\right)^{\alpha} \frac{f(s|z_c)}{f(s|z_a)}$$

is continuous and strictly increasing on $(\underline{S}, \overline{S})$, converges to zero as s converges to $\underline{S}$ from above, and, for a fixed $s \in (\underline{S}, \overline{S})$, grows unboundedly with α.

It is simpler to start by considering the case $K = 1$. Let n_1 be the number of type-1 voters among the first n voters. Define s_1^n as the unique solution (if it exists) to

$$\left(\frac{1-F(s|z_c)}{1-F(s|z_a)}\right)^{n_1-1} \frac{f(s|z_c)}{f(s|z_a)} = \frac{r_1 P\{Z = z_a\}}{q_1 P\{Z = z_c\}}. \tag{2}$$

Since there is an infinite number of type-1 voters, n_1 goes to infinity with n. From this and (*iii*) it follows that for n large enough the above equation has in fact a unique solution. Note that s_1^n is strictly decreasing in n_1 and $\lim_{n\to\infty} s_1^n = \underline{S}$.

Now take n large enough so that s_1^n exists, and consider the following strategies: for every type-0 voter, $p_{ni}(s) = 1$ for every $s \in (\underline{S}, \overline{S})$, and for every type-1 voter,

$$p_{ni}(s) = \begin{cases} 0 & \text{if } s \in (\underline{S}, s_1^n], \\ 1 & \text{otherwise.} \end{cases}$$

We claim that $(p_{n1}, \ldots, p_{nn})$ is a Nash equilibrium profile. To verify this, note that for every vector of strategies followed by other voters, it is a best response for each type-0 voter to adopt the proposed strategy. Suppose that every other voter behaves according to the proposed strategies; then, it is a best response for a type-1 voter to adopt the proposed strategy if

$$q_1 P\{piv|Z = z_c\} f(s|z_c) P\{Z = z_c\} \leq r_1 P\{piv|Z = z_a\} f(s|z_a) P\{Z = z_a\}$$

for every $s \in (\underline{S}, s_1^n]$, and

$$q_1 P\{piv|Z = z_c\} f(s|z_c) P\{Z = z_c\} \geq r_1 P\{piv|Z = z_a\} f(s|z_a) P\{Z = z_a\}$$

for every $s \in (s_1^n, \overline{S})$. These two equations compare the expected payoff of voting for conviction with the expected payoff of voting for acquittal, conditional on receiving the signal s and on being pivotal. In these two equations, the term $P\{piv|Z = z\}$ represents the probability that a single type-1 voter is pivotal, conditional on the strategies followed by the other voters and the state of the world. Under the proposed strategies,

$$P\{piv|Z = z_c\} = (1 - F(s_1^n|z_c))^{n_1-1},\ P\{piv|Z = z_a\} = (1 - F(s_1^n|z_a))^{n_1-1}.$$

But then the desired inequalities follow from the definition of s_1^n and (*iii*).

Finally, we claim that $\lim_{n\to\infty} P\{d_n = C|Z = z_c\} = 1$ along the sequence of Nash equilibrium n-tuples $(p_{n1}, \ldots, p_{nn})$. To verify this, from the definition of s_1^n we obtain that, for large enough n,

$$n_1 = 1 + \log\left(\frac{r_1 f(s_1^n|z_a)P\{Z = z_a\}}{q_1 f(s_1^n|z_c)P\{Z = z_c\}}\right)\left(\log\left(\frac{1 - F(s_1^n|z_c)}{1 - F(s_1^n|z_a)}\right)\right)^{-1}.$$

Replacing in

$$P\{d_n = C|Z = z_c\} \ = \ (1 - F(s_1^n|z_c))^{n_1}$$

and taking logs,

$$\begin{aligned}(3)\qquad \log P\{d_n = C|Z = z_c\} = & \\ & \log(1 - F(s_1^n|z_c)) \\ + \ & \log\left(\frac{r_1 f(s_1^n|z_a)P\{Z = z_a\}}{q_1 f(s_1^n|z_c)P\{Z = z_c\}}\right)\left(1 - \frac{\log(1 - F(s_1^n|z_a))}{\log(1 - F(s_1^n|z_c))}\right)^{-1}.\end{aligned}$$

Recall that $s_1^n \downarrow \underline{S}$ as $n \to \infty$. Thus, the first term in the RHS converges to zero. With respect to the second term, it can be shown to converge to zero by using

$$\lim_{n\to\infty} \log(1 - F(s_1^n|z_a))/F(s_1^n|z_a) = \lim_{n\to\infty} \log(1 - F(s_1^n|z_c))/F(s_1^n|z_c) = -1$$

(by L'Hôpital's rule),

$$\lim_{n\to\infty} \frac{f(s_1^n|z_a)/f(s_1^n|z_c)}{\log(f(s_1^n|z_a)/f(s_1^n|z_c))} = \infty$$

(because $f(s_1^n|z_a)/f(s_1^n|z_c)$ diverges to infinity), and

$$F(s_1^n|z_a)/F(s_1^n|z_c) > f(s_1^n|z_a)/f(s_1^n|z_c)$$

(from (i)). This yields the desired result for the case $K = 1$.

We turn now to the case $K > 1$. Let n_k be the number of type-k voters among the first n voters. Define a *cutoff strategy* to be any strategy of the form

$$p_{ni}(s) \quad = \quad \begin{cases} 0 & \text{if } s \in (\underline{S}, x], \\ 1 & \text{otherwise} \end{cases}$$

for some $x \in [\underline{S}, \overline{S}]$. With a slight abuse of notation, we denote a cutoff strategy by its cutoff x.

Using an analogue of equation (2), define s_k^n for $k = 1, \ldots, K$ as the unique solution (if it exists) to

$$\left(\frac{1 - F(s|z_c)}{1 - F(s|z_a)}\right)^{n_k - 1} \frac{f(s|z_c)}{f(s|z_a)} \quad = \quad \frac{r_k P\{Z = z_a\}}{q_k P\{Z = z_c\}}.$$

By convention, let $s_0^n = \underline{S}$. For the rest of the proof, suppose that n is large enough so that $n_k > 0$ and s_k^n exists for all $k \geq 1$. Now, for a given vector $x^n = (x_0^n, \cdots, x_K^n)$ of cutoff strategies, with $x_k^n \in [\underline{S}, s_k^n]$, let y^n be given by $y_0^n = \underline{S}$ and y_k^n for $k \geq 1$ be given by the unique solution (if it exists) to

$$L(x^n) \left(\frac{1 - F(s|z_c)}{1 - F(s|z_a)}\right)^{n_k - 1} \frac{f(s|z_c)}{f(s|z_a)} \quad = \quad \frac{r_k P\{Z = z_a\}}{q_k P\{Z = z_c\}}$$

with

$$L(x^n) = \prod_{\substack{k'=1,\ldots,K \\ k' \neq k}} \left(\frac{1 - F(x_{k'}^n|z_c)}{1 - F(x_{k'}^n|z_a)}\right)^{n_{k'}}.$$

Note that, from (ii), $L(x^n) \geq 1$. Thus, from (iii), if s_k^n exists, y_k^n exists and, moreover, $y_k^n \in [\underline{S}, s_k^n]$. Define the function

$$G^n : \times_{k=0,\ldots,K} [\underline{S}, s_k^n] \to \times_{k=0,\ldots,K} [\underline{S}, s_k^n]$$

with $G^n(x^n) = y^n$. By Brouwer's fixed point theorem, there exists some vector x^{n*} such that $G^n(x^{n*}) = x^{n*}$. The probability that a type-k voter is

pivotal, conditional in the state being z, if everyone else is behaving according to the strategy vector x^{n*}, is

$$\prod_{\substack{k'=1,\ldots,K \\ k' \neq k}} (1 - F(x_{k'}^{n*}|z))^{n_{k'}} (1 - F(x_k^{n*}|z))^{n_k - 1}.$$

Following the steps of the case $K = 1$, it is simple to prove that x^{n*} is in fact a Nash equilibrium profile.

Finally, note that along the sequence $\{x^{n*}\}$

$$\begin{aligned} P\{d_n = C|Z = z_c\} &= \prod_{k=1,\ldots,K} (1 - F(x_k^{n*}|z_c))^{n_k} \\ &\geq \prod_{k=1,\ldots,K} (1 - F(s_k^n|z_c))^{n_k}, \end{aligned}$$

where the last inequality follows from $x_k^{n*} \leq s_k^n$. Following the steps of the case $K = 1$, we have that $(1 - F(s_k^n|z_c))^{n_k}$ converges to one as $n \to \infty$ for $k = 1, \ldots, K$. It follows that $P\{d_n = C|Z = z_c\}$ also converges to one. ■

The following result is a robustness check on Theorem 2. It tells us that if the clear evidence condition holds approximately, then there is a sequence of Nash equilibria that nearly converges to the perfect information outcome. Suppose without loss of generality that $r_1/q_1 > \max_{k>1} r_k/q_k$ (if $K > 1$) and denote

$$\rho = \frac{r_1 P\{Z = z_a\}}{q_1 P\{Z = z_c\}} \quad \text{and} \quad \epsilon = \lim_{s \downarrow \underline{S}} \frac{f(s|z_c)}{f(s|z_a)}.$$

We have:

Theorem 3 *In the model with two states, continuous signals, and finite number of types, if $0 < \epsilon < \rho$, then there is a sequence of Nash equilibria such that*

$$\begin{aligned} \lim_{n \to \infty} P\{d_n = C|Z = z_c\} &= (\epsilon/\rho)^{\epsilon/(1-\epsilon)}, \\ \lim_{n \to \infty} P\{d_n = C|Z = z_a\} &= (\epsilon/\rho)^{1/(1-\epsilon)}. \end{aligned}$$ □

PROOF Consider the vector of cutoff strategies given by s_1^n for $k = 1$ and $\underline{S}$ for every other k for n large enough. Here s_1^n is defined as in the proof of Theorem 2. Under the proposed strategies,

$$\begin{aligned} P\{d_n = C|Z = z_c\} &= (1 - F(s_1^n|z_c))^{n_1}, \\ P\{d_n = C|Z = z_a\} &= (1 - F(s_1^n|z_a))^{n_1}. \end{aligned}$$

The limits in the statement of this Theorem are obtained from equation (3) in the proof of Theorem 2, and an analogue for the state $Z = z_a$.

We claim that the proposed strategies constitute a Nash equilibrium for n large enough. From the proof of Theorem 2, each type-1 voter is playing a best response to the other voters' strategies. With respect to voters of type $k > 1$, they are playing a best response if

$$q_k P\{piv|Z = z_c\} f(s|z_c) P\{Z = z_c\} \quad \geq \quad r_k P\{piv|Z = z_a\} f(s|z_a) P\{Z = z_a\}$$

for every $s \in S$. Rearranging this expression,

$$\frac{r_k}{q_k} \leq \frac{P\{piv|Z = z_c\}}{P\{piv|Z = z_a\}} \cdot \frac{f(s|z_c)}{f(s|z_a)} \cdot \frac{P\{Z = z_c\}}{P\{Z = z_a\}}.$$

Under the proposed strategies, for every voter of type $k > 1$,

$$\frac{P\{piv|Z = z_c\}}{P\{piv|Z = z_a\}} = \left(\frac{1 - F(s_1^n|z_c)}{1 - F(s_1^n|z_c)}\right)^{n_1}.$$

Using

$$\lim_{n\to\infty} \left(\frac{1 - F(s_1^n|z_c)}{1 - F(s_1^n|z_c)}\right)^{n_1} = \frac{\rho}{\epsilon},$$

the definitions of ρ and ϵ, and $r_1/q_1 > r_k/q_k$ for $k > 1$, we obtain the desired conclusion. ■

Of the additional assumptions imposed in this section, continuity is generally necessary for existence of nontrivial equilibria. (There always exist trivial equilibria in which two or more voters vote to acquit regardless of their signals because each of them knows that someone else is doing the same, so the defendant is acquitted regardless of the state.) Extending Theorems 2 and 3 to an infinite number of types or more than two states requires adding more structure to the preferences and information system of voters in order to ensure that best responses can still be written as cutoff strategies.

5 Examples

5.1 The Gamma Model

In the continuous model of last section, let $f(s|z_c)$ be a gamma density with parameters $\lambda_c, \tau_c > 0$; that is

$$f(s|z_c) = \frac{1}{,(\tau_c)} \lambda_c s^{\tau_c - 1} e^{-\lambda_c s}, \quad s \in (0, \infty).$$

Similarly, let $f(s|z_a)$ be a gamma density with parameters $\lambda_a, \tau_a > 0$. The increasing likelihood ratio assumption implies that $\tau_c \geq \tau_a$, $\lambda_c \leq \lambda_a$, with at least one of the two inequalities holding strictly. It is easy to verify that the clear evidence condition holds except in the case $\tau_c = \tau_a$.

5.2 The Beta Model

In the continuous model of last section, let $f(s|z_c)$ be a beta density with parameters $\alpha_c, \beta_c > 0$; that is

$$f(s|z_c) = \frac{,(\alpha_c + \beta_c)}{,(\alpha_c),(\beta_c)} s^{\alpha_c - 1}(1-s)^{\beta_c - 1}, \quad s \in (0, 1).$$

Similarly, let $f(s|z_a)$ be a beta density with parameters $\alpha_a, \beta_a > 0$. The increasing likelihood ratio assumption implies that $\alpha_c \geq \alpha_a$, $\beta_c \leq \beta_a$, with at least one of the two inequalities holding strictly. The clear evidence condition holds except in the case $\alpha_c = \alpha_a$.

5.3 A Numerical Exercise

Consider the beta model with parameters $\alpha_c = \beta_a = 4$ and $\beta_c = \alpha_a = 1$. There is a type of voters who prefer to convict regardless of the state (type 0), and a type of voters who prefer to acquit in state z_a and to convict in state z_c (type 1). Both states are equally likely *ex ante*, and $r_1/q_1 = 1$, so that type-1 voters experience equal disutility from both mistakes, acquitting in state z_c or convicting in state z_a. There are 12 voters, of which n_1 are of type 1 and $12 - n_1$ are of type 0. Table 1 below shows the probability of a mistaken decision conditional on the two states for different values of

n_1. The second column shows the cutoff strategy followed by type-1 voters in equilibrium. (Type-0 voters vote to convict all the time.) Consistent with the theoretical development in the previous section, the equilibrium cutoff for type-1 voters converges to $\underline{S} = 0$. We can see that, as the number of type-1 voters increases, the probability of a mistaken decision declines noticeably.

Table 1: Probabilities of mistakes under unanimity rule.

n_1	s_1	$P\{A\|z_c\}$	$P\{C\|z_a\}$
1	.50000	6.2500%	6.2500%
2	.35779	3.2506%	2.8935%
3	.28846	2.0628%	1.6842%
4	.24546	1.4442%	1.1039%
5	.21554	1.0745%	0.7788%
6	.19323	0.8336%	0.5781%
7	.17581	0.6669%	0.4454%
8	.16176	0.5464%	0.3530%
9	.15012	0.4562%	0.2863%
10	.14030	0.3868%	0.2365%
11	.13187	0.3321%	0.1985%
12	.12455	0.2884%	0.1687%

Table 2 below shows the probabilities of mistakes for the voting rule that *maximizes* the expected payoff for type-1 voters. This is equivalent to minimizing the sum of the probabilities of mistakes. A rule is now described by the number m of votes it requires to convict. Unanimity is then characterized by $m = 12$. To compute probabilities for rules other than unanimity, we derive a symmetric cutoff equilibrium strategy from an expression similar to equation (2) (for $m > 12 - n_1$):

$$\left(\frac{F(s_1^{n,m}|z_c)}{F(s_1^{n,m}|z_a)}\right)^{12-m} \left(\frac{1-F(s_1^{n,m}|z_c)}{1-F(s_1^{n,m}|z_a)}\right)^{n_1+m-13} \frac{f(s_1^{n,m}|z_c)}{f(s_1^{n,m}|z_a)} \quad = \quad 1.$$

The probabilities of conviction are then given by

$$P^m\{C|z\} = \sum_{j=12-m}^{n_1} \binom{n_1}{j} (1 - F(s_1^{n,m}|z)^j (F(s_1^{n,m}|z)^{n_1-j}.$$

The second column now shows the optimal rule for type-1 voters, and the third column their equilibrium cutoff strategy.

Table 2: Probabilities of mistakes under the rule that minimizes $P^m\{A|z_c\} + P^m\{C|z_a\}$.

n_1	m	s_1^m	$P^m\{A\|z_c\}$	$P^m\{C\|z_a\}$
1	12	.50000	6.2500%	6.2500%
2	12	.35779	3.2506%	2.8935%
3	11	.50000	1.1230%	1.1230%
4	11	.42069	0.5643%	0.5232%
5	10	.50000	0.2218%	0.2218%
6	10	.44490	0.1100%	0.1041%
7	9	.50000	0.0458%	0.0458%
8	9	.45777	0.0226%	0.0216%
9	8	.50000	0.0097%	0.0097%
10	8	.46576	0.0048%	0.0046%
11	7	.50000	0.0021%	0.0021%
12	7	.47120	0.0010%	0.0010%

Considering that type-0 voters always vote to convict, for type-1 voters the optimal rule is the one that requires $(n_1 + 1)/2$ votes from type-1 voters to convict if n_1 is odd and either $n_1/2$ or $(n_1+2)/2$ votes from type-1 voters if n_1 is even (we report the second case in the table). That is, simple majority among type-1 voters is the optimal voting rule. The equilibrium cutoff for type-1 voters under the optimal rule is 1/2 if n_1 is odd and approximately 1/2 if n_1 is even. The reason is that, in this example, $f(s|z_c) = f(1 - s|z_a)$ for $s \in (\underline{S}, \overline{S}) = (0, 1)$, and type-1 voters are equally concerned about both mistakes.

Note that convergence to the perfect information outcome is much faster under the optimal voting rule than under unanimity. However, for n_1 small enough, every rule characterized by a fixed m other than unanimity leads to conviction regardless of the state.

6 Related Literature and Discussion

Unanimous voting has received some attention after a provocative piece by Feddersen and Pesendorfer (1998). In a setting in which a group of jurors with common preferences have two available signals, indicating that culpability or innocence of a defendant is more likely, they show that requiring unanimity to convict the defendant performs worse than other voting rules and fails to exhibit convergence to the complete information outcome. The example produced by Feddersen and Pesendorfer does not satisfy the clear evidence condition described in this paper. In fact, the condition cannot be satisfied if there is only a finite number of (imperfect) signals available to voters. We take this as an indication that a finite number of signals is an inappropriate assumption. The numerical example in section 5 illustrates that convergence may be relatively fast under unanimous voting. But it also illustrates the point that majority rule dominates unanimity if voters have common preferences and show equal concern for both types of mistakes. Unanimity rule makes sense in the example only as a way to make sure that the viewpoint of a minority is respected.

The model in section 4 extends the work of Duggan and Martinelli (1998) to the case of heterogeneous preferences. Meirowitz (1999) offers another model with continuous signals and common preferences. Common preferences does not seem to be the right assumption to discuss unanimity rule: unanimity rule seems unlikely to be the optimal voting rule in the sense of maximizing the expected utility of a group of like-minded voters. Li, Rosen, and Suen (1999) consider a model with continuous signals and heterogenous preferences, but they restrict their attention to a two-member committee.

Of some relevance for the current paper, Li *et al.* show that conflicts of interests among the members of the committee preclude the use of reports defined on an arbitrarily fine partition of the information available. This provides a foundation for the use of voting rules, which require of each voter only a report on a two-partition of the set of possible signals. On a related matter, with a small electorate, it makes sense to ask whether the results are sensitive to introducing a round of debate before voting. Earlier work by Austen-Smith (1990) shows that debate can have effects on information aggregation by a committee only if the distribution of preferences with respect

to outcomes is narrow.

Other recent research on the jury problem includes the work of Chwe (1999) on non-anonymous decision rules that maximize the utility of a type of voter in the majority, and the work of Persico (1999) on information acquisition by voters.

More generally, this paper is related to the literature on information aggregation in elections; some references are Young (1988), Austen-Smith and Banks (1996), and Feddersen and Pesendorfer (1997). Feddersen and Pesendorfer (1997), in particular, provide convergence results for (non-unanimous) voting rules. They impose a bound on how good a signal might be in distinguishing between different circumstances, and show that convergence to the perfect information outcome is obtained along a sequence of equilibria, even though only a vanishing fraction of voters vote according to their private information (their setup includes a continuum of types). In contrast, the current paper shows that unboundedness is a necessary condition for convergence to the perfect information outcome under unanimity rule. With an unbounded likelihood ratio, every voter (other than those of type 0) votes according to their private information even as n grows large. With a bounded likelihood ratio, only approximate convergence can hold, and for large enough n only the more lenient voters vote to acquit or to convict according to the signal they receive.

Last, but most importantly, this paper is related to the literature on information aggregation in auctions: Wilson (1977), Milgrom (1981), Pesendorfer and Swinkels (1997), and in particular Milgrom (1979). The model described in section 2 is a recasting of that of Milgrom (1979) in the context of unanimous voting; the proof of necessity in section 3 (but not that of sufficiency) follows his proof closely. While a requirement equivalent to clear evidence is necessary and sufficient for convergence to the true value in the case of a single-prize auction, in the case of unanimous voting it is only sufficient for convergence conditional on the "right" decision being A. Voting imposes upon voters a coordination problem, and unanimous voting puts all the burden of the coordination problem on the decision to select C.

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