Anonymity in Large Societies

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Abstract

We formalize the notion of coalition measure as a primitive of social choice with a countably infinite number of agents. An example would be a dynamic setting with an infinite horizon, where it may be convenient to consider separately each individual's preferences at every moment of time. In this setting, there are some "equal size" coalitions that a social choice rule should treat in the same manner. We therefore introduce a new property of equal treatment with respect to a coalition size measure and explore its interaction with other common axioms of social choice. We observe that, provided the measure space is sufficiently rich in coalitions of the same measure, the new axiom can play a role similar to that of anonymity.

1 Introduction

It has long been known that social choice problems with a countably infinite number of agents differ in significant ways from their finite-agent analogues. Thus, for instance, countably large societies admit social choice rules satisfying the standard Arrovian axioms of efficiency, independence, nondictatorship and transitivity (see, for instance, Fishburn [9], Kirman and Sondermann [11], Hansson [10], or Chichilnisky [7]). These Arrovian rules

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involve, in the words of Kirman and Sondermann [11], “invisible dictators:” collections of decisive coalitions that recede into infinity so that no particular individual has to be a member of every decisive coalition.

Armstrong [2] extended the earlier results to societies where the set of admissible coalitions is restricted to be an algebra, with the implied measurability restriction on the social choice rules. He also introduced the measure of coalition size with a two-fold objective of describing the size of decisive coalitions and of defining the notion of a “negligible” coalition.

Surprisingly, it seems that no attempt has been made to tie the measure of coalition size with the idea of equal treatment. Instead, Mihara [13] has recently extended the usual notion of anonymity from a finite-agent to the infinite-agent set up. This anonymity axiom states that a social choice rule should be insensitive to permutations of the agent space. Mihara [13] found that, subject to a simple richness condition on the algebra of admissible coalitions, any measurable Arrovian rule must treat differently coalitions of the same cardinality and thus violates anonymity.

However, whenever the set of agents is endowed with a measure-space structure, there seems to be no particular reason why a cardinality-based equal-treatment notion such as anonymity should be the relevant one. In fact, in certain applications it may be rather hard to justify, since it would require equal treatment of intuitively very different coalitions such as those including, respectively, every second and every thousandth agent. A number of restrictions on agent permutations allowed in the definition of anonymity have therefore been proposed. One of these, which seems to avoid the counterintuitive implications mentioned above, has been proposed by Lauwers [12]. This is *bounded anonymity*, which can be viewed as requiring admissible permutations to be measure-preserving for a particular (“frequency”) measure on the set of agents (even though he never explicitly introduces a measure space for the case when there are countably many agents).

We study the consequences of explicitly incorporating the notion of the coalition measure size into the model and requiring equal-treatment of coalitions of equal measure. It turns out that, subject, once again, to a richness condition on the equal-measure coalition classes, imposing such equal treatment results in restoration of Arrow’s impossibility. We then relax the requirement of transitivity by dropping the requirement of transitivity of social indifference, which is known to allow a broader class of social choice rules. Indeed we show that with this adjustment our equal treatment axiom is consistent with the rest. One rule satisfying this set of axioms is what we call *consensus*: a rule which takes as decisive the full-measure coalitions. We proceed to provide a complete axiomatic characterization of this rule.

The rest of this paper is organized as follows. In section 2 we introduce the standard model of measurable social choice. In section 3 we discuss the notion of coalition size. In section 4 we define the notion of anonymity with respect to a measure and proceed to study the consequences of imposing it in the place of the standard cardinality-based notion. Section 5 contains conclusions and possible extenssions.

2 Measurable Social Choice

We define a social choice model with an algebra of admissible coalitions following Armstrong [2].

Let $N = \{1, 2, 3, \ldots\}$ be a countable set of individuals (voters), indexed by the natural numbers. A *coalition* is any subset $L \subset N$. An *algebra* is any class of coalitions $\mathcal{L} \subset 2^N$ such that it contains N itself and it is closed under the formation of complements and finite unions (or, equivalently, finite intersections). The pair $(N, \mathcal{L})$ is a *coalition measurable space.*

An algebra is to be understood as a collection of admissible coalitions that satisfies some minimum requirements: the union of two admissible coalitions should itself be admissible, and the complement of an admissible coalition should also be admissible. The admissibility restriction may arise from the nature of the economic model at hand; alternatively it could be viewed as coming from observability or computability constraints facing the social planner as in [14].

Let X be a set of alternatives, which has at least three elements. For simplicity, we assume throughout that X is finite and, as usual, has at least three elements. Each individual possesses a weak order (i.e., a reflexive, complete, and transitive binary relation) on X. We let R_i denote the order for $i \in N$; the relation R_i represents individual i's weak preference. We let P_i denote the asymmetric part of R_i and we let I_i denote the symmetric part; the relations P_i and I_i represent, respectively, individual i's strict preference and indifference. A *preference profile* is then a list of weak orders $\rho = (R_i)_{i \in N}$ describing the preferences of all individuals. A profile ρ is $\mathcal{L}$*-measurable* if for all $x, y \in X$, the coalitions $\{i : xP_iy\}$ and $\{i : yP_ix\}$ belong to $\mathcal{L}$. Finally, denote by $\mathcal{R}$ the set of all weak orders on X, denote by $\mathcal{R}_{\mathcal{L}}^N$ the set of all $\mathcal{L}$-measurable profiles, and denote by $\mathcal{B}$ the set of all reflexive and complete binary relations on X.

Definition 1 A $\mathcal{L}$*-measurable preference aggregation rule* is a map

$$f : \mathcal{R}_{\mathcal{L}}^N \to \mathcal{B}.$$

A measurable preference aggregation rule assigns to every measurable profile a reflexive and complete binary relation, the *social* preference $R = f(\rho)$. We denote by P and I, respectively, the asymmetric and the symmetric part of the social preference R.

A $\mathcal{L}$-measurable coalition L is *decisive* under a $\mathcal{L}$-measurable rule f if

$$\forall i \in L,\ xP_iy \Rightarrow xPy$$

for all directed pairs $x, y \in X$ for all measurable profiles. Given the set of decisive coalitions $\mathcal{D}_f$ associated to f, we can define the preference aggregation rule $f_{\mathcal{D}}$ as

$$xPy \iff [\exists L \in \mathcal{D}_f : \forall i \in L,\ xP_iy]$$

for all directed pairs $x, y \in X$ for all measurable profiles. We say that f is a *simple rule* if $f = f_{\mathcal{D}}$.

We next consider coalition measurable spaces that are fine enough to admit all preference profiles of interest in some examples of countable societies.

2.1 A few examples

Consider the algebra $\mathcal{L}_c$ which is composed of all finite (including the empty set) and cofinite subsets of N. In many ways this is the "smallest" algebra of interest to us. In particularly, this is the coarsest algebra that admits all the singleton coalitions. We call $(N, \mathcal{L}_c)$ the *cofinite measurable space.*

An *extension* of a measurable space $(N, \mathcal{L})$ is any measurable space $(N, \mathcal{L}')$ such that $\mathcal{L}' \supset \mathcal{L}$. The following examples are suggestive of a way to extend $(N, \mathcal{L}_c)$.

Example 1 (Mihara) Consider a society composed of finitely many people, where there is uncertainty expresses as a countable number of states of the world. Let index the people by $j = 1, \ldots, n$ and the states by $s = 1, 2, \ldots$; we can let the preferences of person j in state s be represented by an individual $i = js$.

Example 2 Consider a society composed of finitely many dynasties, which we index by $j = 1, \ldots, n$; at each period $s = 1, 2, \ldots$ each dynasty is represented by one individual. We can name an individual belonging to dynasty j and living in period s by $i = js$.

In the first example, it seems natural to consider as admissible coalitions the sets $\{i \in N : i = js, s = 1, 2, \ldots\}$ for $j = 1, \ldots, n$, representing the person j in all states of the world (the *n-period sets*), and the sets $\{i \in N : (s-1)n < i \leq sn\}$ for $s = 1, 2, \ldots$ (*finite sets*) representing all the people in a particular state. Similarly, in the second example, it seems natural to consider as admissible coalitions the sets including all members of a dynasty as well as the sets including all individuals living at a given period. Note that an algebra containing both types of sets will include also all the coalitions consisting of a single individual, and hence all finite and cofinite coalitions. We shall denote this $n-$*period coalition* algebra as $\mathcal{L}_{p,n}$.

We may also wish to consider a coalition algebra that recognizes all admissible coalitions for a society such as that in example 1 or in example 2 for arbitrary n. The coarsest such algebra admits all finite unions of n-period sets, which we refer to as *purely periodic sets*, as well as all unions and differences of a purely periodic and a finite set. We denote this algebra by $\mathcal{L}_p$, and refer to $\mathcal{L}_p$-coalitions as *periodic sets*.

Other coalition algebras may be of interest as well. Of course, the finest such algebra is the power set 2^N of N.

2.2 Measurable Preference Aggregation

The following are some desirable criteria an aggregation rule might satisfy:

Definition 2 A $\mathcal{L}$-measurable preference aggregation rule f is

(P1) *weakly Paretian* if, for every $\rho \in \mathcal{R}_{\mathcal{L}}^N$ and for any $x, y \in X$,

$$[\{i \in N : xP_iy\} = N] \;\; \Rightarrow \;\; xPy.$$

(P2) *independent of irrelevant alternatives* if for every $\rho, \rho' \in \mathcal{R}_{\mathcal{L}}^N$ and for any $x, y \in X$,

$$[\rho|_{\{x,y\}} = \rho'|_{\{x,y\}}] \; \Rightarrow \; [f(\rho)|_{\{x,y\}} = f(\rho')|_{\{x,y\}}],$$

where $\rho|_S$ represents the restriction of ρ to the set S.

(P3) *nondictatorial* if there does not exist $i \in N$ such that xP_iy implies xPy for every $\rho \in \mathcal{R}_{\mathcal{L}}^N$ and for any $x, y \in X$.

(P4) *transitive* if for every $\rho \in \mathcal{R}_{\mathcal{L}}^N$ and for all $x, y, z \in X$,

$$xRy \;\&\; yRz \; \Rightarrow \; xRz.$$

Arrow's [4] impossibility theorem shows that, with a finite number of individuals, there is no preference aggregation function satisfying properties (P1) to (P4). Fishburn [9], Kirman and Sondermann [11] and others have shown that, in fact, such functions are possible once an infinite set of voters is considered. Armstrong [2] extended this result to coalition measurable spaces. Our first result is a corollary of Proposition 3.2 in Armstrong [2].

For any extension of the cofinite measurable space (including $(N, \mathcal{L}_c)$ itself) define a social choice rule σ_c by

$$xPy \iff [\{i \in N : xP_iy\} \text{ is a cofinite set}]$$

for all directed pairs $x, y \in X$. That is, σ_c is the simple rule with the cofinite sets as decisive coalitions. We say that a rule f is an *extension* of σ_c if $\mathcal{D}_f \supset \{L \in 2^N : L \text{ is cofinite}\}$.

Proposition 1 *For any extension $(N, \mathcal{L})$ of the cofinite measurable space, there exists at least one $\mathcal{L}$-measurable rule satisfying (P1) to (P4); any such rule is an extension of σ_c.*

Proof It follows immediately from Armstrong's [2] proposition 3.2 (as ammended in [3]) that for every free ultrafilter $\mathcal{U}$ of measurable coalitions there exists a measurable social choice rule satisfying (P1)-(P4) such that all coalitions in $\mathcal{U}$ are decisive, and viceversa, for every measurable social choice rule satisfying (P1)-(P4) the set of decisive coalitions is a free ultrafilter. (For the definitions of filter, ultrafilter, and free filter see, e.g. Aliprantis and Border [1].) It is easy to see that in every extension of the cofinite measurable space the set of cofinite coalitions is a free filter. By an application of Zorn's lemma ([1], p. 32), there exists a free ultrafilter of measurable coalitions that contains the set of cofinite coalitions. Now suppose there exists a measurable social choice rule f satisfying (P1)-(P4), and a cofinite coalition $L_c \in \mathcal{L}$ that is not decisive under f. Then its complement $L_c^c \in \mathcal{U}$ is a finite decisive coalition under f. But (P3) implies that no individual is in every decisive coalition. That is, for every $x \in L_c^c$ there exists a $L_x \in \mathcal{U}$ such that $x \notin L_x$. Hence, there exists a finite intersection of elements of $\mathcal{U}$ with an empty intersection, which contradicts the definition of an ultrafilter. □

In general, explicitly constructing such Arrovian rules may be rather difficult (see [15]). When the coalition algebra $\mathcal{L}$ is sufficiently restricted this may be much easier. Thus, consider a society of two infinitely-lived dynasties. As discussed above a natural algebra of admissible coalitions for such a society could be $\mathcal{L}_{p,2}$. The one-dynasty coalitions in this setting are represented by the sets of even and odd numbers, respectively. The only other admissible infinite purely periodic coalition here is N. All other admissible coalitions are eventually periodic with period 2. A measurable preference-aggregation rule satisfying all the four properties can be defined as follows:

$$xPy \iff [\{j \in \mathbb{N} : xP_{2j}y\} \text{ is a cofinite set}]$$

for all directed pairs $x, y \in X$. In words, this rule says that an alternative is preferred to another if and only if all but finitely many members of the second coalition agree. It is obvious that all cofinite coalitions are decisive (notice, however, that σ_c itself is not Arrovian, since it would violate transitivity).

A striking feature of the above example is that the social choice rule discriminates among dynasties: The "evens" eventually rule. In fact, as we show below, this has to be generally the case for rules satisfying rule satisfying (P1) to (P4). This seems to violate some notion of "equal treatment" of what should be "equals." To formalize this idea, however, we need to introduce a notion of coalition size.

3 Coalition Size

In a finite world, a coalition's size is easy to define as its cardinality. This is the idea behind the standard anonymity axiom in social choice, which says that a social choice rule should be invariant under the agents' permutations.

Mihara [13] has shown that, subject to a richness condition on the algebra of admissible coalitions, invariance under permutations of agents is inconsistent with (P1) - (P4). Unfortunately, in an infinite society such axiom may be hard to justify. Consider for instance the "dynastic society" example. It is straightforward to show that if the number of dynasties n is larger than 2, there exists a permutation of the agent space that transforms a single dynasty into its union with another dynasty. It seems rather hard to insist on equal treatment of such clearly different coalitions.

We could, of course, recall that our dynastic society is endowed with an additional $\mathcal{L}-$measurability structure. A natural question to ask is, therefore, if it alone can help resolve this problem. Unfortunately, referring to the example above, it is possible to construct periodically measurable permutations that change the period of a coalition. To sum up, while cardinality-based anonymity notions are undoubtedly of interest, in order to discuss coalition size in important applications they may be inadequate. We therefore proceed to define explicitly coalition size as its measure, and to study the consequences of requiring equal treatment of equal measure (rather than equal cardinality) coalitions.

3.1 Coalition Measure Spaces

Definition 3 Given a coalition measurable space $(N, \mathcal{L})$, a set function μ on the algebra $\mathcal{L}$ is a (finitely-additive) *probability measure* if:

(i) $\mu(L) \in [0, 1]$ for $L \in \mathcal{L}$;

(ii) $\mu(\emptyset) = 0$ and $\mu(N) = 1$;

(iii) If $L_1, \ldots, L_n$ are disjoint $\mathcal{L}$-coalitions, then

$$\mu\left(\bigcup_{k=1}^{n} L_k\right) = \sum_{k=1}^{n} \mu(L_k).$$

The triple $(N, \mathcal{L}, \mu)$ is a *coalition measure space*. An *extension* of a measure space $(N, \mathcal{L}, \mu)$ is any measure space $(N, \mathcal{L}', \mu')$ such that $\mathcal{L}' \supset \mathcal{L}$ and $\mu'(L) = \mu(L)$ for every $L \in \mathcal{L}$.

Example 3 On the cofinite measurable space $(N, \mathcal{L}_c)$ we may define a measure μ_c by

$$\mu_c(L) = \begin{cases} 0 & \text{if } L \text{ is finite} \\ 1 & \text{if } L \text{ is cofinite} \end{cases}$$

(note that this is the only probability measure that assigns equal weight to all singleton coalitions in the cofinite measurable space).

Example 4 Extending the cofinite measure space, consider the n-period coalition measurable space $(N, \mathcal{L}_{p,n})$. Treating all $n-$period coalitions as having equal size implies the following probability measure on $\mathcal{L}_{p,n}$:

$$\mu_{p,n}(L) = \lim_{k \uparrow \infty} \frac{1}{k} \# [m \in L : 1 < m \leq k].$$

This measure assigns 0 to every finite set, $\frac{1}{n}$ to every n-period set, and 1 to every cofinite set. We refer to $(N, \mathcal{L}_{p,n}, \mu_{p,n})$ as the *n-period measure space.*

Example 5 Consider the periodic measurable space $(N, \mathcal{L}_p)$. Extending $\mu_{p,n}$ leads to the following measure defined on $\mathcal{L}_p$:

$$\mu_p(L) = \lim_{k \uparrow \infty} \frac{1}{k} \#[m \in L : 1 < m \leq k].$$

Like $\mu_{p,n}$, the measure μ_p is *not* countably additive. Consider the following example. Let L_1 be the periodic set containing all individuals named with odd natural numbers. For each $n \geq 2$, define $L_n = L_1 \setminus \{k \in \mathbb{N} : 1 \leq k \leq n\}$. Then $\forall n$, $L_{n+1} \subset L_n$. Now $L_n \downarrow \emptyset$, since, given any $k \in \mathbb{N}$, $n > k \Rightarrow k \notin L_n$. But $\mu_p(L_n) \to 1/2$, since for all n, $\mu_p(A_n) = 1/2$. The reason why μ_p is not countably additive is that μ_p is very different from the regular probability measures on complete metric spaces. All such probability measures are *tight*, that is, most of their mass is concentrated on a finite or a compact set. On the other hand, as our previous example illustrates, the value of μ_p is independent of what happens in any finite set, that is, μ_p concentrates most of its mass "at infinity."

We refer to $(N, \mathcal{L}_p, \mu_p)$ as the *periodic measure space*. Obviously, the periodic measure space is an extension of every n -period measure space. Billingsley ([6], p. 577) contains an example showing that $(N, \mathcal{L}_p, \mu_p)$ cannot be uniquely extended to include all sets such that μ_p is well-defined.

Going back to our original examples, the periodic measures are not the only ones of interest. For instance, in example 1, there may be an available (countably additive) probability measure $\pi(\cdot)$ over states; we may use it to construct a probability measure over $\mathcal{L}_{p,n}$ that assigns measure $1/n$ to every n-period set, and measure $\pi(s)/n$ to every individual $i = js$. Similarly, in example 2, we may wish to "discount" the welfare of future generations according to a discount factor $\beta \in (0, 1)$; in this case, we can construct a probability measure over $\mathcal{L}_{p,n}$ that assigns measure $1/n$ to every n-period set, and measure $\beta^{s-1}/(n(1-\beta))$ to every individual $i = js$. In example 1, we may think of the periodic measure space as appropriate for social choice "under a veil of ignorance" with respect to the likelihood of different states. In example 2, we may think of the periodic measure space as appropriate from a normative perspective if discounting of future generations is disallowed.

Unlike the examples two and three, the last one admits atoms – in fact, it is purely atomic. Other reasons to allow atoms include making explicit the idea that some individuals (we could call them "politicians") may have a non-negligible weight even in a large society, whereas others may be negligible on their own. Alternatively, atoms may arise if we want to accommodate political organizations as "indivisible" coalitions, while also allowing for the presence of unorganized agents.

4 Measures and anonymity

Once the appropriate measure of coalition size is adopted we want to explore consequences of adopting the corresponding equal treatment notion.

Definition 4 Given a measure space $(N, \mathcal{L}, \mu)$, a $\mathcal{L}$-measurable preference aggregation rule f is

(P6) *μ-anonimous* if for every $\rho, \rho' \in \mathcal{R}_{\mathcal{L}}^{N}$,

$$\forall \hat{R} \in \mathcal{R},\ \mu(\{i \in N : R_i = \hat{R}\}) = \mu(\{i \in N : R'_i = \hat{R}\})$$

implies $f(\rho) = f(\rho')$.

This requirement has no "bite" unless the equal-measure coalition classes are sufficiently large. The following example is particularly nasty in that there are no two equal-measure coalitions:

Example 6 *The measure space* $\left(N, 2^N, \mu_\beta\right)$ *where* $\mu_\beta(A) = \frac{1}{1-\beta} \sum_{i \in A} \beta^i$ *for any* $0 < \beta < \frac{1}{2}$.

This and similar examples can be avoived if a richness condition on coalition algebras is satisfied.

Assumption R1 For every free ultrafilter $\mathcal{U}$ on the coalition algebra $\mathcal{L}$ there is a coalition $A \in \mathcal{U}$ such that there exists a coalition $B \in \mathcal{L} \setminus \mathcal{U}$ with $\mu(A) = \mu(B)$.

Theorem 1 *There is no measurable rule satisfying* (P1), (P2), (P4), *and* (P6) *if and only if assumption* (R1) *holds.*

Proof This proof is similar to the one Mihara [13] provides for a cardinality-based anonymity axiom. Suppose first that there is a social choice rule f satisfying (P1), (P2), (P4) and (P6). ¿From (P1), (P2) and (P4), the set of decisive coalitions is an ultrafilter $\mathcal{U}$. (This result holds generally, regarldess of whether the society is finite or infinite; see e.g. [5], p. 47.) If (R1) holds, there exist $A \in \mathcal{U}$ and $B \in \mathcal{L} \backslash \mathcal{U}$ such that $\mu(A) = \mu(B)$. If $A \cap B =$

$\emptyset$, compare the outcomes of the preference profile ρ with $\{i : xP_iy\} = A$ and $\{i : yP_ix\} = A^c$ and the preference profile ρ' with $\{i : xP_iy\} = B$ and $\{i : yP_ix\} = B^c$ for some $x, y \in X$ (and complete indifference elsewhere). Since, by (P6), $f(\rho) = f(\rho')$ it follows that both A and B are members of $\mathcal{U}$, a contradiction. If $A \cap B \neq \emptyset$, consider the complement of B. Since $\mathcal{U}$ is an ultrafilter, $B^c \in \mathcal{U}$. Therefore, $B^c \cap A \in \mathcal{U}$ while $A^c \cap B \notin \mathcal{U}$ (otherwise $B \in \mathcal{U}$). Since $(B^c \cap A) \cap (A^c \cap B) = \emptyset$ and $\mu(B^c \cap A) = \mu(A^c \cap B)$ the result follows.

Now suppose that there is a free ultrafilter $\mathcal{U}$ that violates assumption (R1). The simple rule whose decisive coalitions are all members of $\mathcal{U}$ satisfies (P1), (P2), and (P4) (see proposition 3.1 in [2]), while (P6) is trivially satisfied. □

A corollary of this result is that for any extension of $(N, \mathcal{L}_{n,p}, \mu_{n,p})$ with $n \geq 2$ there is no measurable rule satisfying (P1), (P2), (P4), and (P6). More generally, there is no measurable rule satisfying (P1), (P2), (P4), and (P6) in every measure space in which there is a partition of the set of agents in equal-measure coalitions.

Transitivity, however, may be a rather strong requirement. We next explore the possiblity of dropping transitivity and strengthening independence or irrelevant alternatives and weak Paretianism to, respectively, neutrality and monotonicity.

Definition 5 A $\mathcal{L}$-measurable preference aggregation rule f is

(P7) *neutral* if, for all $\rho, \rho' \in \mathcal{R}^N_{\mathcal{L}}$ and all $x, y, a, b \in X$,

$$[\rho|_{\{x,y\}} = \rho'|_{\{a,b\}}] \Rightarrow [f(\rho)|_{\{x,y\}} = f(\rho')|_{\{a,b\}}].$$

(P8) *monotonic* if for all $\rho, \rho' \in \mathcal{R}^N_{\mathcal{L}}$ and all $x, y \in X$,

$$[\{i : xP_iy\} \subset \{i : xP'_iy\}, \ \{i : xR_iy\} \subset \{i : xR'_iy\} \ \textit{and}\ xPy] \ \textit{imply}\ xP'y.$$

For the finite case it is known that a class of aggregation rules defined by (cardinality-based) anonymity, neutrality and monotonicity involves disregarding all information other than the head-count of members agreeing in their rankings. These are called *counting rules* (see e.g. [5]). In the infinite-agent case we can analogously define *measuring* rules as follows:

Definition 6 An $\mathcal{L}$-measurable preference aggregation rule f is a *(proper) measuring rule* if and only if there exist a non-empty collection of ordered pairs of decisive numbers $\left\{(p,a) \in \mathbb{R}_{+}^{2} : p + a \leq 1; p > a\right\}$ *such that* xPy if and only if $\mu\left(\{i : xP_iy\}\right) \geq p$ and $\mu\left(\{i : yP_ix\}\right) \leq a$.

(The non-emptyness requirement in the definition is to avoid the considering the case when the social preference is always full indifference. Such a rule would be neutral, anonymous and monotonic, but not weakly Paretian.) In general, we require assumptions on μ to ensure that $\mu-$ anonymity, neutrality and monotonicity characterize measuring rules. The reason for this is that if a measure has atoms it may be possible to find coalitions that can not be matched in measure and, consequently, $\mu-$ anonymity is too weak an assumption to have any impact. An extreme example of this is example 6. Indeed, defining individual 2 as a dictator we obtain a preference aggregation rule, that, somewhat counterintuitively, is not only monotonic and neutral, but also $\mu-$anonymous. While this example actually violates assumption (R1), the latter can easily be seen as insufficient. We therefore need another richness condition:

Assumption R2 Let $A, A' \in \mathcal{L}$ be disjoint measurable coalitions. Consider any pair $B, B' \in \mathcal{L}$ $(B \cap B' = \emptyset)$ such that $\mu(B) \geq \mu(A)$ and $\mu(B') \leq \mu(A')$. Then $\mathcal{L}$ contains disjoint pairs C and C', D and D' such that $A \subseteq C$, $C' \subseteq A'$, $D \subseteq B, B' \subseteq D'$ and $\mu(C) = \mu(D), \mu(C') = \mu(D')$.

While this assumption is undoubtedly stringent, there are many measure spaces that satisfy it. Indeed, if the standard finite-agent case is viewed as assigning equal measure to every agent, it holds trivially. Similarly, every distribution that is non-atomic, or in which atoms together contain less than a half of the total weight, satisfies it. When (R2) holds, the following characterization is straightforward:

Proposition 2 *If assumption* (R2) *holds, an* $\mathcal{L}$ *-measurable preference aggregation rule* f *satisfies* (P1), (P6), (P7) *and* (P8) *if and only if it is a measuring rule.*

Proof The proof follows that of the theorem 3.6 in Austen-Smith and Banks [5]. That any measuring rule satisfies $\mu-$anonymity and neutrality is immediate (the rule neither depends on identities of individuals nor it is specific to alternatives being compared). Monotonicity can likewise be easily shown to hold, once we observe that for any preference profiles $\rho, \rho' \in \mathcal{R}_{\mathcal{L}}^{N}$ and any pair of alternatives $x, y \in X$ $\{i : xP_iy\} \subset \{i : xP_i'y\}$ implies $\mu(\{i : xP_iy\}) \leq \mu(\{i : xP_i'y\})$.

Now, suppose f is a weakly Paretian, μ-anonymous, monotonic and neutral rule. Suppose for some preference profile $\rho \in \mathcal{R}_{\mathcal{L}}^{N}$ and some pair of alternatives $x, y \in X$ we have xPy (weak Paretianism ensures that we may actually select these). Define $\mu\{i : xP_iy\} = p$ and $\mu\{i : yP_ix\} = a$. We have to show that for any preference profile $\rho' \in \mathcal{R}_{\mathcal{L}}^{N}$ and any pair of alternatives $z, w \in X$ such that $\mu\{i : zP_i'w\} \geq p$ and $\mu\{i : wP_i'z\} \leq a$ we will have $zP'w$. Since assumption (R2) holds, we know that there exists a measurable superset C of $\{i : xP_iy\}$ and the measurable subset D of $\{i : zP_i'w\}$ such that $\mu(C) = \mu(D)$. Furthermore, there must exist a measurable subset C' of $\{i : yP_ix\}$ and measurable superset D' of $\{i : wP_i'z\}$ such that $\mu(C') = \mu(D')$ and $C \cap C' = D \cap D' = \emptyset$. Consider a measurable preference profile $\hat{\rho}$ with $\{i : x\hat{P}_iy\} = C$ and $\{i : y\hat{P}_ix\} = C'$ and another measurable profile ρ^* with $\{i : zP_i^*w\} = D$ and $\{i : wP_i^*z\} = D'$. (Such measurable profiles can be obtained from ρ and ρ', respectively.) By monotonicity $x\hat{P}y$. Therefore, by $\mu-$anonymity and neutrality zP^*w and by monotonicity $zP'w$. That $p > a$ follows immediately, since otherwise both xPy and yPx. □

Among measuring rules, an important one is "almost sure unanimity," which we call *consensus:*

Definition 7 Given a measure space $(N, \mathcal{L}, \mu)$ the *consensus social choice rule* σ_{cs} is defined by

$$xPy \Longleftrightarrow [\mu\{j \in \mathbb{N} : xP_jy\} = 1]$$

It turns out that we can characterize consensus by adding the requirement of quasitransitivity.

Definition 8 A $\mathcal{L}$-measurable preference aggregation rule f is

(P9) *quasitransitive* if for every $\rho \in \mathcal{R}_{\mathcal{L}}^{N}$ and for all $x, y, z \in X$,

$$xPy \;\&\; yPz \Rightarrow xPz.$$

Assumption R3 : Every coalition A such that $0 < \mu(A) < 1$ has a subcoalition B such that $0 < \mu(B) \leq \mu(A^c)$.

Note, that like assumptions (R1) and (R2), (R3) is satisfied by any measure which allows for a partition of the set of agents into equal measure coalitions.

Theorem 2 *For any measure space* $(N, \mathcal{L}, \mu)$ *satisfying assumptions* (R2) *and* (R3)*, the consensus rule is the unique* $\mathcal{L}$*-measurable rule satisfying* (P1), (P6), (P7), (P8) *and* (P9)*.*

Proof It is easy to see that the consensus rule satisfies (P9), in addition to the other properties common to measuring rules. Now consider any another measuring rule with $p < 1$ and $a > 0$. This rule is effectively different from consensus if there is a profile ρ such that for some pair of alternatives x, y, the sets $A = \{i : xP_iy\}$ has measure $p \leq \mu(A) < 1$. Furthermore, without loss of generality we can select $A \in \mathcal{L}$ such that $p < \mu(A) < p + \varepsilon$ (if this is impossible, selecting a smaller p would not change the rule). Assumption (R3) implies that there exists also a $B \in \mathcal{L}$ such that $\mu(A) \geq p$, $\mu(B) \geq p$ but $\mu(A \cap B) < p$. Consider a preference profile such that xP_iyP_iz for any $i \in A \cap B$, but zI_ixP_iy for any $i \in B^c \cap A$ and yP_izI_ix for any $i \in B \cap A^c$. It is immediate that for any a xPy, yPz but not xPz. □

Note that when the number of voters is finite and we use an equal treatment coalition algebra (all individual coalitions are recognized with the same measure) consensus reduces to unanimity. The difference between the finite society and the infinite society is that in the last one consensus may also satisfy the following desirable strengthening of (P3):

Definition 9 A $\mathcal{L}$-measurable preference aggregation rule f is

(P10) *veto-proof* if there does not exist $i \in N$ such that xP_iy implies xRy for every $\rho \in \mathcal{R}_{\mathcal{L}}^{N}$ and for any $x, y \in X$.

In the finite case consensus implies that every individual has a veto, while in the non-atomic case no measure zero set of individuals has a veto.

5 Conclusions and Future Research

This paper considers implications of altering the usual anonymity axiom of social choice in a setting with a measure defined over a potentially infinite agent space. Rather than concentrating on cardinality of coalitions, we choose to subject to an equal-treatment regime coalitions of the same measure. When the number of agents is finite, our results include the standard ones as a special case. In general, we observe that the usual results in the theory can be replicated with $\mu-$anonymity, provided the measures define sufficiently large classes of coalitions that have to be treated equally. It is of interest that the "richness" assumptions involved are entirely phrased in terms of properties of "equal-size" coalition collections. This suggests that it is possible (and perhaps desirable) to re-frame the analysis in terms of properties of classes of equally treated coalitions (not necessarily defined with respect to a measure *or* cardinality).

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