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Group Formation and Voter Participation

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M E X I C O

GROUP FORMATION AND VOTER PARTICIPATION

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Abstract. We present a model of participation in large elections with endogenous formation of voter groups. Citizens decide whether to be followers or become leaders (activists) and try to bring other citizens to vote for their preferred party. In the (unique) pure strategy equilibrium, the number of leaders favoring each party is a function of the cost of activism and the importance of the election. Expected turnout and winning margin in the election are, in turn, a function of the number of leaders and the strength of social interactions. The model predicts a non monotonic relationship between expected turnout and winning margin in large elections.

Keywords: Voter's Paradox, Endogenous Leaders, Turnout, Winning Margin. *JEL* D72.

"Si nos habitudes naissent de nos propres sentiments dans la retraite, elles naissent de l'opinion d'autrui dans la société. Quand on ne vit pas en soi mais dans les autres, ce sont leurs jugements qui règlent tout . . . "
Jean-Jacques Rousseau, Lettre à M. d'Alembert (1758)

"Most people are other people. Their thoughts are someone else's opinions, their lives a mimicry, their passions a quotation."
Oscar Wilde, De Profundis (1905)

1. Introduction

The decision to cast a vote in a large election, it has been observed, is strongly correlated with indicators of how well integrated is the individual in society. Empirical research on the US and other countries has found that the better educated, the eldest, the most religious, the

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married, and the less mobile are more likely to vote.[1] The decision to cast a vote, it has also been observed, has a weak relation with the individual cost and benefit of voting, if the benefit is proportional to the likelihood of a single vote deciding the election. Election closeness is correlated with aggregate turnout in different countries, but the relationship completely or almost completely vanishes at the individual level, especially if direct measures of the probability of casting a decisive vote are employed.[2] And variations in the cost of voting due to, say, weather conditions, have been shown to be of marginal significance.[3]

Overall, the evidence suggests that in order to account for voluntary participation in large elections we must look beyond the individual voters to include in the picture the groups they belong to. This is, perhaps, unsurprising. If voters are motivated only by the effect of their actions in the result of the election, and there is but a slight cost involved in the act of voting, game-theoretic models predict a dismally low turnout, as long as voters are somewhat uncertain about the preferences of others – a prediction clearly at odds with mass participation in elections.[4] Thus, for empirical and theoretical reasons, attention has turn to modelling participation in elections as a group activity.

The earliest group-based models of electoral turnout (e.g. Uhlaner [23] and Morton [15]) effectively substitute a game between relatively few players (groups or rather their leaders) for a game between many players (voters at large). Early models emphasize either side-payments, social pressure, or "group identity" as explanations of why individual voters would follow the group leaders. More recently, Feddersen and Sandroni [7], following Harsanyi [9], have proposed an ethical theory of group behavior in elections. (See also Coate and Conlin [4] for a different take on the same proposal.)

Group-based models of turnout proposed so far have in common the idea that the electorate is divided into distinct, prearranged groups. In this paper we attempt to explain how the electorate is divided in

[1]The landmark study of Wolfinger and Rosenstone [24] in the US shows the importance of education, age, marriage and mobility. The recent study by Blais [3] in nine democratic countries, including the US, maintains education and age at the top of the list and adds religiosity, income and marriage.

[2]Matsusaka and Palda [14] and Matsusaka [12] provide evidence of the insensitivity of individual voters to election closeness. Blais [3] provides evidence of a marginal effect of closeness and of no effect of direct measures of the probability of casting a decisive vote.

[3]Knack [10] shows that the impact of rain on turnout in the US is minimal. See again the excellent discussion in Blais [3].

[4]See e.g. Palfrey and Rosenthal [18, 19], who build on earlier work by Ledyard [11]. An elegant treatment of the subject is provided by Myerson [16].

groups and what are the implications for electoral participation in large elections. We propose a model in which there is a continuum of citizens distributed uniformly over a circle. Citizens must decide whether to become leaders, and mobilize groups of voters in support of the leader's preferred candidate in an election, or remain as followers. If a citizen becomes a leader, the citizen is assigned an interval of influence in the circle. The length of the influence interval of each leader is random and depends on the number of other leaders and possibly on some exogenous breaks in the circle, representing weak spots in social interactions. Leaders are able to sway citizens in their interval of influence to the voting booth. Becoming a leader is costly, but carries the potential benefit of swinging the election in the direction favored by the leader.

The model has a unique pure strategy equilibrium, described by a simple formula comparing the probability of a marginal leader being decisive with the cost/benefit ratio of political activism. This in itself is noteworthy since, as stated in a recent survey by Feddersen [6], existence of equilibria has been a central problem for group-based models of turnout. After showing the existence and uniqueness of equilibrium, we analyze the effects of changes in the parameters of the model on the endogenous variables, namely the expected turnout and the expected margin of victory in the election. As expected, a reduction in the cost/benefit ratio of activism, by increasing the number of leaders for each party, results in a larger expected turnout. More surprisingly, reductions in the cost/benefit ratio of activism have non monotonic effects on the expected margin of victory. Increasing the number of leaders for each party increases the expected winning margin if expected turnout is below 50% and decreases the expected winning margin if expected turnout is above 50%. As a result, the relationship between the winning margin and turnout is non monotonic. For commonly observed turnout levels, though, the model predicts a positive association between turnout and election closeness, in accordance with the evidence.

For relatively low cost/benefit ratios of activism, that is when there are many leaders in equilibrium, it is possible to relate the parameters of the model quite precisely to the "observable" variables, winning margin and turnout. We illustrate this using data from US presidential elections before and after the well-known fall in turnout in the sixties and seventies. Of course, this is a just a suggestive exercise, meant to show the advantage of having at least a simple, highly stylized model to help make sense of the data.

Our model has some resemblance to the citizen-candidate models pioneered by Osborne and Slivinsky [17] and Besley and Coate [2]. We

have in common with that literature the idea of endogenizing political activism. The type of political activism we try to capture in our model is radically different, however. In that literature, political activists are candidates themselves. The issue of turnout is not addressed and the focus is on candidate platforms and party formation. By contrast, we take party platforms as given and focus on the issue of turnout. In our model, political activists are not candidates themselves; rather, they are citizens interested in influencing the election outcome. We can also compare our model with the work on voter mobilization by Shachar and Nalebuff [22]. In our model, as opposed to Shachar and Nalebuff's, leaders are not exogenously given. The margin of analysis we focus on is that of the decision to influence others, rather than the effort invested in influencing others.

Our work is also related to the social interactions literature pioneered, *inter alia*, by Glaeser, Sacerdote and Scheinkman [8].[5] We borrow from Glaeser *et al.* the arrangement of agents in a circle and the idea that some agents imitate their group behavior while others act independently, depending on their voting cost. We deviate in that the number of agents that act as leaders is derived endogenously in the model. When the costs and benefits of an activity, such as casting a vote or issuing an opinion, are very low, most people may be content to follow the lead of a few, as expressed, perhaps not without certain pessimism, by Rousseau and Wilde in our introductory quotations.

2. The Model

We consider an election with two alternatives or parties, A and B. There is a continuum of citizens with measure one. A (positive measure) subset of citizens are A-partisans, another (positive measure) subset of citizens B-partisans, and the remainder are non-partisans. Partisans enjoy a net gain of $G > 0$ if their preferred party wins the election, while non-partisans are indifferent with respect to the result of the election.

Citizens make decisions in three stages. In the first stage, every citizen chooses whether to become an active supporter of a party or not. We refer to an active supporter as an opinion *leader*, and to a citizen who is not an active supporter as a *follower*. Becoming a leader in favor of one party or the other involves a utility cost $c > 0$. In the second stage, followers are assigned to one of the leaders or to no leader at all, according to a random process that is described in detail

[5]See Becker and Murphy [1] and Durlauf and Young [5] for more recent contributions.

below. Followers under the range of influence of some leader are offered by their leader a compensation of $v + \varepsilon$ in exchange for committing to vote for the party of that leader, where ε is arbitrarily small. These followers have the option of accepting this compensation, or rejecting it. In case of rejection, they receive no compensation but are free to abstain or vote for who they want, just like the followers who are not assigned to any leader. In the third stage, every citizen decides whether to vote for one party or abstain. Voting involves a cost $v > 0$, while abstaining involves no cost. The party favored by the larger fraction of votes wins the election; if both parties receive the same measure of votes, there is a tie and it is resolved by a fair coin toss.

The fraction of followers that a given leader gets assigned in the second stage of the game is random and depends negatively on how many other citizens become leaders. We assume that leaders are dropped uniformly on a circle of measure one, which represents the population of citizens. A fixed number of exogenous interruptions O is also dropped uniformly on the circle. Each leader gets assigned the interval of citizens of agents to the right of the leader, until the interval is interrupted by another leader or by an exogenous interruption. The inverse of the number of exogenous interruptions is meant to represent the strength of social interactions. When social interactions are strong, the influence of an opinion leader is likely to extend over the circle representing society until it is contested by some other opinion leader. *Per contra*, when social interactions are weak, the influence of an opinion leader is likely to die out at some point even before it is contested. The assumption that the number of exogenous interruptions is fixed is done only for simplicity; as will be explained later on (in Section 5), our results hold if the number of exogenous interruptions is random.

It is easy to check that best response behavior implies that uncommitted citizens or citizens who are not assigned to any leader will abstain in the third stage of the game. Moreover, citizens under the influence of one leader in the second stage of the game will always commit to vote for the party of that leader. With respect to leaders, we can assume that they have to commit to vote for the party they support, so that the cost of becoming a leader includes the cost of voting. (This assumption is really unnecessary since in equilibrium there will be only a finite – i.e. measure zero – set of leaders in favor of either party.) Note that if no citizen volunteers to become a leader in the first stage of the game, all citizens remain uncommitted in the third stage. In that case, since the influence of a single vote is negligible, no one shows up to vote and there is a tied election.

It is also easy to check that only A-partisans will want to become leaders in favor of party A, and similarly for B. Let L_A and L_B represent the (finite) number of A and B-partisans that become leaders. Using the model, we can calculate the probability that A wins the election, say $P(L_A, L_B)$. Define

$$P_A(L_A, L_B) = P(L_A, L_B) - P(L_A - 1, L_B),$$

$$P_B(L_A, L_B) = P(L_A, L_B - 1) - P(L_A, L_B)$$

as the probabilities that a leader for A and for B, respectively, are decisive in favor of their party. We refer to these probabilities as *decisiveness in favor of* A and *decisiveness in favor of* B, respectively, with the mention of the party suppressed in symmetric situations.

For L_A or L_B (countably or uncountably) infinite, we simply assume that the probability of being decisive is zero. This guarantees that there is no equilibrium in which more than a finite number of citizens become leaders.

A (subgame perfect, pure strategy Nash) *equilibrium* is a pair of nonnegative integers L_A^*, L_B^* such that

$$\begin{aligned}
&P_A(L_A^* + 1, L_B^*) \leq c/G, \\
&P_A(L_A^*, L_B^*) > c/G \text{ if } L_A^* \geq 1, \\
&P_B(L_A^*, L_B^* + 1) \leq c/G, \\
&P_B(L_A^*, L_B^*) > c/G \text{ if } L_B^* \geq 1.
\end{aligned}$$

The strict inequalities in the definition are due to the fact that if either L_A^* or L_B^* is positive, a citizen who remains a follower makes a (small) utility gain ε with positive probability.

Note that the definition of equilibrium makes no reference to the identities of the partisans who become leaders; that is, we make no distinction between any two situations in which different citizens become leaders, as long as the number of leaders for each party remains constant.

Our definition of equilibrium is restricted to pure strategy equilibria, even though it is possible to show that our model admits some mixed strategy equilibria. As pointed out by Feddersen [6], the interpretation of mixed strategy equilibria presents conceptual problems in group-based models. In a mixed strategy equilibrium, partisan citizens who become leaders are supposed to start conveying instructions to followers without being overheard by other potential leaders.

We describe the equilibria of the model in the next section.

3. Decisiveness and Equilibrium

We have

Theorem 1. *There is a unique equilibrium. If $c/G < 1/2$, in equilibrium $L_A^* = L_B^* = L^*$, where $L^* \geq 1$ is the largest integer solution to the inequality*

$$\frac{1}{2^{2L-1}} \frac{(2L-2)!}{(L-1)!\,(L-1)!} \quad > \quad c/G.$$

If $c/G \geq 1/2$, $L_A^* = L_B^* = 0$. Moreover, if $c/G < 1/2$, the expected turnout in equilibrium is

$$\frac{L^*}{L^* + O/2},$$

and the expected winning margin (the difference in the fraction of votes for the two parties) in equilibrium is

$$\frac{L^*}{L^* + O/2} \left(\frac{1}{2^{2L^*}} \frac{(2L^*)!}{L^*!L^*!} \right).$$

As it will be clear from the proofs, the expression

$$\frac{1}{2^{2L-1}} \frac{(2L-2)!}{(L-1)!\,(L-1)!}$$

represents decisiveness when there are L leaders for each party. Note that this expression is strictly decreasing in L, and takes the value $1/2$ when $L = 1$. Note also that decisiveness approaches asymptotically (using the Stirling formula) $1/\sqrt{4\pi\,(L-1)}$. Since this last expression declines relatively slowly with L, the equilibrium number of leaders is not necessarily small.

The absence of asymmetric equilibria is due to the symmetry property of decisiveness, as described below. Intuitively, if a party has more leaders than the other party, and their partisans have no incentive to deviate increasing or decreasing the number of their leaders, then necessarily some partisan of the party in the minority has incentives to become an additional leader.

Before the proof of the theorem, we state a series of lemmas that are proved in the Appendix.

First, we consider the question of how a given number of leaders for each party L_A, L_B maps into the distribution of votes.

Lemma 1. *If L_A, L_B and O are positive, the joint probability density function of the fraction of votes a for party A and b for party B is*

$$h_{L_A,L_B}(a,b) = \frac{(L_A+L_B+O-1)!}{(L_A-1)!\,(L_B-1)!\,(O-1)!}\,a^{L_A-1}b^{L_B-1}\,(1-a-b)^{O-1}$$

for $0 \leq a+b \leq 1$.

If L_A and L_B are positive and $O=0$, the probability density function of a is

$$h^0_{L_A,L_B}(a) = \frac{(L_A+L_B-1)!}{(L_A-1)!\,(L_B-1)!}\,a^{L_A-1}\,(1-a)^{L_B-1}$$

for $0 \leq a \leq 1$, with $b = 1-a$.

Lemma 1 establishes that the joint distribution of the fraction of votes going to party A and to party B is equal to the joint distribution of the L_A-th order statistic and the difference between the (L_A+L_B)-th and the L_A-th order statistic of a sample of size L_A+L_B+O-1 drawn from a uniform distribution over the unit interval.[6] The idea of the proof is the following. We can pick the location in the circle of any leader for party A and consider that location point 0 (from the left) and point 1 (from the right). Thus, the remainder of leaders and exogenous breaks is a uniform sample of size L_A+L_B+O-1. Due to a symmetry property of uniform order statistics, we can calculate the fraction of votes for A and for B as if all the remaining leaders for A would come first in the unit interval, and all the leaders for B would come next.

Next, we calculate the probability that party A wins the election using the distributions described by Lemma 1.

Lemma 2. *If L_A and L_B are positive, the probability of party A winning the election is*

$$P(L_A,L_B) = 1-\sum_{k=1}^{L_B}\left(\frac{1}{2}\right)^{L_A+L_B-1}\frac{(L_A+L_B-1)!}{(L_A+k-1)!\,(L_B-k)!}.$$

Note that the probability of party A winning the election is independent of the number of exogenous interruptions. Following the line of the previous intuitive argument, the probability that the L_A-th order

[6]We follow the convention of naming the smallest element in the sample the first-order statistic, and so on.

statistic is larger than the difference between the $(L_A + L_B)$-th and the L_B-th order statistic is independent of the size of the sample.

Next, we use Lemma 2 to calculate the decisiveness in favor of A.

Lemma 3. *If L_A and L_B are positive, the decisiveness in favor of A is*

$$P_A(L_A, L_B) = \frac{1}{2^{L_A+L_B-1}} \frac{(L_A + L_B - 2)!}{(L_A - 1)!\,(L_B - 1)!}.$$

The expression for decisiveness provided by Lemma 3 is exceedingly simple and plays a key role in the proof of the theorem.

Finally, we derive some useful properties of decisiveness from the expression above.

Lemma 4. *If L_A and L_B are positive, decisiveness satisfies the following properties:*

(i) (*near single-peakedness*)

$$P_A(L_A, L_B) \gtreqless P_A(L_A + 1, L_B) \iff L_A + 1 \gtreqless L_B.$$

(ii) (*symmetry*)

$$P_A(L_A, L_B) = P_A(L_B, L_A) = P_B(L_B, L_A).$$

Lemma 4 establishes, in particular, that decisiveness in favor of a party achieves its maximum value when this party has either the same number of leaders than the other party or one fewer leader.

We are now ready to prove the theorem.

Proof of Theorem 1. Using the equilibrium conditions, any equilibrium L_A^*, L_B^* with a positive number of leaders for party A and for party B must be such that

$$P_A(L_A^*, L_B^*) > c/G \geq P_A(L_A^* + 1, L_B^*),$$
$$P_B(L_A^*, L_B^*) > c/G \geq P_B(L_A^*, L_B^* + 1).$$

Using symmetry (Lemma 4*ii*), the equilibrium conditions are equivalent to

$$P_A(L_A^*, L_B^*) > c/G \geq P_A(L_A^* + 1, L_B^*),$$
$$P_A(L_B^*, L_A^*) > c/G \geq P_A(L_B^* + 1, L_A^*).$$

Using near single-peakedness (Lemma 4*i*), we get

$$L_A^* + 1 > L_B^* \quad \text{and} \quad L_B^* + 1 > L_A^*,$$

which together imply

$$L_A^* = L_B^*.$$

Thus, any equilibrium with a positive number of leaders for party A and for party B must be such that $L_A^* = L_B^* = L^*$, where L^* satisfies

$$P_A(L^*, L^*) > c/G \geq P_A(L^* + 1, L^*).$$

Using symmetry, this condition is equivalent to

$$P_A(L^*, L^*) > c/G \geq P_A(L^*, L^* + 1).$$

Using near single-peakedness (Lemma 4i), we get that the equilibrium condition is

$$P_A(L^*, L^*) > c/G \geq P_A(L^* + 1, L^* + 1).$$

Now, from Lemma 3 we get

$$P_A\left(L, L\right) = \frac{1}{2^{2L-1}} \frac{(2L-2)!}{(L-1)!\,(L-1)!}.$$

Note that this expression is strictly decreasing in L and it takes the value $1/2$ when $L = 1$. Thus, there are equilibria with a positive number of leaders for party A and for party B if and only if $c/G < 1/2$, and they are as described by the statement of the theorem.

Consider now an equilibrium in which there is a positive number L_A^* of leaders for party A and no leaders for party B. The equilibrium conditions are

$$P_A(L_A^*, 0) > c/G \geq P_A(L_A^* + 1, 0),$$
$$c/G \geq P_B(L_A^*, 1).$$

Recall that if both parties receive the same fraction of votes, the election is resolved by a fair coin toss. Since neither party receives votes when there are no leaders, we have $P(0, 0) = 1/2$. Since only party A receives a positive fraction of votes when L_A^* is positive and L_B^* is zero, we have $P(L_A^*, 0) = 1$ for $L_A^* \geq 1$. Thus, $P_A(1, 0) = 1/2$ and $P_A(L_A^*, 0) = 0$ for $L_A^* \geq 2$. Using the first equilibrium condition above, we get $L_A^* = 1$. From symmetry and Lemma 3, we get $P_B(1, 1) = P_A(1, 1) = 1/2$. Thus, for $L_A^* = 1$ the first equilibrium above implies $1/2 > c/G$ and the second equilibrium condition implies $c/G \geq 1/2$, a contradiction. It follows that there are no equilibria in which only party A obtains a positive number of leaders. A similar argument shows that there are no equilibria in which only party B obtains a positive number of leaders.

Consider, finally, an equilibrium in which neither party obtains a positive number of leaders. The equilibrium conditions are

$$c/G \geq P_A(1, 0),$$
$$c/G \geq P_B(0, 1).$$

We established in the previous paragraph that $P_A(1,0) = 1/2$. A similar argument shows that $P_B(0,1) = 1/2$. Thus, there is an equilibrium in which neither party obtains a positive number of leaders if and only if $c/G \geq 1/2$.

For the remainder of the proof we assume that there is a positive number of leaders L for each party. With respect to voter participation, for $O = 0$ it is easy to see that turnout is one, as results from the formula provided in the statement of the theorem. For $O \geq 1$, we have that the expected turnout in equilibrium is

$$\mathrm{E}(a+b) \quad = \quad \int_0^1 \int_0^{1-a} (a+b)\, h_{L,L}\,(a,b)\; db\, da.$$

Using Lemma 1,

$$\mathrm{E}(a+b) = \frac{(2L+O-1)!}{(L-1)!\,(L-1)!\,(O-1)!} \times \int_0^1 \int_0^{1-a} (a+b)\, a^{L-1} b^{L-1}\,(1-a-b)^{O-1}\; db\, da.$$

Or equivalently,

$$\mathrm{E}(a+b) = \frac{(2L+O-1)!}{(L-1)!\,(L-1)!\,(O-1)!} \times \int_0^1 \int_0^{1-a} \left(a^L b^{L-1}\,(1-a-b)^{O-1} + a^{L-1} b^L\,(1-a-b)^{O-1} \right)\; db\, da.$$

Using Lemma 1 again,

$$\mathrm{E}(a+b) = \frac{(2L+O-1)!}{(L-1)!\,(L-1)!\,(O-1)!} \times \frac{(L-1)!L!\,(O-1)!}{(2L+O)!} \times \left(\int_0^1 \int_0^{1-a} h_{L+1,L}(a,b)\, db\, da + \int_0^1 \int_0^{1-a} h_{L,L+1}(a,b)\, db\, da \right).$$

Since $h_{L+1,L}(a,b)$ and $h_{L,L+1}(a,b)$ are bivariate probability density functions with support $0 \leq a+b \leq 1$, we get

$$\mathrm{E}\,(a+b) \quad = \quad \frac{(2L+O-1)!}{(L-1)!\,(L-1)!\,(O-1)!} \times \frac{(L-1)!L!\,(O-1)!}{(2L+O)!} \times 2.$$

The result on turnout in the statement of the theorem follows.

With respect to the closeness of the election, suppose that $O \geq 1$. (The proof for $O = 0$ is similar.) Using Lemma 1, the expected winning

margin is

$$\begin{aligned}
\mathrm{E}\left(|a-b|\right) &= 2\mathrm{E}\left(a-b \,|\, a>b\right) \\
&= 2\int_0^1 \int_0^{\min\{a,1-a\}} (a-b)\, h_{L,L}\left(a,b\right)\, db\, da \\
&= \frac{2\,(2L+O-1)!}{(L-1)!\,(L-1)!\,(O-1)!} \times \\
&\qquad \int_0^1 \int_0^{\min\{a,1-a\}} (a-b)\, a^{L-1} b^{L-1} \left(1-a-b\right)^{O-1}\, db\, da \\
&= \frac{2\,(2L+O-1)!}{(L-1)!\,(L-1)!\,(O-1)!} \times \\
&\qquad \int_0^1 \int_0^{\min\{a,1-a\}} \Big(a^L b^{L-1} \left(1-a-b\right)^{O-1} \\
&\qquad\qquad -a^{L-1} b^L \left(1-a-b\right)^{O-1}\Big)\, db\, da.
\end{aligned}$$

Using Lemma 1 again,

$$\begin{aligned}
\mathrm{E}\left(|a-b|\right) = \frac{2\,(2L+O-1)!}{(L-1)!\,(L-1)!\,(O-1)!} \times \frac{(L-1)!L!\,(O-1)!}{(2L+O)!} \times \\
\left(\int_0^1 \int_0^{\min\{a,1-a\}} h_{L+1,L}(a,b)\, db\, da \right. \\
\left. - \int_0^1 \int_0^{\min\{a,1-a\}} h_{L,L+1}(a,b)\, db\, da\right).
\end{aligned}$$

Note that

$$P\left(L_A, L_B\right) = \int_0^1 \int_0^{\min\{a,1-a\}} h_{L_A,L_B}(a,b)\, db\, da.$$

Thus, we get

$$\mathrm{E}\left(|a-b|\right) = \frac{2L}{2L+O}\left(\mathrm{P}\left(L+1,L\right) - \mathrm{P}\left(L,L+1\right)\right).$$

Using the definitions of decisiveness in favor of A and B and symmetry (Lemma 4ii),

$$\begin{aligned}
\mathrm{E}\left(|a-b|\right) &= \frac{2L}{2L+O}\left(P_A\left(L+1,L+1\right) + P_B\left(L+1,L+1\right)\right) \\
&= \frac{4L}{2L+O} P_A\left(L+1,L+1\right).
\end{aligned}$$

Using Lemma 3,

$$\mathrm{E}\left(|a-b|\right) = \frac{L}{L+O/2}\left(\frac{1}{2^{2L}}\frac{(2L)!}{L!L!}\right),$$

as stated by the theorem. □

4. Turnout and Winning Margin

In this section we analyze the relation between the parameters of the model, namely the cost/benefit ratio of political activism (c/G) and the strength of social interactions ($1/O$), and the endogenous, observable variables of the model, namely the expected turnout and the expected winning margin.

4.1. **Comparative Statics.** Theorem 1 calculates the equilibrium number of leaders for each party as a (decreasing) function of the cost/benefit ratio of activism. The effect of changes in the cost/benefit ratio of activism and the strength of social interactions on the equilibrium expected turnout follows immediately from Theorem 1.

Corollary 1. *A reduction in the cost/benefit ratio of activism (weakly) increases the expected turnout. An increase in the strength of social interactions increases the expected turnout and reduces the expected winning margin.*

While the effect of the cost/benefit ratio of activism on electoral turnout is unambiguous, its effect on the expected closeness of the election is more complex.

Corollary 2. *If a reduction in the cost/benefit ratio of activism leads to an increase in one in the number of leaders for each party, then the expected winning margin decreases if and only if initially the expected turnout is above 1/2 and it increases if and only if initially the expected winning margin is below 1/2.*

Proof. From Theorem 1, we have that the expected winning margin with $L+1$ leaders is

$$\frac{L+1}{L+1+O/2}\left(\frac{1}{2^{2L+2}}\frac{(2L+2)!}{(L+1)!(L+1)!}\right),$$

and with L leaders is

$$\frac{L}{L+O/2}\left(\frac{1}{2^{2L}}\frac{(2L)!}{L!L!}\right).$$

The ratio of these two expressions is

$$\frac{1+1/(2L)}{1+1/(L+O/2)},$$

which is smaller than one if and only if $O < 2L$ (or equivalently, if and only if expected turnout is smaller than 1/2) and is larger than one if and only if $O > 2L$. □

Intuitively, if the fraction of voters is smaller in expectation than the fraction of abstainers, then extra leaders are more likely to bring abstainers to vote rather than to steal voters from other leaders, thereby increasing the variance of the difference of votes between the two parties. If, instead, the fraction of voters is larger in expectation than the fraction of abstainers, then new leaders tend to steal voters from each other rather than bringing abstainers to vote, thereby reducing the variance of the difference of votes.

Let ET be the expected turnout and EW be the expected winning margin. Using Theorem 1, we can obtain an approximation for the parameters of the model when there are many activists (that is, when the cost/benefit ratio of activism is small):

Corollary 3. *For fixed* $1/O$ *and* $c/G \to 0$,

$$c/G \approx \frac{EW}{2ET} \quad \textit{and} \quad 1/O \approx \frac{\pi EW^2}{2ET(1-ET)}.$$

(See the proof in the Appendix.) We use these expressions in the context of a discussion about empirical evidence below.

4.2. **Empirical Evidence.** Average turnout in elections for national-level office in most democracies is above 50% (see e.g. Blais [3]). In those circumstances, our model predicts that movements in the cost/benefit ratio of activism lead to movements with the same sign in the winning margin. If, as seems likely, costs and benefits of activism experience more short term variations than the strength of social interactions, we should expect to see a negative short term correlation between winning margin and turnout. Long term trends in turnout, however, are likely to be affected by trends in both the cost/benefit ratio of activism and the strength of social interactions.

The relation between winning margin (or rather, election closeness) and turnout has been the subject of an empirical literature reviewed recently by Blais [3]. In this author's words,

> [...] the verdict is crystal clear with respect to closeness: closeness has been found to increase turnout in 27 out

> of the 32 studies that have tested the relationship, in many different settings and with diverse methodologies. ([3], p. 60)

Blais goes on to state that the importance of closeness is not captured by direct measures of the (very small) probability of a single vote being decisive.

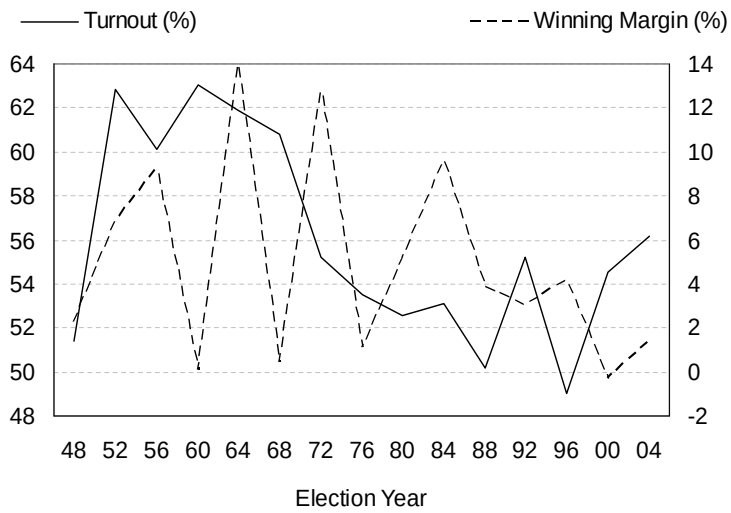
> [...] what affects the voting decision is not [...] the strict probability that the individual could cast a decisive vote, but rather the probability that the outcome could be decided by a relatively small number of votes. ([3], p. 139)

Closeness remains statistically significant even controlling for party spending ([3], p. 62), which is at least consistent with decentralized, group-based models of turnout such as ours.

Figure 1 provides some data from US presidential elections since 1948. Turnout is defined as the total of valid popular votes casted in the election as a percentage of the voting age population. Winning margin is defined as the difference between the votes received by the winning presidential candidate and the votes received by the second presidential candidate, as a percentage of the voting age population. Of course, presidential elections in the US are decided by the electoral college and not by the popular vote, so that the "winning margin" may in fact be negative, as it happened (barely) in the 2000 election. Although there seems to be some negative correlation between turnout and winning margin from 1976 on, the most salient feature of the data is the dramatic fall in turnout in the late sixties and early seventies, accompanied by a decrease in the winning margin. For the period 1952-1972, the average turnout is 60.66% and the average winning margin is 7.23%. The corresponding numbers for 1976-2004 are, respectively, 53.03% and 3.51%.

Matsusaka [13] discusses some of the literature dealing with the decline in turnout in the US. Matsusaka contends that demographic and legal changes alone do not explain the fall in turnout, and instead proposes that the decline in party identification rendered voters less capable of making an informed decision at the voting booth. A simple calculation using Corollary 3 suggests that the cost/benefit ratio of political activism fell from about 0.060 to 0.033 during the early 1970s, while at the same time the social interactions term fell from about 0.034 to 0.008. That is, perhaps in agreement with Matsusaka's contention, society become more fragmented even as barriers to political activity were lowered.

Figure 1: Turnout and Winnin Margin in US Presidential Elections



source: Leips' Atlas of US Presidential Elections; US Census Bureau

5. Extensions

Randomness in the number of exogenous interruptions in the circle is easily incorporated into the model. Though the model assumes ex-ante identical agents except for their party preferences, it can be extended to allow for heterogeneous voting costs and cost/benefit ratios of activism under some conditions.

5.1. **Random Interruptions.** Assume that the number of exogenous interruptions in the circle is a random variable $\tilde{O}$ with bounded support whose realization is not known by partisan citizens at the moment of deciding whether to become leaders or not. For any given number of leaders for party A and for party B, the probability of party A winning the election is the same regardless of the realization of $\tilde{O}$ and is given by Lemma 2. Thus, the decisiveness in favor of A is as given by Lemma 3 and therefore the equilibrium number of leaders is as given by Theorem 1. Expected turnout is now the expectation (conditional on the realization of $\tilde{O}$) of the expression for turnout in Theorem 1, and similarly for expected winning margin.

5.2. **Heterogeneous Voting Costs.** Assume that voting costs are heterogeneous across citizens and distributed according to some continuous probability density with support bounded away from zero, and that the fixed compensation offered by opinion leaders to voters is larger than the lower endpoint in the support of the density. In this case, followers who are not assigned to any leader still abstain, and followers who are assigned to a leader commit to vote as long as the (fixed) compensation offered by the leader is larger than their idiosyncratic voting cost. As a result, the decisiveness calculation of leaders is the same and so is their equilibrium number of leaders. Note that the citizens with low voting cost that become followers assigned to a leader may obtain a positive net benefit from voting. This raises their expected reservation value from being followers and biases the first stage leader-follower decision towards being followers. Citizens with low voting costs are more likely to be followers than leaders, unless we assume that leaders reward themselves too for voting. In either case these different reservation values do not change the results.

5.3. **Heterogeneous Leadership Costs.** Assume that the cost of activism (or equivalently, the importance of the election) is heterogeneous across citizens and distributed according to come continuous probability density with lower endpoint of the support $\underline{c} > 0$. Recall that the decision to become a leader depends on the decisiveness cost-benefit calculation. The equilibrium number of leaders L^* will be given by the largest integer solution to the inequality

$$P_A(L, L) > \underline{c}/G.$$

There is really no difference with the homogeneous case. There may be some inefficiency, though, since the leaders in equilibrium are not necessarily the citizens with lower costs. This is due to the discrete nature of the number of leaders. Only the citizens with cost/benefit ratio smaller than $P_A(L^*, L^*)$ may become leaders in this heterogeneous case.

5.4. **Different Effectiveness of Leaders.** We assumed so far that the interval of influence or cluster of a leader was random but the effectiveness of the leaders of each party was the same. That is, we assumed the leader could attract all the agents in the leader's interval of influence to vote for his party. More generally, we can assume that leaders of party A and party B can attract only a fraction $\alpha \in (0, 1]$ and $\beta \in (0, 1]$, respectively, of the potential voters of their interval of influence. This setup is general and allows for proportions α and β to

depend on the original preferences of the voters (provided that the proportions of A-partisans and B-partisans are the same in each interval). For instance, a leader for party A could attract all A-partisans and none of the B-partisans. For $\alpha = \beta$, the equilibrium number of leaders of the model is the one we previously obtained: only the turnout should be scaled down accordingly if $\alpha = \beta < 1$. If $\alpha \neq \beta$, then the equilibrium is no longer necessarily symmetric. To extend the model we need to recalculate the decisiveness in favor of A and B.

Lemma 5. *Let $\gamma = \alpha/(\alpha+\beta)$. If L_A and L_B are positive, the decisiveness in favor of A and B are, respectively,*

$$P_A(L_A, L_B) = (1-\gamma)^{L_A-1}\gamma^{L_B}\frac{(L_A+L_B-2)!}{(L_A-1)!\,(L_B-1)!} \quad \textit{and}$$

$$P_B(L_A, L_B) = ((1-\gamma)^{L_A}\gamma^{L_B-1}\frac{(L_A+L_B-2)!}{(L_A-1)!\,(L_B-1)!}.$$

(See the proof in the Appendix.)

The constant ratio of decisiveness

$$P_B\left(L_A, L_B\right)/P_A\left(L_A, L_B\right) = (1-\gamma)/\gamma = \beta/\alpha$$

is an indicator of the advantage of B relative to A. We can easily check

$$P_A\left(L_A, L_B\right) \gtreqless P_A\left(L_A-1, L_B\right) \iff \left(L_A-1\right) \lesseqgtr \left(\beta/\alpha\right)\left(L_B-1\right)$$

and

$$P_B\left(L_A, L_B\right) \gtreqless P_B\left(L_A, L_B-1\right) \iff \left(L_A-1\right) \gtreqless \left(\beta/\alpha\right)\left(L_B-1\right).$$

That is, both decisiveness in favor of A and decisiveness in favor of B peak when the number of leaders is at or near the ratio

$$\frac{L_A-1}{L_B-1} = \frac{\beta}{\alpha}.$$

As in the case of symmetric effectiveness, the equilibrium conditions

$$P_A(L_A, L_B) \;>\; \frac{c}{G} \;\geq\; P_A(L_A+1, L_B)$$

and

$$P_B(L_A, L_B) \;>\; \frac{c}{G} \;\geq\; P_B(L_A, L_B+1)$$

must be satisfied simultaneously, which implies that decisiveness must be at its peak. Note that for decisiveness to be at its peak, the party at disadvantage must have more leaders than the other party. That is, decisiveness peaks when the probabilities of winning the election are near 50%.

Existence of pure strategy equilibria becomes an issue when leaders have different effectiveness. If, for instance, α/β is not rational, then there is no equilibrium for $c/G < 1/2$, since the condition $(L_A - 1)/(L_B - 1) = \beta/\alpha$ cannot be satisfied. However, it can be shown that there are almost pure strategy equilibria in which only one potential leader per party randomizes between becoming a leader or not.

6. Final Remarks

There is no generally accepted model of such pervasive social phenomenon as massive voluntary participation in large elections. As pointed out in a recent survey article by Feddersen [6], the literature appears to be converging toward a group-based model of turnout, in which members participate in elections because they are directly coordinated and rewarded by leaders. This paper is intended to be a contribution to the literature on groups in elections. We treat all citizens as ex ante identical (except with respect to their political inclination) and have leaders self-selecting endogenously out of this homogeneous population. Thus, at the substantive level, our model sheds some light on how groups of voters can be formed out of the general population, and how voting behavior is affected by the underlying parameters of the model, namely the cost of political activism, the importance of the election and the strength or weakness of social interactions, even if the vast majority of voters do not behave strategically at the ballot.

We carry out the analysis in a highly stylized environment. At the technical level, we are able to obtain an attractive closed form expression for the equilibrium of the model. The equilibrium pins down uniquely the number of leaders for each party, the expected turnout, and the closeness of the election. We carry out comparative statics exercises, obtaining intuitive results with respect to expected turnout, and some unexpected results with respect to the expected winning margin. The fact that we deal successfully with issues of existence and uniqueness of pure strategy equilibria is, we believe, an encouraging step in the direction of a satisfactory group-based model of elections.

We believe our model is flexible enough to be extended in a number of dimensions beyond those discussed in the previous section. The version of the model with different effectiveness of leaders may be useful as a building block to incorporate variable effort of leaders. Though we focus on policy motivated leaders, incorporating a private interest for leaders – e.g. a reward from the party proportional to the fraction of voters carried to the voting booth – is clearly feasible. Other

extensions that may be of interest are allowing leaders to have overlapping intervals of influence and considering elections with three or more candidates.

Appendix: Proofs

Before proving Lemma 1 we need some statistical results. Let $n+1 = L_A + L_B + O$ be the total number of leaders and exogenous interruptions that are distributed uniformly on the circle. Pick any leader or exogenous interruption and call that point 0 (from the right) and 1 (from the left). From 0 to 1 (going to the right) the remaining n leaders and exogenous interruptions are distributed uniformly. Let $y_1, \ldots, y_n$ (with $y_1 \leq \cdots \leq y_n$) represent the (random) location of these points. Then the interval of influence of each leader or interruption is $x_1 = y_1, \ldots, x_k = y_k - y_{k-1}, \ldots, x_{n+1} = 1 - y_n$.

Theorem A.1. *The joint distribution of the intervals*

$$(x_1 = y_1, \ldots, x_k = y_k - y_{k-1}, \ldots, x_{n+1} = 1 - y_n)$$

of the uniform order statistic $0 \leq y_1 < y_2 < \cdots < y_n \leq 1$ *is invariant under the permutation of its components.*

Proof. See Reiss [21], p. 40. □

This implies, in particular,

Corollary A.1. *All marginal distributions of* $(x_1, \ldots, x_k, \ldots, x_{n+1})$ *of equal dimension are equal.*

Proof of Lemma 1. Given a sample size n, the joint density function of two order statistics for a uniform underlying distribution on the unit interval is

$$f(a_i, a_j) = \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!} (a_i)^{i-1} (a_j - a_i)^{j-i-1} (1 - a_j)^{n-j}$$

for $0 \leq a_i < a_j \leq 1$. Reordering the intervals of influence (see Corollary A.1) so that all the A leaders come first and then all the B leaders come next, and letting $L_A = i$, $L_A + L_B = j$, $L_A + L_B + O = n+1$, $a = a_i$ and $b = a_j - a_i$, we get the bivariate distribution of a and b when $O \geq 1$.

Similarly, the density function of an order statistic for a uniform underlying distribution on the unit interval is

$$f(a_i) = \frac{n!}{(i-1)!\,(n-i)!} (a_i)^{i-1} (1 - a_i)^{n-i}$$

for $0 \leq a_i \leq 1$. Reordering the intervals of influence so that there are first A leaders then B leaders, and letting $L_A = i$, $L_A + L_B = n+1$ and $a = a_i$, we get the distribution of a when $O = 0$. □

Before proving Lemmas 2 to 5, we need a couple of hypergeometric identities.[7]

Lemma A.1. *For all $M \in \mathbb{N}$, $N \in \mathbb{N}$ and $d \in (0,1)$,*

$$\sum_{k=1}^{N} (1-d)^{M+N-k} \frac{(M+N-k-1)!}{(M-1)!\,(N-k)!} = \sum_{k=1}^{N} (1-d)^{M+k-1} d^{N-k} \frac{(M+N-1)!}{(M+k-1)!\,(N-k)!}.$$

Proof. We proceed by induction. For any positive M, the equality is easily shown to be satisfied for $N = 1$. We claim that if the equality is satisfied for $N-1$, then it is satisfied for N, for any $N \geq 2$. To see this, evaluate the expression in the left-hand side at N and at $N-1$. The difference is

$$\sum_{k=1}^{N} (1-d)^{M+N-k} \frac{(M+N-k-1)!}{(M-1)!\,(N-k)!} - \sum_{k=1}^{N-1} (1-d)^{M+N-1-k} \frac{(M+N-k-2)!}{(M-1)!\,(N-1-k)!} = (1-d)^{M+N-1} \frac{(M+N-2)!}{(M-1)!\,(N-1)!}.$$

Similarly, evaluate the expression in the right-hand side at N and at $N-1$. The difference is

$$\sum_{k=1}^{N} (1-d)^{M+k-1} d^{N-k} \frac{(M+N-1)!}{(M+k-1)!\,(N-k)!} - \sum_{k=1}^{N-1} (1-d)^{M+k-1} d^{N-k-1} \frac{(M+N-2)!}{(M+k-1)!\,(N-1-k)!} = (1-d)^{M+N-1} (M+N-2)! \times \sum_{k=1}^{N} H(k),$$

[7] We are very grateful to Aaron Robertson for explaining to us the Wilf-Zeilberger method [20] to solve hypergeometric identities.

where

$$\begin{aligned} H(k) &= \frac{(1-d)^{k-N}d^{N-k}(M+N-1)}{(M+k-1)!(N-k)!} - \frac{(1-d)^{k-N}d^{N-k-1}}{(M+k-1)!(N-k-1)!} \\ &= \frac{(1-d)^{k-N}d^{N-k}(M+N-1) - (1-d)^{k-N}d^{N-k-1}(N-k)}{(M+k-1)!(N-k)!} \\ &= \frac{(1-d)^{k-N}d^{N-k}(M+k-1) - (1-d)^{k+1-N}d^{N-k-1}(N-k)}{(M+k-1)!(N-k)!} \\ &= \frac{(1-d)^{k-N}d^{N-k}}{(M+k-2)!(N-k)!} - \frac{(1-d)^{k+1-N}d^{N-k-1}}{(M+k-1)!(N-k-1)!}. \end{aligned}$$

for $k = 1, \ldots, N-1$ and

$$H(N) = \frac{1}{(M+N-2)!}.$$

Now define

$$J(k) = \frac{(1-d)^{k-N}d^{N-k}}{(M+k-2)!(N-k)!}.$$

Since $H(k) = J(k) - J(k+1)$ for $k = 1, \ldots, N-1$ and $H(N) = J(N)$, we get

$$\sum_{k=1}^{N} H(k) = J(1) = \frac{1}{(M-1)!(N-1)!}.$$

It follows that the difference between the expression in the right-hand side evaluated at N and at $N-1$ is also equal to

$$(1-d)^{M+N-1} \frac{(M+N-2)!}{(M-1)!\,(N-1)!}.$$

□

Lemma A.2. *For all $M \in \mathbb{N}$, $N \in \mathbb{N}$ and $d \in (0,1)$,*

$$\begin{aligned} &\sum_{k=1}^{N} \frac{(M+N-2)!}{(M+k-2)!\,(N-k)!} (1-d)^{M+k-2} d^{N-k} \\ &\qquad - \sum_{k=1}^{N} \frac{(M+N-1)!}{(M+k-1)!\,(N-k)!} (1-d)^{M+k-1} d^{N-k} \\ &\qquad\qquad = (1-d)^{M-1} d^{N} \frac{(M+N-2)!}{(M-1)!\,(N-1)!}. \end{aligned}$$

Proof. Note that

$$\sum_{k=1}^{N} \frac{(M+N-2)!}{(M+k-2)!\,(N-k)!}\,(1-d)^{M+k-2}\,d^{N-k}$$
$$-\sum_{k=1}^{N} \frac{(M+N-1)!}{(M+k-1)!\,(N-k)!}\,(1-d)^{M+k-1}\,d^{N-k}$$
$$= (1-d)^{M+N-1}\,(M+N-2)!\,\times \sum_{k=1}^{N} \hat{H}(k),$$

where

$$\begin{aligned}
&\hat{H}(k)\\
&\quad= \frac{(1-d)^{k-N-1}\,d^{N-k}}{(M+k-2)!(N-k)!} - \frac{(1-d)^{k-N}\,d^{N-k}(M+N-1)}{(M+k-1)!(N-k)!}\\
&\quad= \frac{(1-d)^{k-N-1}\,d^{N-k}(M+k-1) - (1-d)^{k-N}\,d^{N-k}(M+N-1)}{(M+k-1)!(N-k)!}\\
&\quad= \frac{(1-d)^{k-N-1}\,d^{N-k+1}(M+k-1) - (1-d)^{k-N}\,d^{N-k}(N-k)}{(M+k-1)!(N-k)!}\\
&\quad= \frac{(1-d)^{k-N-1}\,d^{N-k+1}}{(M+k-2)!(N-k)!} - \frac{(1-d)^{k-N}\,d^{N-k}}{(M+k-1)!(N-k-1)!}
\end{aligned}$$

for $k = 1, \ldots, N-1$, and

$$\begin{aligned}
\hat{H}(N) &= \frac{(1-d)^{-1}}{(M+N-2)!} - \frac{1}{(M+N-2)!}\\
&= \frac{(1-d)^{-1}d}{(M+N-2)!}.
\end{aligned}$$

Now define

$$\hat{J}(k) = \frac{(1-d)^{k-N-1}\,d^{N-k+1}}{(M+k-2)!(N-k)!}$$

for $k = 1, \ldots, N$, and note that

$$\hat{H}(k) = \hat{J}(k) - \hat{J}(k+1)$$

for $k = 1, \ldots, N-1$, and

$$\hat{H}(N) = \hat{J}(N).$$

Thus,

$$\sum_{k=1}^{N} \hat{H}(k) = \hat{J}(1) = \frac{(1-d)^{-N} d^N}{(M-1)!(N-1)!}.$$

The statement in the Lemma follows. □

Proof of Lemma 2. Suppose first that $O = 0$ (the easier case). We have

$$\begin{aligned} P(L_A, L_B) &= 1 - \int_0^{1/2} h^0_{L_A,L_B}(a)\, da \\ &= 1 - \int_0^{1/2} \frac{(L_A + L_B - 1)!}{(L_A - 1)!\,(L_B - 1)!}\, a^{L_A - 1}(1-a)^{L_B - 1}\, da. \end{aligned}$$

Integrating by parts we obtain

$$\begin{aligned} P(L_A, L_B) = 1 &- \frac{(L_A + L_B - 1)!}{L_A!\,(L_B - 1)!} \left(\frac{1}{2}\right)^{L_A + L_B - 1} \\ &- \frac{(L_A + L_B - 1)!}{L_A!\,(L_B - 2)!} \int_0^{1/2} a^{L_A}(1-a)^{L_B - 2}\, db. \end{aligned}$$

Proceeding iteratively we obtain, for $O = 0$,

$$P(L_A, L_B) = 1 - \sum_{k=1}^{L_B} \left(\frac{1}{2}\right)^{L_A + L_B - 1} \frac{(L_A + L_B - 1)!}{(L_A + k - 1)!\,(L_B - k)!}.$$

Now suppose that $O \geq 1$. We have

$$P(L_A, L_B) = \int_0^{1/2} \int_0^{a} h_{L_A,L_B}(a,b)\, db\, da + \int_{1/2}^{1} \int_0^{1-a} h_{L_A,L_B}(a,b)\, db\, da.$$

Using Lemma 1,

$$\begin{aligned} P(L_A, L_B) = &\frac{(L_A + L_B + O - 1)!}{(L_A - 1)!\,(L_B - 1)!\,(O - 1)!} \\ &\left(\int_0^{1/2} \int_0^{a} a^{L_A - 1} b^{L_B - 1} (1 - a - b)^{O-1}\, db\, da \right. \\ &\left. + \int_{1/2}^{1} \int_0^{1-a} a^{L_A - 1} b^{L_B - 1} (1 - a - b)^{O-1}\, db\, da \right). \end{aligned}$$

Consider the first inner integral. Integrating by parts we obtain

$$\int_0^a b^{L_B-1}\left(1-a-b\right)^{O-1}\,db = \frac{-a^{L_B-1}(1-2a)^O}{O} + \frac{L_B-1}{O}\int_0^a b^{L_B-2}\left(1-a-b\right)^{O}\,db.$$

Proceeding iteratively we obtain

$$\int_0^a b^{L_B-1}\left(1-a-b\right)^{O-1}\,db = -\sum_{k=1}^{L_B}\frac{(L_B-1)!(O-1)!}{(L_B-k)!(O+k-1)!}a^{L_B-k}(1-2a)^{O+k-1} + \frac{(L_B-1)!(O-1)!}{(O+L_B-1)!}(1-a)^{O+L_B-1}.$$

Consider the second inner integral. Integrating by parts we obtain

$$\int_0^{1-a} b^{L_B-1}\left(1-a-b\right)^{O-1}\,db = \frac{L_B-1}{O}\int_0^{1-a} b^{L_B-2}\left(1-a-b\right)^{O}\,db.$$

Proceeding iteratively we obtain

$$\int_0^{1-a} b^{L_B-1}\left(1-a-b\right)^{O-1}\,db = \frac{(L_B-1)!(O-1)!}{(O+L_B-1)!}(1-a)^{O+L_B-1}.$$

Substituting both inner integrals back in the previous expression for $P\left(L_A, L_B\right)$, we get

$$P\left(L_A, L_B\right) = \int_0^1 \frac{(L_A+L_B+O-1)!}{(L_A-1)!\,(L_B+O-1)!}a^{L_A-1}(1-a)^{L_B+O-1}\,da - \frac{(L_A+L_B+O-1)!}{(L_A-1)!}\sum_{k=1}^{L_B}\int_0^{1/2}\frac{a^{L_A+L_B-k-1}(1-2a)^{O+k-1}}{(L_B-k)!(O+k-1)!}\,da.$$

The first term in this expression for $P\left(L_A, L_B\right)$ is equal to

$$\int_0^1 h^0_{L_A,L_B+O}(a)\,da,$$

which is equal to one because $h^0_{L_A,L_B+O}$ is a probability density with support $(0,1)$. With respect to the second term, integrating by parts

$$\int_0^{1/2} a^{L_A+L_B-k-1}\,(1-2a)^{O+k-1}\;da = \\ 2\frac{O+k-1}{L_A+L_B-k}\int_0^{1/2} a^{L_A+L_B-k}\,(1-2a)^{O+k-2}\;da.$$

Proceeding iteratively we obtain

$$\int_0^{1/2} a^{L_A+L_B-k-1}\,(1-2a)^{O+k-1}\;da = \\ \left(\frac{1}{2}\right)^{L_A+L_B-k}\frac{(O+k-1)!(L_A+L_B-k-1)!}{(L_A+L_B+O-1)!}.$$

Thus, for $O \geq 1$,

$$P\left(L_A,L_B\right) \;=\; 1-\sum_{k=1}^{L_B}\left(\frac{1}{2}\right)^{L_A+L_B-k}\frac{(L_A+L_B-k-1)!}{(L_A-1)!\,(L_B-k)!}.$$

To show that the expressions for $P\left(L_A,L_B\right)$ for the cases $O=0$ and $O\geq 1$ are equivalent, we need to verify the identity

$$\sum_{k=1}^{L_B}\left(\frac{1}{2}\right)^{L_A+L_B-k}\frac{(L_A+L_B-k-1)!}{(L_A-1)!\,(L_B-k)!} = \\ \sum_{k=1}^{L_B}\left(\frac{1}{2}\right)^{L_A+L_B-1}\frac{(L_A+L_B-1)!}{(L_A+k-1)!\,(L_B-k)!}.$$

This identity follows from Lemma A.1, letting $M=L_A$, $N=L_B$ and $d=1/2$. □

Proof of Lemma 3. Using the definition of $P_A(L_A,L_B)$ and Lemma 2,

$$\begin{aligned} P_A(L_A,L_B) &= P(L_A,L_B)-P(L_A-1,L_B) \\ &= \sum_{k=1}^{L_B}\left(\frac{1}{2}\right)^{L_A+L_B-2}\frac{(L_A+L_B-2)!}{(L_A+k-2)!\,(L_B-k)!} \\ &\quad -\sum_{k=1}^{L_B}\left(\frac{1}{2}\right)^{L_A+L_B-1}\frac{(L_A+L_B-1)!}{(L_A+k-1)!\,(L_B-k)!}. \end{aligned}$$

Thus, we need to verify

$$\sum_{k=1}^{L_B}\left(\frac{1}{2}\right)^{L_A+L_B-2}\frac{(L_A+L_B-2)!}{(L_A+k-2)!\,(L_B-k)!}$$
$$-\sum_{k=1}^{L_B}\left(\frac{1}{2}\right)^{L_A+L_B-1}\frac{(L_A+L_B-1)!}{(L_A+k-1)!\,(L_B-k)!}$$
$$=\frac{1}{2^{L_A+L_B-1}}\frac{(L_A+L_B-2)!}{(L_A-1)!\,(L_B-1)!}.$$

This identity follows from Lemma A.2, letting $M = L_A$, $N = L_B$ and $d = 1/2$. □

Proof of Lemma 4. With respect to near single-peakedness, from Lemma 3,

$$\frac{P_A\left(L_A+1,L_B\right)}{P_A\left(L_A,L_B\right)}=\frac{L_A+L_B-1}{2L_A}.$$

Thus,

$$\frac{P_A\left(L_A+1,L_B\right)}{P_A\left(L_A,L_B\right)}\gtreqless 1 \quad\Longleftrightarrow\quad L_B\gtreqless L_A+1.$$

With respect to symmetry, from the definition of decisiveness,

$$P_A(L_A,L_B)=P_B(L_B,L_A),$$

and from Lemma 3,

$$P_A(L_A,L_B)=P_A(L_B,L_A).$$

□

Proof of Lemma 5. Since parties A and B attract a fraction α and β, respectively, of the voters in the intervals corresponding to leaders for A and for B, party B wins the election if αa is smaller than βb.

Suppose first that $O = 0$. In this case, B wins the election if $a < \beta/(\alpha+\beta)$. Thus

$$P\left(L_A,L_B\right)=1-\frac{(L_A+L_B-1)!}{(L_A-1)!\,(L_B-1)!}\int_0^{1-\gamma}a^{L_A-1}\left(1-a\right)^{L_B-1}\,da,$$

where $\gamma=\alpha/(\alpha+\beta)$. Integrating by parts iteratively, as in the proof of Lemma 2,

$$P\left(L_A,L_B\right)=1-\sum_{k=1}^{L_B}\frac{(L_A+L_B-1)!}{(L_A+k-1)!\,(L_B-k)!}\left(1-\gamma\right)^{L_A+k-1}\gamma^{L_B-k}.$$

Now suppose that $O \geq 1$. Note that B loses the election if $b < \alpha a/\beta$. Note also that $b \leq 1-a$, and $1-a \leq \alpha a/\beta$ if and only if $a \geq \beta/(\alpha+\beta) = 1-\gamma$. Thus,

$$P(L_A, L_B) = \int_0^{1-\gamma} \int_0^{\alpha a/\beta} h_{L_A,L_B}(a,b)\, db\, da + \int_{1-\gamma}^{1} \int_0^{1-a} h_{L_A,L_B}(a,b)\, db\, da.$$

Using Lemma 1,

$$P(L_A, L_B) = \frac{(L_A+L_B+O-1)!}{(L_A-1)!\,(L_B-1)!\,(O-1)!} \left(\int_0^{1-\gamma} \int_0^{\alpha a/\beta} a^{L_A-1} b^{L_B-1} (1-a-b)^{O-1}\, db\, da + \int_{1-\gamma}^{1} \int_0^{1-a} a^{L_A-1} b^{L_B-1} (1-a-b)^{O-1}\, db\, da \right).$$

Consider the inner integrals. Integrating by parts iteratively, as in the proof of Lemma 2, we obtain

$$\int_0^{\alpha a/\beta} b^{L_B-1} (1-a-b)^{O-1}\, db = -\sum_{k=1}^{L_B} \frac{(L_B-1)!(O-1)!}{(L_B-k)!(O+k-1)!} (\alpha a/\beta)^{L_B-k} (1-a-\alpha a/\beta)^{O+k-1} + \frac{(L_B-1)!(O-1)!}{(O+L_B-1)!} (1-a)^{O+L_B-1}.$$

As shown in the proof of Lemma 2,

$$\int_0^{1-a} b^{L_B-1} (1-a-b)^{O-1}\, db = \frac{(L_B-1)!(O-1)!}{(O+L_B-1)!} (1-a)^{O+L_B-1}.$$

Substituting both inner integrals back in the previous expression for $P(L_A, L_B)$, we get

$$P(L_A, L_B) = 1 - \frac{(L_A+L_B+O-1)!}{(L_A-1)!} \times \sum_{k=1}^{L_B} \int_0^{1-\gamma} \frac{(\alpha a/\beta)^{L_A+L_B-k-1} (1-a-\alpha a/\beta)^{O+k-1}}{(L_B-k)!(O+k-1)!}\, da.$$

With respect to the second term, integrating by parts iteratively

$$\int_0^{1/2} a^{L_A+L_B-k-1}\,(1-2a)^{O+k-1}\;da = \\ (1-\gamma)^{L_A+L_B-k}\,\frac{(O+k-1)!(L_A+L_B-k-1)!}{(L_A+L_B+O-1)!}.$$

Thus, for $O \geq 1$,

$$P\left(L_A, L_B\right) \;=\; 1-\sum_{k=1}^{L_B}\frac{(L_A+L_B-k-1)!}{(L_A-1)!\,(L_B-k)!}\,(1-\gamma)^{L_A+L_B-k}\,.$$

To show that the expressions for $P\left(L_A, L_B\right)$ for the cases $O=0$ and $O \geq 1$ are equivalent, we need to verify the identity

$$\sum_{k=1}^{L_B}(1-\gamma)^{L_A+L_B-k}\,\frac{(L_A+L_B-k-1)!}{(L_A-1)!\,(L_B-k)!} = \\ \sum_{k=1}^{L_B}(1-\gamma)^{L_A+k-1}\,\gamma^{L_B-k}\,\frac{(L_A+L_B-1)!}{(L_A+k-1)!\,(L_B-k)!}.$$

This identity follows from Lemma A.1, letting $M=L_A$, $N=L_B$ and $d=\gamma$.

Finally, using the definition of $P_A(L_A, L_B)$ and one of the (equivalent) expressions for $P\left(L_A, L_B\right)$,

$$\begin{aligned} P_A(L_A, L_B) &= P(L_A, L_B) - P(L_A-1, L_B) \\ &= \sum_{k=1}^{L_B}\frac{(L_A+L_B-2)!}{(L_A+k-2)!\,(L_B-k)!}\,(1-\gamma)^{L_A+k-2}\,\gamma^{L_B-k} \\ &\quad -\sum_{k=1}^{L_B}\frac{(L_A+L_B-1)!}{(L_A+k-1)!\,(L_B-k)!}\,(1-\gamma)^{L_A+k-1}\,\gamma^{L_B-k}. \end{aligned}$$

Thus, to prove the statement of the Lemma with respect to $P_A(L_A, L_B)$ we need to verify the identity

$$\sum_{k=1}^{L_B}\frac{(L_A+L_B-2)!}{(L_A+k-2)!\,(L_B-k)!}\,(1-\gamma)^{L_A+k-2}\,\gamma^{L_B-k} \\ -\sum_{k=1}^{L_B}\frac{(L_A+L_B-1)!}{(L_A+k-1)!\,(L_B-k)!}\,(1-\gamma)^{L_A+k-1}\,\gamma^{L_B-k} \\ = (1-\gamma)^{L_A-1}\,\gamma^{L_B}\,\frac{(L_A+L_B-2)!}{(L_A-1)!\,(L_B-1)!}.$$

This identity follows from Lemma A.1, letting $M = L_A$, $N = L_B$ and $d = \gamma$. The proof of the statement with respect to $P_B(L_A, L_B)$ proceeds along similar lines. □

Proof of Corollary 3. Using the expression for L^* in Theorem 1 we get

$$\frac{1}{2^{2L^*+1}}\frac{(2L^*)!}{L^*!L^*!} \leq c/G < \frac{1}{2^{2L^*-1}}\frac{(2L^*-2)!}{(L^*-1)!\,(L^*-1)!}.$$

Since

$$\frac{1}{2^{2L+1}}\frac{(2L)!}{L!L!}$$

is positive and converges monotonically to zero as L goes to infinity, we get that $L^* \to +\infty$ as $c/G \to 0$. Moreover, since

$$\frac{1}{2^{2L+1}}\frac{(2L)!}{L!L!} - \frac{1}{2^{2L-1}}\frac{(2L-2)!}{(L-1)!\,(L-1)!} \to 0$$

as $L \to +\infty$, we get that

$$\frac{c/G}{\frac{1}{2^{2L^*+1}}\frac{(2L^*)!}{L^*!L^*!}} \to 1$$

as $c/G \to 0$. Using the expressions for expected turnout and expected winning margin in Theorem 1,

$$\mathrm{WM/ET} = \frac{1}{2^{2L^*}}\frac{(2L^*)!}{L^*!L^*!}.$$

Thus

$$\frac{c/G}{\mathrm{WM/(2ET)}} \to 1$$

as $c/G \to 0$, which is the approximation result for c/G stated in the Corollary.

Now, using the Stirling formula,

$$\frac{1/\sqrt{4\pi L^*}}{\frac{1}{2^{2L^*+1}}\frac{(2L^*)!}{L^*!L^*!}} \to 1.$$

Thus,

$$\frac{1/\sqrt{4\pi L^*}}{\mathrm{WM/(2ET)}} \to 1$$

or equivalently

$$\frac{L^*}{(\mathrm{ET/WM})^2/\pi} \to 1$$

as $c/G \to 0$. The approximation result for $1/O$ follows from this and the expression for ET in Theorem 1. $\square$

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