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M E X I C O

# Equilibrium Indeterminacy in an Endogenous Growth Model: Debt as a Coordination Device

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Abstract

This paper presents a two-sector endogenous growth model where public spending {which is endogenous and productive{ may generate equilibrium indeterminacy. Under certain mild conditions, there exists a continuum of expectations-driven equilibrium paths approaching a common balanced growth path. We show that the welfare-maximizing equilibrium path is associated with a labor supply as large as possible at time zero. Furthermore, the welfare cost of indeterminacy can represent more than a 2:1% of total consumption. It is also shown that public debt may be used to coordinate private expectations on current and future prices, and therefore, may break down the indeterminacy result. The equilibrium selection mechanism works through the amount of debt issued at time zero.

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# 1. Introduction

As it is well known, the existence of complementarities between individual decisions may lead to situations in which the competitive equilibrium is not determinate, i.e., there may be multiple equilibria supported by di®erent expectations on equilibrium prices. A rich variety of models with these kinds of complementarities has been proposed so far to study the empirical plausibility of such expectations-driven equilibria. In representative agent in¯nite horizon models of capital accumulation, it is generally assumed that the marginal productivity of private inputs is a function of the aggregate or average level of a speci¯ed endogenous variable. In models with one state variable, Boldrin and Rustichini (1994) consider a positive externality from the aggregate stock of physical capital; Benhabib and Farmer (1994) assume that the externalities spring from the average levels of physical capital and labor; Perli (1998) assumes capital and labor externalities in a model with home production. In two sector models with physical and human capital accumulation, Chamley (1993) considers a positive labor externality in the production of human capital; Xie (1994) studies the case of a positive externality from the average level of human capital in the production of goods; Benhabib and Perli (1994) analyze the consequences of having di®erent combinations of externalities together with the assumption of an endogenous determination of the leisure time.

The success of these models in yielding the indeterminacy result under plausible parameter values has been mixed. While in one sector models the required externality from physical capital is unrealistically large, in two sector models, a too high elasticity of intertemporal substitution for consumption is needed. On the other hand, models with home production or with a human capital sector and valued leisure seem to be more appropriate to explain the issue of equilibrium indeterminacy. From our viewpoint, there are two important points which are still unsolved in this literature. First, there is not a systematic study of the welfare properties of the equilibria set. Second, the idea of designing an equilibrium selection mechanism which operates on the formation of expectations has been unfruitful so far.

In this paper, we present a two-sector endogenous growth model where linkages between agents come from the existence of a public sector which conducts taxation and spending policies. One premise in our model is that some categories of public spending a®ect the productivity of private inputs. Since public spending depends on taxation revenues, and since these revenues are a function of the labor and saving decisions taken by the agents, it is easy to trace the source of the complementarity in the agents' actions.

The premise of productive public spending has been extensively used in theoretical models of optimal growth as an important determinant of the rate of growth, and in empirical studies as a possible explanation of the productivity slowdown observed in the last decades. The first models to include public spending in the production function of goods were developed by Shell (1967) and Arrow and Kurz (1970). More recently, Barro (1990), Barro and Xavier Sala-i-Martin (1992), Baxter and King (1993), Benhabib et al. (1996), Cassou and Lansing (1998), Cazavillan (1996) and Glomm and Ravikumar (1994), among others, present models in which public spending is also productive. Empirical evidence in favor of this premise has been reported in a large number of papers [e.g., Aschauer (1989), Evans and Karras (1994), Morrison and Schwartz (1996), Lynde and Richmond (1993), Munnell (1990), Ratner (1983), among others].

Our model overcomes the two major criticisms to the existing literature on equilibrium indeterminacy. First, we start by calibrating our model by making use of estimates from the empirical literature. Then, we show that equilibrium indeterminacy arises for that calibrated economy. In contrast to previous models, indeterminacy does not rely on unrealistically high returns to scale. Second, we carry out a complete welfare analysis of the equilibria set, and propose a simple equilibrium selection mechanism, which comes out naturally from the model.

The intuition underlying the high plausibility of our model to generate a multiplicity of equilibria follows from the type of complementarities generated by the public sector. When capital and labor income is taxed at a flat rate, the level of productive public spending is a function of a large set of exogenous and endogenous variables. This set includes taxation and spending policy parameters, the stocks of physical and human capital, hours worked, and the interest and wage rates. Indeed, it is the expected equilibrium values for these latter variables what makes the existence of a continuum of self-fulfilling equilibria possible.

At the initial period, if the representative agent expects low interest and wage rates to prevail in equilibrium, she will choose a low saving rate and a high allocation of time in the human capital sector. By doing so, taxation revenues, and consequently, public spending will be low. Since the marginal productivity of capital and labor are affected by the public expenditure, the interest and wage rates will be correspondingly low in equilibrium. Starting from this equilibrium, imagine now an increase in the expected values for the interest and wage rates. In order to reach a new equilibrium from these corrected expectations, the saving rate and hours worked must go up. Due to the increase in the labor supply, and in the level of public expenditure, output also increases. Therefore, a higher saving rate is possible in equilibrium if the increase in consumption is sufficiently low; in other words, if the elasticity of intertemporal substitution for consumption is low.

The fact that private expectations on equilibrium prices are self-fulfilling, along with the externality-type complementarities considered in the literature has hindered the study of equilibrium selection mechanisms. [Christiano and Harrison (1996) and Guo and Lansing (1998) propose coordination mechanisms based on quite ad hoc taxation schemes, in which the government sets tax rates as complicated functions of economic variables such as employment or production.] In our model, we take advantage of the role of the government as the source of complementarities in order to break down the self-fulfillingness of expectations. All we need is to permit the government to run deficits. Moreover, the possibility of choosing the level of debt to be issued at the initial period can be used to target the interest and wage rates in equilibrium, thus canceling the effects of private expectations. It should be emphasized that the use of public debt as a coordination device is not restricted to the model with productive public spending. We illustrate below its applicability in a model with externalities from the average level of human capital. Furthermore, our claim is that public debt may be an effective coordination device in more general setups, such as models with imperfect capital markets, and models with money in the utility or production functions [e.g., Benhabib and Farmer (1998), Benhabib et al. (1998)].

The rest of the paper is organized as follows. In Section 2, we outline the model and present two indeterminacy results. Section 3 contains the welfare analysis of equilibrium indeterminacy. In Section 4, we show the ability of public debt to work as a coordination device, and present some computations of the welfare gain of coordination. In Section 5, we extend the analysis of coordination to a model with human capital externalities. Finally, in Section 6 we draw the main conclusions, and Section 7 contains the appendixes.

## 2. The Model

We consider an economy populated by a continuum of identical dynasties who derive utility from the consumption of an aggregate good. The size of the population is normalized to unity, and we abstract from population growth. The instantaneous utility function has a constant elasticity of intertemporal substitution for consumption. Future utility is discounted at the positive rate $\rho$. The problem of a representative household is then to maximize the lifetime discounted utility given by

$$\int_0^{\infty} e^{-\rho t}\frac{c(t)^{1-\sigma}-1}{1-\sigma}dt \tag{2.1}$$

The household owns the stock of physical capital which has a gross rental price of $r(t)$. Labor is supplied at the gross wage rate $\omega(t)$. The government collects taxes from capital and labor income at the constant rate $\tau$. Thus, the budget constraint for the representative household is,

$$c(t)+\dot{k}(t)=(1-\tau)[r(t)k(t)+\omega(t)h(t)u(t)]+T(t) \tag{2.2}$$

where $\dot{k}(t)$ denotes investment in physical capital, $u(t)$ is the amount of raw time devoted to work, and $h(t)$ is the level of human capital or efficiency. $T(t)$ denotes transfers from the government. The efficiency of labor can be increased by devoting time to the human capital sector. The technology in this sector is

$$\dot{h}(t)=B(1-u(t))h(t) \tag{2.3}$$

In this model the government has two types of activities: collects taxes from the household sector, and spends total revenues in infrastructure and transfers. The budget constraint must be balanced at every period, i.e.

$$\tau[r(t)k(t)+\omega(t)h(t)u(t)]=g(t)+T(t) \tag{2.4}$$

where $g(t)$ denotes expenditure in infrastructure. The spending policy is assumed to be constant, a fixed share, say $\beta$, of total revenues is devoted to infrastructure. Therefore,

$$g(t)=\beta\tau[r(t)k(t)+\omega(t)h(t)u(t)] \tag{2.5}$$

The output sector is competitive, and each firm has access to the same technology. Public spending in infrastructure is assumed to affect the production set. For analytical convenience, we assume a Cobb-Douglas production function,

$$F(k;hu;x)=Ak(t)^{\alpha}(h(t)u(t))^{1-\alpha}x(t)^{\alpha_2} \tag{2.6}$$

where $x(t)$ denotes the services derived from public expenditure in infrastructure[1]. The determination of these services will be specified below under two different assumptions on

[1] At this point, $x(t)$ can also have the interpretation of technology or basic knowledge which, given its nature of public good, has to be publicly financed [see Shell (1966), (1967) and Antinolfi et al. (1998) for a deeper description of this interpretation of $x(t)$].

the nature of public investment. The elasticity of per capita output with respect to the services from public investment is given by $\alpha_2$:

Profits maximization implies that rental prices for capital and labor equalize the respective marginal products,

$$r(t) = F_k[k(t); h(t)u(t); x(t)] \tag{2.7}$$

$$\omega(t) = F_{hu}[k(t); h(t)u(t); x(t)] \tag{2.8}$$

The first order conditions for the consumer problem involve the optimal allocation of income between consumption and investment, and the allocation of time between working and education activities. For income allocation, the consumer must equalize the marginal rate of substitution in consumption with its relative price, that is,

$$e^{-\rho t}\frac{c(t)^{-\sigma}}{c(0)^{-\sigma}} = e^{-\int_0^t (1-\tau) r(s) ds} \tag{2.9}$$

As for time allocation, the consumer must equalize the marginal revenue of time devoted to the two competing activities, that is,

$$\omega(t)h(t) = B\int_t^{\infty} e^{-\int_t^s (1-\tau) r(\psi) d\psi}\,\omega(s)u(s)h(s)ds \tag{2.10}$$

The following transversality condition must be fulfilled in a competitive equilibrium,

$$\lim_{t\to\infty}\left\{k(t)e^{-\int_0^t (1-\tau) r(s) ds} + h(t)e^{-Bt}\right\} = 0 \tag{2.11}$$

In order to complete the characterization of a competitive equilibrium, we need to write out an equation for the determination of per capita services from public investment. We will distinguish two different cases depending on the nature of public investment: Rival and excludable public infrastructure, and congestionable public infrastructure.

## 2.1 Rival and Excludable Public Infrastructure

We start our analysis of public infrastructure assuming that it is rival –there is a quantity allocated to each firm– and excludable. As a consequence, there is no congestion in the services derived by each firm, that is,

$$x(t) = g(t) \tag{2.12}$$

where $g(t)$ denotes public investment per firm.

## Balanced Growth Path

A balanced growth path is defined as an equilibrium solution such that, for some initial conditions, consumption, investment, capital stocks, and government spending grow at constant rates, and hours worked remain constant. By imposing these conditions, it follows from (2.2) and (2.3) that the rate of growth for physical capital, $\gamma_k$, and human capital, $\gamma_h$, must satisfy the following relationship,

$$\gamma_k = \left(\frac{1-\alpha}{1-\alpha-\alpha_2}\right)\gamma_h \tag{2.13}$$

Moreover, evaluating (2.9) and (2.10) along a balanced growth path, we obtain,

$$\gamma_k = \frac{(B-\rho)(1-\alpha)}{(1-\alpha)\sigma-\alpha_2} \tag{2.14}$$

It is clear from (2.2) that production and consumption grow also at $\gamma_k$. The long-run value for hours worked, $u^*$, can then be easily obtained from (2.3). It can be readily shown that if $\alpha \neq \alpha_2$ and $(1-\alpha)\sigma \neq \alpha_2$, then there is a unique interior balanced growth path. We assume throughout the paper that these latter conditions are satisfied, and restrict our analysis to the case of a single balanced growth equilibrium.

We offer here two alternative calibrations of the model based on different estimates of parameter $\alpha_2$. The empirical literature on the productivity effects of public inputs has yielded a wide variety of estimates. Leaving out differences regarding econometric techniques, data sets, and the definition of the public input, one can find estimates as high as 0.39 [Aschauer (1989)], and estimates as low as 0.06 and 0.1 [Ratner (1983) and Munnel (1990), respectively]. We will consider two benchmark economies, one with $\alpha_2 = 0.39$ and the second with $\alpha_2 = 0.1$. Values for fiscal policy parameters are constructed as follows. Using a tax rate of 0.36 on capital income, a 0.40 on labor, and a capital share of 0.36, we obtain an estimate for the common tax rate on capital and labor income, $\tau$, of 0.3856. For the percentage of public revenues spent in infrastructure, $\beta$, we use data from the National Income and Product Accounts (NIPA) published by the Department of Commerce. We considered local, state and federal investment in transportation, energy, natural resources, civilian safety and housing and community services as a proxy for public investment in infrastructure. Next, using $\tau = 0.3856$, we calculated the value of $\beta$ for the last 30 years. >From 1981 to 1996, this value was roughly constant at 0.045. For the remaining parameters, we use standard values in the growth literature. The discount factor, $\rho$, is set at 0.05, the elasticity of intertemporal substitution for consumption at 0.5, and $B$ is chosen so that total production grows at 2.9%.

Benchmark economy 1: $\rho = 0.05$; $\sigma = 2$; $A = 3.5$; $\alpha = 0.36$; $\alpha_2 = 0.39$; $B = 0.0903$; $\tau = 0.3856$ and $\beta = 0.045$.

Benchmark economy 2: $\rho = 0.05$; $\sigma = 2$; $A = 3.5$; $\alpha = 0.36$; $\alpha_2 = 0.1$; $B = 0.1034$; $\tau = 0.3856$ and $\beta = 0.045$.

Transitional Dynamics

In order to study the dynamic properties of the balanced growth path, we define the following variables,

$$\hat{k}(t) \equiv \frac{k(t)}{h(t)^{\phi}} \text{ and } \hat{c}(t) \equiv \frac{c(t)}{h(t)^{\phi}}$$

where $\phi \equiv (1-\alpha)/(1-\alpha-\alpha_2)$. The dynamic system in $\hat{k}(t)$, $\hat{c}(t)$ and $u(t)$ is then

$$\begin{aligned}
\frac{\dot{\hat{k}}(t)}{\hat{k}(t)} &= (1-\beta\tau)A(\beta\tau A)^{\frac{\alpha_2}{1-\alpha_2}}\hat{k}(t)^{\frac{\alpha+\alpha_2-1}{1-\alpha_2}}u(t)^{\frac{1-\alpha}{1-\alpha_2}} - \frac{\hat{c}(t)}{\hat{k}(t)} - \phi B(1-u(t))\\
\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} &= \frac{1}{\sigma}\left[(1-\tau)\alpha A(\beta\tau A)^{\frac{\alpha_2}{1-\alpha_2}}\hat{k}(t)^{\frac{\alpha+\alpha_2-1}{1-\alpha_2}}u(t)^{\frac{1-\alpha}{1-\alpha_2}} - \rho\right] - \phi B(1-u(t))\\
\frac{\dot{u}(t)}{u(t)} &= \eta A(\beta\tau A)^{\frac{\alpha_2}{1-\alpha_2}}\hat{k}(t)^{\frac{\alpha+\alpha_2-1}{1-\alpha_2}}u(t)^{\frac{1-\alpha}{1-\alpha_2}} + B\left(\frac{1-\alpha}{\alpha-\alpha_2}\right) - \left(\frac{\alpha}{\alpha-\alpha_2}\right)\frac{\hat{c}(t)}{\hat{k}(t)} + Bu(t)
\end{aligned}$$

where $\eta \equiv \frac{(1-\tau)\alpha\alpha_2+(1-\beta)\alpha\tau}{\alpha-\alpha_2}$. By definition, these new variables are constant along a balanced growth path, $(\hat{k}^*, \hat{c}^*, u^*)$. In our benchmark economy 1, the balanced growth path is defined by the following values: $\hat{k}^* = 1.09728$; $\hat{c}^* = 0.49466$; $u^* = 0.87458$, $\nu_k = 0.029$ and $\nu_h = 0.01132$. For the benchmark economy 2, we get: $\hat{k}^* = 11.53684$; $\hat{c}^* = 5.19751$; $u^* = 0.76366$, $\nu_k = 0.029$ and $\nu_h = 0.02443$.

Our first result shows that the existence of a continuum of equilibria is very likely under completely standard parameter values.

**Proposition 1**: Consider the economy described by (2.1)–(2.12). Then

a) There is a continuum of equilibria converging to the balanced growth path if and only if,

$$\text{(i)} \quad \alpha < \alpha_2 < (1-\alpha)\sigma$$

b) There is no equilibrium converging to the balanced growth path if and only if,

$$\text{(ii)} \quad \alpha > \alpha_2 > (1-\alpha)\sigma$$

c) There is a unique equilibrium converging to the balanced growth path if and only if neither (i) nor (ii) hold.

**Proof:** See Appendix I.

According to this Proposition, our benchmark economy 1 satisfies condition (i), and therefore the equilibrium is indeterminate. For our benchmark economy 2, neither (i) nor (ii) are satisfied, and consequently the equilibrium is determinate. In the next section, we show however that by assuming congestion in public infrastructure the equilibrium may also be indeterminate in this latter economy.

### 2.2 Public Infrastructure Subject to Congestion

We consider now the existence of some degree of congestion in the use of public expenditure. The services derived by each firm from a given level of aggregate public investment decrease with the aggregate stock of physical capital and increase with the firm's level of efficient labor, that is,

$$x(t) = \left( \frac{h(t)u(t)}{K(t)} \right)^{\theta} G(t) \tag{2.15}$$

where $\theta > 0$ gives the degree of congestion, and $G(t)$ denotes aggregate public investment.

**Proposition 2:** Consider the economy described by (2.1)–(2.11) and (2.15). Then

a) There is a continuum of equilibria converging to the balanced growth path if and only if,

$$\text{(i)} \quad \frac{\alpha}{1+\theta} < \alpha_2 < \frac{(1-\alpha)\sigma}{1+\theta\sigma}$$

b) There is no equilibrium converging to the balanced growth path if and only if,

$$\text{(ii)} \quad \frac{\alpha}{1+\theta} > \alpha_2 > \frac{(1-\alpha)\sigma}{1+\theta\sigma}$$

c) There is a unique equilibrium converging to the balanced growth path if and only if neither (i) nor (ii) hold.

**Proof:** See Appendix I.

According to Proposition 2, under parameter values defining our benchmark economy 2, the equilibrium is indeterminate if $2.6 < \theta < 5.9$.

In general, it is clear from Proposition 2 that, under congestion, the indeterminacy result may arise for very small values of $\alpha_2$. This result stands in sharp contrast with those obtained in the literature with human capital spillovers, where a high external effect is required to produce equilibrium indeterminacy. It should also be clear that the size of the population has scale effects under this formulation of the services derived from public investment. These scale effects do not affect however our indeterminacy result presented in Proposition 2.

For analytical convenience, we will assume from now on that public investment is rival and excludable. This assumption, however, does not affect any of our results qualitatively.

## 3. Welfare Analysis of the Equilibria Set

The indeterminacy result presented in Section 2 immediately raises a question about the rank of expectations-driven equilibria in terms of welfare. Xie (1994), using the Lucas (1988) model, is able to rank the equilibria only after assuming that the inverse of the elasticity of intertemporal substitution equals the physical capital income share (in our notation, $\sigma = \alpha$), However, according to the empirical estimates of these two parameters, this restriction turns out to be quite unlikely. In this section, we resort to numerical methods to carry out the welfare analysis without imposing further restrictions on parameter values. Before starting the discussion of our numerical results, it might be helpful to see the set of interior equilibria as indexed by the initial labor supply, $u(0)$. The main effects of $u(0)$ on the level of welfare are the following. First, consumption at time zero is an increasing function of $u(0)$. Second, a larger $u(0)$ implies a lower investment rate in human capital along the transitional period, and therefore, a lower level of human capital along the balanced growth path. As $\hat{c}^* = \frac{c(t)}{h(t)^A}$ does not depend on $u(0)$, it must be true that the long-run consumption path is a decreasing function of $u(0)$. These two opposite effects of $u(0)$ can be shown in a more transparent manner. We write the change in the lifetime utility with respect to changes in $u(0)$ as a function of the change in the initial consumption, and in the interest rate path. From (2.1) and (2.9), we can write the lifetime utility as

$$W(k(0);h(0);u(0)) = \int_0^\infty e^{-\rho t}\left(\frac{1}{1-\sigma}c(0)^{1-\sigma}e^{-\frac{1-\sigma}{\sigma}\int_0^t[\rho-(1-\tau)r(s)]ds} - \frac{1}{1-\sigma}\right)dt$$

Differentiating the value function with respect to $u(0)$ we obtain,

$$\begin{aligned}\frac{W(k(0);h(0);u(0))}{\partial u(0)} &= \frac{\partial c(0)}{\partial u(0)}\int_0^\infty e^{-\rho t}c(0)^{-\sigma}e^{-\frac{1-\sigma}{\sigma}\int_0^t[\rho-(1-\tau)r(s)]ds}dt + \\ &+ \int_0^\infty\int_0^t \frac{\partial r(s)}{\partial u(0)}e^{-\rho t}(1-\tau)\frac{c(0)^{1-\sigma}}{\sigma}e^{-\frac{1-\sigma}{\sigma}\int_0^t[\rho-(1-\tau)r(s)]ds}dsdt\end{aligned} \qquad (3.1)$$

Although it is clear that $\frac{\partial c(0)}{\partial u(0)} > 0$, it is not always true that $\frac{\partial r(t)}{\partial u(0)} \geq 0$ for all t, and therefore, the sign of (3.1) cannot be neatly determined. In order to shed some light on this problem, we present in Figure 1 the interest rates associated with two different initial values of $u(0)$[2]. Given that interest rates cross at some t, it seems to be very hard to establish any general result on the welfare properties of the equilibria set.

Our welfare computations are based on the solution to the linearized system. One concern which arises from using linear approximations is the extent to which the linearized system mimics the nonlinear dynamics. Figure 2 displays the interest rate as obtained from the linear and non linear systems, for our benchmark economy 1. It should be clear from this figure that the linear dynamics is a good approximation in a significantly large neighborhood of the steady state.

Thus, after linearizing the dynamical system $\hat{k}(t), \hat{c}(t), u(t)$ around the steady state equilibrium, we can write the consumption path associated with a given $u_0$ as

$$c_{u_0}(t) = \left[ \left[ \hat{c}^* + \zeta_1 v_{12} e^{\lambda_1 t} + \zeta_2 v_{22} e^{\lambda_2 t} \right] h(0) e^{\gamma_h t} e^{-\zeta_1 v_{13} B \left( \frac{e^{\lambda_1 t} - 1}{\lambda_1} \right)} e^{-\zeta_2 v_{23} B \left( \frac{e^{\lambda_2 t} - 1}{\lambda_2} \right)} \right]^{\phi} \tag{3.2}$$

which converges to the balanced consumption path

$$c^*_{u_0}(t) = \left[ \hat{c}^* \; h(0) e^{\gamma_h t} e^{\frac{\zeta_1 v_{13} B}{\lambda_1}} e^{\frac{\zeta_2 v_{23} B}{\lambda_2}} \right]^{\phi} \tag{3.3}$$

where $v_i = (v_{i1}, v_{i2}, v_{i3})'$ for $i = 1, 2$ denotes the eigenvector associated with the negative eigenvalue $\lambda_i$, and $\zeta_1$ and $\zeta_2$ are integration constants whose values depend on $u_0$.

Our procedure to compute the lifetime utility function[3], $\int_0^{\infty} e^{-\rho t} \frac{c_{u_0}(t)^{1-\sigma} - 1}{1-\sigma} dt$, along the equilibrium consumption path given by (3.2), yields an expression which can be numerically computed at a relatively low cost.

Our computations show that lifetime utility increases with $u(0)$. (Figure 3 presents the lifetime utility as a function of $u(0)$.) Thus, the welfare-maximizing equilibrium is associated with a labor supply as large as possible at time zero. Furthermore, we found the same result for alternative calibrations of the model rendering equilibrium indeterminacy. This result contrasts with that of Xie (1994). Under his version of the Lucas (1988) model, the welfare-maximizing equilibrium has the lowest possible labor supply at time zero.

[2] We follow a Euler method to compute numerically the interest rates using the nonlinear system of differential equations.

[3] See Appendix II for a description of the numerical procedure.

### 3.1 The Welfare Cost of Indeterminacy

As a first step before considering how to introduce a coordination mechanism in order to select the best equilibrium, it seems natural to inquire about the cost of miscoordination. The magnitude of this cost should be used to assess the benefits from intervention policies which are addressed to accomplish a particular equilibrium. Moreover, as such policies generally involve some implementation costs, one should assure that these latter costs are worth paying.

Making use of the approximation to the lifetime utility presented in the previous subsection, we can calculate the welfare cost of starting at $u_0'$, instead at $u_0 > u_0'$, as the value of $\gamma_{u_0'}^{u_0}$ that solves the following equation,

$$\int_0^\infty e^{-\rho t}\frac{1}{1-\sigma}\left(c_{u_0'}(t)(1+\gamma_{u_0'}^{u_0})\right)^{1-\sigma}dt = \int_0^\infty e^{-\rho t}\frac{1}{1-\sigma}c_{u_0}(t)^{1-\sigma}dt$$

The welfare cost of miscoordination is given, thus, as a percentage of consumption. As interior equilibrium paths are indexed by $u(0) \in (0;\infty)$, the maximum welfare cost from indeterminacy is $\gamma_0^\infty$.

Table 1 presents the magnitude of $\gamma_0^\infty$ and $\gamma_{0.9}^\infty$ for different initial values of $k(0)$. Since our computations are heavily based on linear approximations, we try to minimize approximation errors by setting an initial condition in a close interval of the steady state. Throughout this paper we shall set $k(0) = 0.95\hat{k}^*$. For our benchmark economy 1, we obtained $\gamma_0^\infty = 2.1\%$, and $\gamma_{0.9}^\infty = .196\%$.

Table 1

| | $k(0) = 0.75\hat{k}^*$ | $k(0) = 0.8\hat{k}^*$ | $k(0) = 0.85\hat{k}^*$ | $k(0) = 0.9\hat{k}^*$ | $k(0) = 0.95\hat{k}^*$ |
|---|---|---|---|---|---|
| $\gamma_0^\infty$ | .02380 | .02305 | .02234 | .02167 | .02103 |
| $\gamma_{0.9}^\infty$ | .00220 | .00214 | .00208 | .00202 | .00196 |

Notes. These computations were carried out under benchmark economy 1, and $h(0) = 1$.

The inverse of the elasticity of intertemporal substitution for consumption, $\sigma$, is a key parameter in the determination of the magnitude of these welfare costs. The initial level of consumption, and consequently, the welfare cost of indeterminacy, strongly depend on this parameter. Table 2 presents the value of $\gamma_0^\infty$, $\gamma_{0.9}^\infty$, and initial consumption for different values of $\sigma$.

Table 2

| | $\sigma = 1$ | $\sigma = 2$ | $\sigma = 3$ | $\sigma = 4$ |
|---|---|---|---|---|
| $\gamma^1_0$ | .02342 | .02103 | .02043 | .02013 |
| $\gamma^1_{0.9}$ | .00208 | .00196 | .00192 | .00190 |
| $c_{u_0=0}(0)$ | .08283 | .46770 | .61923 | .69540 |
| $c_{u_0=0.9}(0)$ | .08918 | .48225 | .63484 | .71125 |
| $c_{u_0=1}(0)$ | .08989 | .48387 | .63658 | .71301 |

Notes. These computations were carried out under benchmark economy 1, and $k(0) = 0.95\hat{k}^*$ and $h(0) = 1$. (Notice that $\hat{k}^*$ refers to the steady state value associated with the corresponding value of $\sigma$.)

It is worth mentioning that a subset of equilibria, indexed by $(0; u_0)$, for some $0 < u_0 < 1$, entails a negative gross investment for a finite number of periods. Therefore, adding to the model a restriction of non reversible capital, the set of interior equilibria is reduced to $[u_0; 1)$. For our benchmark economy 1, $u_0 = 0.82625$. The welfare consequences of restricting the analysis to equilibrium paths with a positive gross investment may be then sizable.

## 4. Debt as a Coordination Device

The problem of equilibrium selection has captured the attention of researchers in many different fields. Game theorists use dynamic coordination games describing a disequilibrium process along which players learn about their opponents and adjust their strategies over time. The challenge in this framework is to show under which conditions the long-run equilibrium corresponds to the Pareto-efficient equilibrium. Hence, addressing the problem of equilibrium selection by considering a learning process entails departing from the assumption of perfect foresight. Besides, its applicability is strongly restricted to models with a finite number of equilibria. Evans, Honkapohja and Romer (1998) study a growth model in which there may be a finite number of perfect foresight equilibria which are characterized by constant interest and growth rates from $t = 0$ on. The authors, then, introduce a learning process to select one of those equilibria. In our model, however, it should be clear that the existence of a continuum of equilibria, along with the transitional dynamics for endogenous variables, prevent us from using learning in the same way. Furthermore, Duffy (1994) questions the usefulness of disequilibrium learning mechanisms to pick out locally unique solutions. This author presents an example where the use of a learning process leads agents to believe in a continuum of solutions paths converging to an indeterminate steady state.

The approach we follow in this paper is to design an equilibrium selection mechanism which operates directly on the formation of expectations. More precisely, given the assumption of perfect foresight, the mechanism works on the expected initial value for the interest and wage rates. In the model of Section 2, the expected values for these prices are determined solely by the self-fulfilling expected value for the equilibrium labor supply. In contrast, when the government is allowed to run deficits, it is possible to break down the self-fulfillingness of different expected values for prices by issuing the appropriate amount of debt. The interest and wage rates at time zero can thus be targeted by choosing conveniently the debt policy at time zero. The idea of using public debt to peg the interest rate in order to achieve equilibrium determinacy is, to some extent, related to the idea used by Calvo (1979) and Woodford (1994) in a monetary economy. Woodford studies a cash-in-advance economy and shows how pegging the nominal interest rate by using debt allows for price determinacy and a higher level of welfare with respect to the indeterminate case.[4]

If we denote the level of public debt by $b(t)$, then the total assets held by the household are now $a(t) = k(t) + b(t)$. At this point, we leave aside questions regarding the optimal determination of $b(0)$, and we study the consequences of having different initial levels of debt. The consumer budget constraint is,

$$c(t) + \dot{a}(t) = (1 - \tau)[r(t)a(t) + \omega(t)h(t)u(t)] + T(t) \tag{4.1}$$

First order conditions for utility maximization are still given by (2.9) and (2.10). The restriction on $a(t)$ to rule out Ponzi games for the household sector is

$$\lim_{t \to \infty} a(t) e^{-\int_0^t (1-\tau) r(s) ds} \geq 0 \tag{4.2}$$

[4] Our simple equilibrium selection mechanism differs from the ones presented by Christiano and Harrison (1999) and Guo and Lansing (1998) among others, in some crucial aspects. These authors postulate that the government sets the tax rate following a certain tax schedule which depends on state and control variables. Then, conditions on parameters values defining the tax schedule are established in order to obtain a steady-state equilibrium exhibiting saddle-path stability. Consequently, when the government sets conveniently these parameters values and applies the tax schedule, the equilibrium path converging to the steady state is also unique. However, there is no possibility of selecting an alternative equilibrium path converging to the same steady-state solution. The coordination mechanism presented in this paper, on the contrary, does not rely on any pre-established control schedule. The dynamics of public debt is simply determined by the budget constraint for the government. In addition, the government is able to select a particular equilibrium path converging to the steady-state equilibrium.

The transversality condition is

$$\lim_{t \to \infty} \left\{ a(t)e^{-\int_0^t (1-\tau) r(s) ds} + h(t)e^{-Bt} \right\} = 0 \tag{4.3}$$

As the government is allowed to issue debt, the budget constraint is,

$$\dot{b}(t) + \tau[r(t)a(t) + \omega(t)h(t)u(t)] = g(t) + T(t) + r(t)b(t) \tag{4.4}$$

The left hand side is the sum of taxation proceeds and the change in the level of debt. The right-hand side is the total spending on infrastructure, transfers and interest payments on debt. The corresponding condition to rule out Ponzi games for the government is that along the balanced growth path, debt cannot grow as fast as the interest rate. Furthermore, a necessary condition for the existence of a balanced growth path is that the share of debt in the portfolio be constant in the long-run. Hence, we shall restrict our analysis to public policies consistent with long-run positive levels of debt growing at the rate $\gamma_k$ given by (2.14).

The lifetime budget constraint for the government can be derived from (4.4) and the no-Ponzi-game condition as,

$$b(0) = \int_0^\infty e^{-\int_0^t (1-\tau) r(s) ds} \tau y(t) dt - \int_0^\infty e^{-\int_0^t (1-\tau) r(s) ds} (g(t) + T(t)) dt \tag{4.5}$$

where $y(t)$ denotes output per capita. In equilibrium, the present value of taxation revenues equals the present value of public expenditure plus the initial value of debt.

As in the model without public debt, we assume that infrastructure spending is a constant share, $\beta$, of taxation revenues,

$$g(t) = \beta\tau[r(t)a(t) + \omega(t)h(t)u(t)] \tag{4.6}$$

The share $1 - \beta$ of revenues collected from debt taxation is returned back to the household as lump-sum transfers. It is worth noticing that, since public investment in infrastructure is now a function of the ongoing level of debt, the interest and wage rates at time zero are a function of $b(0)$. We show below that this dependence makes possible the coordination of private expectations on initial equilibrium prices by choosing appropriately the initial public indebtedness.

Balanced Growth Path. It is readily shown that along the balanced growth path, growth rates for physical capital, $\gamma_k$, human capital, $\gamma_h$, hours worked, $u^*$, and the interest rate are not affected by the presence of public debt. The ratio of debt to physical capital can be obtained from (4.4) as

$$\frac{b}{k} = \frac{(1-\beta)\tau r^*}{(r^* - \gamma_k)\alpha} \tag{4.7}$$

where $r^*$ denotes the before-tax interest rate which is given by,

$$r^* = \frac{(1-\alpha)B\sigma - \alpha_2\rho}{((1-\alpha)\sigma - \alpha_2)(1-\tau)} \tag{4.8}$$

From (4.6) we obtain public investment in infrastructure as a percentage of production as, $\frac{g}{y} = \left(1 + \alpha\frac{b}{k}\right)\beta\tau$.

For our benchmark economy 1, the balanced growth path has $\gamma_k = 0.029$, $\gamma_h = 0.011328125$ and $u^* = 0.8745891714$; the long-run ratio of debt to physical capital is $\frac{b}{k} = 1.225010645$, and $\frac{g}{y} = 0.02504934268$. The Jacobian matrix of the dynamical system evaluated at this balanced growth path has two negative eigenvalues $\lambda_1 = -0.4688175423$, $\lambda_2 = -1.223172648$, and two positive eigenvalues, $\lambda_3 = 0.079$, and $\lambda_4 = 0.1833238911$, yielding, thus, equilibrium determinacy. Therefore, for any given initial values for $k(0)$, $h(0)$ and $b(0)$, there is a unique equilibrium path converging to the balanced growth path. As the government is assumed to set the initial level of debt, the set of equilibria can then be parameterized by $b(0)$.

For our benchmark economy 2, Proposition 1 showed that when debt is not allowed, the balanced growth path is determinate. Given the initial value for the state variable, there exists a unique value for $u(0)$ under which the economy converges to the balanced growth path. Consequently, when we allow for public debt, there must be a unique value for $b(0)$ such that the equilibrium converges to the balanced growth path. As expected, the Jacobian matrix has one negative eigenvalue $\lambda_1 = -0.2051215393$ and three positive eigenvalues $\lambda_2 = 0.079$, $\lambda_3 = 0.1523445047$ and $\lambda_4 = 0.5149016759$. Since under equilibrium determinacy the initial value of debt does not affect the equilibrium value of $u(0)$, there is at most one $b(0)$ which fulfills the lifetime budget constraint for the government, (4.5).

In the next section we analyze the welfare consequences of using debt as a coordination mechanism.

## 4.1 Welfare Gains from Coordination

In this section, we study the ability of debt to enhance utility when the equilibrium is indeterminate. For this purpose, we change slightly the previous public policy without altering the nature of the economy. We concentrate on a particular spending policy which yields debt growing at the rate $\gamma_k$ from $t = 0$ on. The reason to consider this scenario is to have an indeterminate level of debt along the balanced growth path. As a consequence, the long-run share of debt in the portfolio, $\frac{b}{k}$; becomes a parameter in the model. We can therefore study the welfare implications of policies setting $\frac{b}{k}$ and $\frac{b(0)}{k(0)}$. Assuming that the government has no debt at time zero, we can say that debt is effective as a coordination device, if there is at least one debt policy which renders higher utility than the worst equilibrium that may arise in the economy without debt.

In order to make the composition of the portfolio indeterminate in the balanced growth path, we consider the same infrastructure policy as in the previous section, that is (4.6). Regarding transfers, we establish now the following policy,

$$T(t) = (1 - \beta)\tau[r(t)a(t) + \omega(t)h(t)u(t)] + (\gamma_k - r(t))b(t) \tag{4.9}$$

where $\gamma_k$ is the long-run physical capital growth rate. It follows then from (4.4) that the law of motion for debt under this policy is

$$\dot{b}(t) = \gamma_k b(t) \tag{4.10}$$

which gives a continuum of balanced growth paths indexed by $\frac{b(t)}{k(t)}$. Moreover, consider the balanced growth path associated with a zero level of debt. It follows then from (4.10) that we must have $b(t) = 0$ for all $t \geq 0$, which corresponds to the model presented in Section 3. Thus, the welfare analysis of coordination reduces to the determination of the optimal long-run share of debt in the portfolio, and of the optimal initial issue of debt, $b(0)$: A simplifying assumption is that government revenues from selling the initial debt are wasted. However, the revenues derived from its taxation are productively spent. Regarding the distortionary effects of issuing $b(0)$ we consider two different scenarios.

Initial debt crowds out private investment in physical capital. After substituting the budget constraint for the government, and the equilibrium interest and wage rates in the budget constraint for the household, we get

$$\dot{k}(0) + b(0) = (1 - \beta\tau)y(0) - \beta\tau r(0)b(0) - c(0) \tag{4.11}$$

and

$$\dot{k}(t) = (1 - \beta\tau)y(t) - \beta\tau r(t)b(t) - c(t) \quad \text{for } t > 0 \tag{4.12}$$

As is clear from (4.11), initial investment in physical capital and consumption are reduced in order to buy public debt. After period zero, investment jumps to the level given by the stable manifold approaching the balanced growth path.

Because the discontinuity in investment at $t = 0$, the computation of the welfare gains of coordination for a given $\frac{b}{k}$ proceeds in two steps. First, given $k(0)$ and $b(0)$, we compute $c(0)$ and $u(0)$ such that the value of endogenous variables at an arbitrarily small $\epsilon \simeq 0$, $(k(\epsilon); b(\epsilon); h(\epsilon); c(\epsilon); u(\epsilon))$ satisfy the following conditions: They are implied by equilibrium conditions at time zero, and belong to the stable manifold of the system from $\epsilon$ onward. Second, we set $b(0)$ in order to maximize the lifetime utility (i.e., $b(0)$ is chosen such that $u(0) = 1$).[5]

Table 3 presents the following information for different values of $\frac{b}{k}$: The initial level of debt that induces $u(0) = 1$ in equilibrium, initial consumption, initial investment in physical capital, and the welfare gain from coordination, $\gamma_{co}$ (measured as the percentage increase in consumption with respect to the equilibrium with $u(0) = 0$ in the economy without debt). Some comments are in order. For very low values of initial debt, the welfare gain from coordination represents a 2.1% of consumption. A high initial debt may crowd out initial investment to the point of having a negative net investment. When non reversible investment is assumed, the maximum welfare gain is in the order of 6%.

Table 3

| | $\frac{b}{k} = 10^{-3}$ | $\frac{b}{k} = 10^{-2}$ | $\frac{b}{k} = 0.05$ | $\frac{b}{k} = 0.093$ | $\frac{b}{k} = .1$ |
|---|---|---|---|---|---|
| $b(0)$ | $1.081\ 10^{-3}$ | $1.084\ 10^{-2}$ | .05508 | .10419 | .11233 |
| $c(0)$ | .48401 | .48522 | .49069 | .49672 | .49772 |
| $\dot{k}(0)$ | .10293 | .09318 | .04902 | $2.527\ 10^{-4}$ | $-.00810$ |
| $\gamma_{co}$ | .02103 | .02490 | .04219 | .06093 | .06400 |

Notes: These computations were carried out under benchmark economy 1, and $k(0) = 0.95\hat{k}^*$ and $h(0) = 1$. (Notice that $\hat{k}^*$ refers to the steady state value in our benchmark economy 1 without public debt.)

No crowding-out effects from initial debt. In this alternative scenario, we need to make a somehow artificial assumption: timing within period zero is such that the initial debt is bought out of initial consumption, and therefore investment is not distorted. Under this assumption, the computation of the welfare gain of coordination becomes straightforward.

[5] See Appendix III for a more detailed explanation of the computational procedure.

Once $\left(\frac{b}{k}\right)$ has been set, we choose $b(0)$ in order to induce $u(0) = 1$ in equilibrium. Then, as initial debt must be bought out of initial consumption, the lifetime utility is given by,[6]

$$W(k(0); h(0)) = W(k(0); h(0); b(0)) \left[1 - (1 - \sigma)\frac{b(0)}{c(0)}\right] \tag{4.13}$$

where $W(k(0); h(0); b(0))$ is the lifetime utility for an economy with initial conditions given by $k(0); h(0)$ and $b(0)$. Using the approximation to the lifetime utility presented in Section 3, we can compute the welfare levels associated with different values of $\frac{b}{k}$.

Table 4 summarizes some of the consequences of coordinating expectations with debt under non crowding-out effects. The most relevant feature is that the welfare gain becomes negative for values of $b(0)$ larger than $0.11$.

Table 4

| | $\frac{b}{k} = 10^{-5}$ | $\frac{b}{k} = 10^{-4}$ | $\frac{b}{k} = 10^{-3}$ | $\frac{b}{k} = 10^{-2}$ | $\frac{b}{k} = .1$ |
|---|---|---|---|---|---|
| $b(0)$ | $1.0805\ 10^{-5}$ | $1.0806\ 10^{-4}$ | $1.0810\ 10^{-3}$ | .01084 | .11233 |
| $c(0)$ | .48386 | .48378 | .48292 | .47437 | .38538 |
| $k(0)$ | .10401 | .10401 | .10401 | .10403 | .10426 |
| $\gamma_{nco}$ | .02058 | .02042 | .01884 | .00310 | -.14725 |

Notes: These computations were carried out under benchmark economy 1, and $k(0) = 0.95\hat{k}^*$ and $h(0) = 1$. (Notice that $\hat{k}^*$ refers to the steady state value in our benchmark economy 1 without public debt.)

It can be concluded from Table 4 that the use of debt to select the best equilibrium may bring out non-negligible increases in welfare. Indeed, issuing debt worth $10^{-5}$ at time zero represents a welfare gain in the order of $2.05\%$ of consumption. Under our assumption that initial debt is bought from initial consumption, the optimal debt policy consists of issuing the minimal amount of debt required to induce $u(0) = 1$ in equilibrium.

As we discus above, besides its coordinating role, public debt has direct effects on the productivity in the output sector by influencing public expenditure in infrastructure.

[6] Using the first order condition given by (2.9), we can write $W(k(0); h(0); b(0)) = \int_0^\infty e^{-\rho t}\left(\frac{1}{1-\sigma}c(0)^{1-\sigma}e^{-\frac{1-\sigma}{\sigma}\int_0^t[\rho-(1-\tau)r(s)]ds} - \frac{1}{1-\sigma}\right)dt$. Differentiating this "value function" with respect to $c(0)$, and noticing that $W(k(0); h(0)) = W(k(0); h(0); b(0)) + \frac{d}{dc(0)}W(k(0); h(0); b(0))dc(0)$ and $dc(0) = -b(0)$, equation (4.10) is directly obtained.

## 5. Indeterminacy and Coordination in a Model with Human Capital Externalities

This section addresses the issue of the coordinating role of public debt in a framework where indeterminacy is due to productive externalities from the average level of human capital [see, Benhabib and Perli (1994) for a complete characterization of the conditions leading to equilibrium indeterminacy in this kind of model]. We therefore assume that the elasticity of output with respect to public infrastructure is zero, and show that the use of debt is still an effective device to coordinate private beliefs.

In order to carry out this exercise, we modify the model in Section 2 by allowing for the positive externality from the average level of human capital in the goods production function, and by assuming $\alpha_2 = 0$. Consider then the production function of a firm as given by

$$F(k; hu; h_a) = Ak(t)^{\alpha}(h(t)u(t))^{1-\alpha}h_a(t)^{\chi} \tag{5.1}$$

where $h_a$ denotes average human capital and $\chi$ is the externality parameter. We assume the same taxation and expenditure policies for the government, that is, the raising of taxes from capital and labor income at a rate given by $\tau$, and the expenditure of a fraction $\beta$ of total revenues in a now non-productive public infrastructure. Remaining revenues are given back to the household as lump sum transfers. (Thus, the only equations that change with respect to the model in Section 2 are (2.6), (2.7) and (2.8).)

A striking feature of this model is that equilibrium indeterminacy may arise even for very small values of $\chi$: As an example, consider the following parameter values[7]: $\rho = 0.065; \sigma = 0.01; A = 3.5; \alpha = 0.36; \chi = 0.075; B = 0.06; \tau = 0.3856$ and $\beta = 0.045$. The Jacobian matrix evaluated at the stationary equilibrium has one positive real eigenvalue and two complex eigenvalues with negative real parts, thus yielding a continuum of equilibria converging to a common balanced growth path.

We proceed now to analyze the welfare properties of the equilibria set in order to assess the use of debt as a coordination device. Our benchmark economy in this section is:

$$\rho = 0.065; \sigma = 0.3; A = 3.5; \alpha = 0.36; \chi = 0.39; B = 0.06; \tau = 0.3856 \text{ and } \beta = 0.045$$

[7] The rates of growth for physical and human capital along the balanced growth path associated with this economy are, respectively: $\gamma_k = 0.05268$ and $\gamma_h = 0.04716$.

which yields $\nu_k = 0.06358$ and $\nu_h = 0.03950$, and the Jacobian matrix has the following eigenvalues: $\lambda_1 = -0.00956$, $\lambda_2 = -0.13711$, and $\lambda_3 = 0.30544$. As in Section 3, we make use of numerical methods to evaluate the lifetime utility associated with different initial values of $u(0)$. Our computations show that the equilibrium path starting at $u(0) = 0.1$ renders the highest lifetime utility.

The analysis of the welfare cost of miscoordination becomes especially relevant in this model. The equilibrium path starting at $u(0) = 1$ amounts to a welfare cost in the order of 186.74% of consumption. This magnitude decreases to 11.94% for $u(0) = 0.2$. These large costs suggest that the use of debt as an equilibrium selection mechanism may render a significant increase in welfare.

### Coordination with Debt

In order to illustrate the coordinating role of debt, we permit the government to issue debt at time zero and to implement the transfers policy given by eq. (4.9). Under this policy, debt grows at the rate of $\nu_k$ from $t = 0$ on, and the ratio of debt to physical capital is not endogenously determined.

Table 5 presents the welfare gains from coordinating with different levels of debt. As in the previous section, $\gamma_{co}$ denotes the welfare gain when initial debt crowds out investment in physical capital; and $\gamma_{nco}$ denotes the welfare gain under no crowding-out effects. Both $\gamma_{co}$ and $\gamma_{nco}$ are expressed as the percentage increase of the consumption in the equilibrium with debt and with $u(0) = 0.1$, with respect to the worst equilibrium in the economy without debt.

Table 5

| | $\frac{b}{k} = 10^{-3}$ | $\frac{b}{k} = 10^{-2}$ | $\frac{b}{k} = 0.025$ | $\frac{b}{k} = 0.05$ | $\frac{b}{k} = .1$ |
|---|---|---|---|---|---|
| $b(0)$ | .04914 | .49153 | 1.22926 | 2.45997 | 3.44558 |
| $\gamma_{co}$ | .65123 | .65102 | .65067 | .65010 | .64891 |
| $\gamma_{nco}$ | .64539 | .58404 | .43091 | -.20697 | -2.83125 |

Notes: These computations were carried out under $\rho = 0.065; \sigma = 0.3; A = 3.5; \alpha = 0.36; \chi = 0.39; B = 0.06; \tau = 0.3856$ and $\beta = 0.045$; and $k(0) = 0.95\hat{k}^*$, $h(0) = 1$.

Regardless of the effects of debt on initial investment, it is clear from Table 5 that the maximum welfare gain is obtained for $\frac{b}{k} = 10^{-3}$. Therefore, we can conclude that the optimal debt policy is to issue the minimal level that induces $u(0) = 0.1$ in equilibrium.

Since a fraction ¯ of taxation revenues is devoted to non productive public infrastructure, public debt has a negative income e®ect by increasing the amount of wasted resources. Table 5 shows that the welfare bene¯ts derived from correcting expectations exceed the welfare costs from those negative income e®ects for low values of $b(0)$.

## 6. Concluding Remarks

In this paper, we presented a model in which the presence of a government providing productive services and collecting taxes from capital and labor income may generate a continuum of equilibria. As we explained above, it is the uncertainty on the level of e®ort chosen by other agents in equilibrium which makes the existence of a continuum of self-ful¯lling equilibria possible. Considering an elastic labor supply is thus crucial for our results. Although we endogenized labor supply by introducing a human capital sector, it should be clear that similar results could be obtained by assuming leisure as an argument in the utility function.

One important feature in our model is that the government can use an active budgetary policy to break down the indeterminacy result. Furthermore, the use of public debt allows the government to select one equilibrium path by targeting the interest rate. The mechanism is simple: By ¯xing the long-run level of debt (as a percentage of capital or total income), and the initial level of debt, there is a unique initial value for the labor supply such that the market equilibrium satis¯es that imposed condition. This role of debt can thus be seen as a stabilizer against extrinsic uncertainty. Both the optimal debt policy and the welfare gains from eliminating this kind of uncertainty, depend on the considered model and on the distortionary e®ects of the initial debt. Some implications for empirical analysis can be drawn from our results.

# 7. Appendixes

## Appendix I

Proof of Proposition 1: For analytical convenience we define the following new variables:

$$m(t) \equiv A(\beta\tau A)^{\frac{\alpha}{1-\alpha_2}}\hat{k}(t)^{\frac{\alpha+\alpha_2-1}{1-\alpha_2}}u(t)^{\frac{1-\alpha}{1-\alpha_2}} \text{ and } p(t) \equiv \frac{\hat{c}(t)}{\hat{k}(t)}$$

The dynamical system in m(t), p(t) and u(t) is then,

$$\begin{aligned}
\frac{\dot{m}(t)}{m(t)} &= \left(\frac{\alpha_2(1-\beta\tau)-(1-\alpha)(1-\tau)\alpha}{\alpha-\alpha_2}\right)m(t) - \left(\frac{\alpha_2}{\alpha-\alpha_2}\right)p(t) + \left(\frac{1-\alpha}{\alpha-\alpha_2}\right)B \\
\frac{\dot{p}(t)}{p(t)} &= \left(\frac{(1-\tau)\alpha}{\sigma} - (1-\beta\tau)\right)m(t) + p(t) - \frac{\rho}{\sigma} \\
\frac{\dot{u}(t)}{u(t)} &= \left(\frac{(1-\tau)\alpha\alpha_2+(1-\beta)\alpha\tau}{\alpha-\alpha_2}\right)m(t) - \left(\frac{\alpha}{\alpha-\alpha_2}\right)p(t) + Bu(t) + \left(\frac{1-\alpha}{\alpha-\alpha_2}\right)B
\end{aligned}$$

The steady state equilibrium is $m^* = \frac{\alpha_2\rho-(1-\alpha)B\sigma}{(1-\tau)\alpha(\alpha_2-(1-\alpha)\sigma)}$, $p^* = (1-\beta\tau)m^* - \frac{(1-\tau)\alpha}{\sigma}m^* + \frac{\rho}{\sigma}$ and $u^* = \frac{\alpha_2\rho-(1-\alpha)B\sigma+(1-\alpha)(B-\rho)}{(\alpha_2-(1-\alpha)\sigma)B}$.

By linearizing this system around this stationary solution we get a $3 \times 3$ matrix, say J, with the following coefficients,

$$\begin{aligned}
a_{11} &= \left(\frac{\alpha_2(1-\beta\tau)-(1-\alpha)(1-\tau)\alpha}{\alpha-\alpha_2}\right)m^* \\
a_{12} &= -\left(\frac{\alpha_2}{\alpha-\alpha_2}\right)m^* \\
a_{13} &= 0 \\
\\
a_{21} &= \left(\frac{(1-\tau)\alpha}{\sigma} - (1-\beta\tau)\right)p^* \\
a_{22} &= p^* \\
a_{23} &= 0 \\
\\
a_{31} &= \left(\frac{(1-\tau)\alpha\alpha_2+(1-\beta)\alpha\tau}{\alpha-\alpha_2}\right)u^* \\
a_{32} &= -\left(\frac{\alpha}{\alpha-\alpha_2}\right)u^* \\
a_{33} &= Bu^*
\end{aligned}$$

Since $a_{13}$ and $a_{23}$ are zero, $Bu^*$ is an eigenvalue of J. Therefore, the sign of the determinant formed by $a_{11}$, $a_{12}$, $a_{21}$ and $a_{22}$ gives the sign of the remaining eigenvalues of J. This determinant is given by

$$\frac{(1-\tau)\alpha[\alpha_2-(1-\alpha)\sigma]}{(\alpha-\alpha_2)\sigma}$$

It is readily seen that under (i) or (ii) the determinant is positive and thus both eigenvalues must have the same sign, otherwise it is negative. Then, under (i) or (ii), a necessary and sufficient condition for indeterminacy –that is, two eigenvalues be negative– is that the trace of the matrix formed by $a_{11}$, $a_{12}$, $a_{21}$ and $a_{22}$ be negative. Simple algebra shows that the trace is given by

$$\frac{\alpha+(1-\alpha-\beta)\tau}{(\alpha-\alpha_2)(1-\tau)}-\frac{1}{\sigma}\left(\frac{\alpha_2\rho-(1-\alpha)B\sigma}{\alpha_2-(1-\alpha)\sigma}\right)+\frac{\rho}{\sigma}$$

Taking into account the restrictions in parameters values to guarantee $m^*>0$, $p^*>0$ and $0<u^*<1$, it is readily shown that (i) is a necessary and sufficient condition for equilibrium indeterminacy. Likewise, (ii) is necessary and sufficient for the existence of three positive eigenvalues.

Proof of Proposition 2: It follows the procedure used in the proof of Proposition 1, after the model is conveniently modified.

## Appendix II

We offer here a brief explanation of our procedure to compute the lifetime utility. The consumption path originated at $u_0$ can be written as,

$$c_{u_0}(t)=\frac{c_{u_0}(t)}{h_{u_0}(t)^{\phi}}h_{u_0}(t)^{\phi}=\hat{c}_{u_0}(t)h_{u_0}(t)^{\phi} \tag{7.1}$$

In a neighborhood of the steady state, the value of $\hat{c}_{u_0}(t)$ is approximated by $\hat{c}^*+\zeta_1 v_{12}e^{\lambda_1 t}+\zeta_2 v_{22}e^{\lambda_2 t}$: Likewise, the human capital path can be written as $h_{u_0}(t)=h(0)e^{\int_0^t B(1-u(s))ds}$. Now, taking into account that $u(t)=u^*+\zeta_1 v_{13}e^{\lambda_1 t}+\zeta_2 v_{23}e^{\lambda_2 t}$, then (3.2) follows directly from (7.1).

In order to ascertain the welfare implications of equilibrium indeterminacy we need to evaluate the lifetime utility function along (3.2). As to get an expression which can be easily computed we proceed as follows. First, performing the change of variable $e^{\lambda_1 t}=s$, and defining $a_1=-\frac{\rho-(1-\sigma)\gamma_h\phi}{\lambda_1}$, $b_1=\frac{\zeta_1 v_{13}(1-\sigma)B\phi}{\lambda_1}$ and $b_2=\frac{\zeta_2 v_{23}(1-\sigma)B\phi}{\lambda_2}$, we can write the lifetime utility as

$$-\frac{h(0)^{(1-\sigma)\phi}e^{b_1+b_2}}{(1-\sigma)\lambda_1}\int_0^1\frac{s^{a_1-1}e^{-b_1 s-b_2 s^{\frac{\lambda_2}{\lambda_1}}}}{[\hat{c}^*+\zeta_1 v_{12}s+\zeta_2 v_{22}s^{\frac{\lambda_2}{\lambda_1}}]^{\sigma-1}}ds \tag{7.2}$$

For the computation of the definite integral in (7.2) we divide the range of integration, $[0,1]$, in $n$ intervals of equal size. For each interval we calculate the value of the function in its midpoint, then the value of the integral can be approximated by the sum of the areas of the $n$ rectangles. If we denote the function inside the integral sign by $f(s)$, then this approximation is written as

$$\frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{1+2i}{2n}\right) \tag{7.3}$$

In all our computations we take $n = 5000$.

## Appendix III

Here, we present the procedure to compute the lifetime utility when the government issue $b_0$ at time zero, and crowds out investment. We denote by $\{c_{b_0}(t)\}_{t=0}^{\infty}$ the consumption path originated in this economy. Since the law of motion for physical capital is not continuous at $t = 0$, the values for $\hat{c}(0)$ and $u(0)$ do not correspond to the ones given by the linear stable manifold. Instead, given the initial values for state variables, $\hat{k}(0), \hat{b}(0)$, the values for $\hat{c}_{b_0}(0)$ and $u(0)$ have to be optimally chosen taking into account that the implied values for $\hat{k}(\varepsilon), \hat{b}(\varepsilon), \hat{c}(\varepsilon)$ and $u(\varepsilon)$ (where $\varepsilon$ is an arbitrarily small number), must lie on the linear stable manifold. That is, the initial values for control variables must solve,

$$\begin{aligned}
\hat{k}(\varepsilon) - \hat{k}^* &= \zeta_1 v_{11} + \zeta_2 v_{21}\\
\hat{b}(\varepsilon) - \hat{b}^* &= \zeta_1 v_{12} + \zeta_2 v_{22}\\
\hat{c}(\varepsilon) - \hat{c}^* &= \zeta_1 v_{13} + \zeta_2 v_{23}\\
\hat{u}(\varepsilon) - \hat{u}^* &= \zeta_1 v_{14} + \zeta_2 v_{24}
\end{aligned}$$

where $v_{1j}, v_{2j},\ j = 1,2,3,4$ are the eigenvectors associated with the negative eigenvalues, and $\hat{k}(\varepsilon) = \hat{k}(0) + \varepsilon\dot{\hat{k}}(0)$; $\hat{b}(\varepsilon) = \hat{b}(0) + \varepsilon\dot{\hat{b}}(0)$; $\hat{c}(\varepsilon) = \hat{c}(0) + \varepsilon\dot{\hat{c}}(0)$ and $\hat{u}(\varepsilon) = \hat{u}(0) + \varepsilon\dot{\hat{u}}(0)$.

The lifetime utility associated with this path can be written as

$$U(c_{b_0}(t)) = \int_0^{\varepsilon} e^{-\rho t} c_{b_0}(t)dt + \int_{\varepsilon}^{\infty} e^{-\rho t} c_{b_0}(t)dt \tag{7.4}$$

Assuming that $c_{b_0}(t)$ is constant between 0 and $\varepsilon$, the first integral becomes,

$$\frac{c_{b_0}(0)^{1-\sigma}}{(1-\sigma)\rho}\left[1 - e^{-\rho\varepsilon}\right] \tag{7.5}$$

In order to compute the second integral in (7.4) we proceed as we did in Appendix II. Again, we perform the change of variable $e^{\lambda_1 t} = s$ to obtain,

$$-\frac{h(\varepsilon)^{(1-\sigma)\phi} e^{b_1 e^{\lambda_1\varepsilon} + b_2 e^{\lambda_2\varepsilon}}}{(1-\sigma)\lambda_1} \int_0^{e^{\lambda_1\varepsilon}} \frac{s^{a_1-1} e^{-b_1 s - b_2 s^{\frac{\lambda_2}{\lambda_1}}}}{\left[\hat{c}^* + \zeta_1 v_{13} s + \zeta_2 v_{23} s^{\frac{\lambda_2}{\lambda_1}}\right]^{\sigma-1}} ds \tag{7.6}$$

where $\zeta_1$ and $\zeta_2$ are integration constants which depend on $\hat{k}(\varepsilon)$ and $\hat{b}(\varepsilon)$. The lifetime utility is thus the sum of (7.5) and (7.6).

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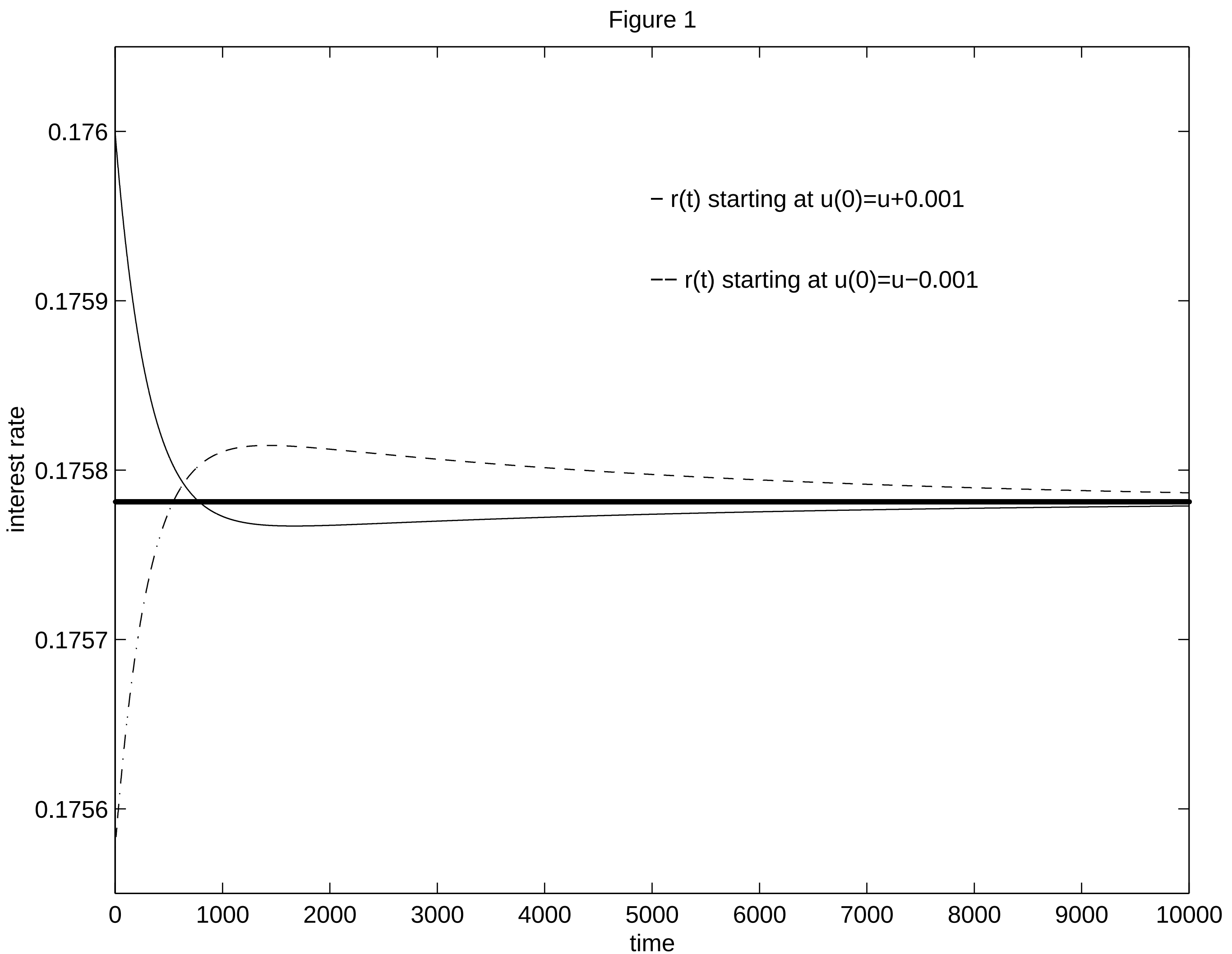
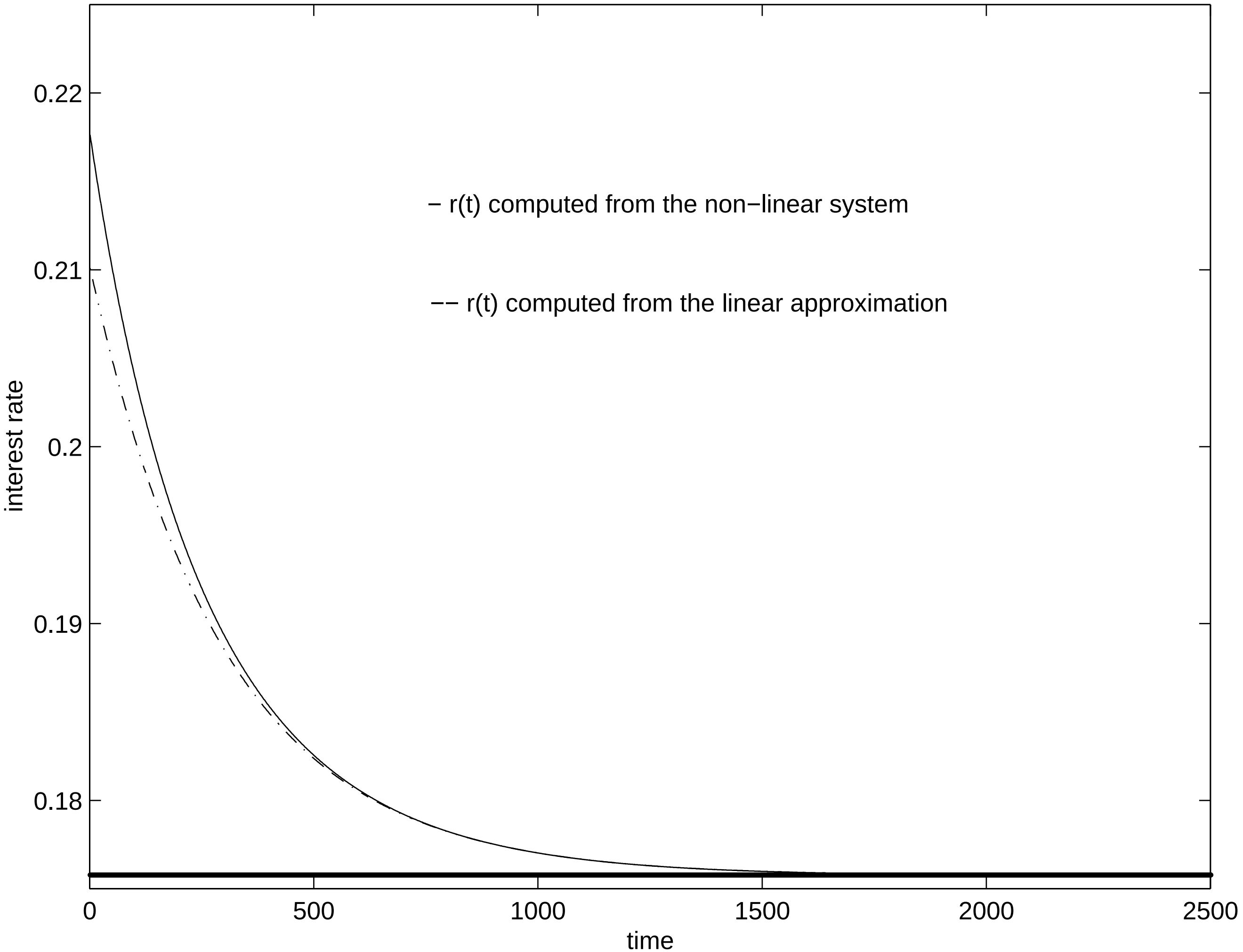


Figure 1
0.176
0.1759
0.1758
0.1757
0.1756
interest rate
− r(t) starting at u(0)=u+0.001
−− r(t) starting at u(0)=u−0.001
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000
time

Figure 2



0.22
0.21
0.2
0.19
0.18
interest rate
− r(t) computed from the non−linear system
−− r(t) computed from the linear approximation
0
500
1000
1500
2000
2500
time