Admissibility in Games*

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Abstract

Suppose that each player in a game is rational, each player thinks the other players are rational, and so on. Also, suppose that rationality is taken to incorporate an admissibility requirement—i.e., the avoidance of weakly dominated strategies. Which strategies can be played? We provide an epistemic framework in which to address this question. Specifically, we formulate conditions of "rationality and mth-order assumption of rationality" (RmAR) and "rationality and common assumption of rationality" (RCAR). We show: (i) RCAR is characterized by a solution concept called a "self-admissible set;" (ii) in a "complete" type structure, RmAR is characterized by the set of strategies that survive $m+1$ rounds of elimination of inadmissible strategies; (iii) under a non-triviality condition, RCAR is impossible in a complete structure.

1 Introduction

What is the implication of supposing that each player in a game is rational, each player thinks the other players are rational, and so on? The natural first answer to this question is that the players will choose **iteratively undominated** (**IU**) strategies–i.e., strategies that survive iterated deletion of strongly dominated strategies. Bernheim [11, 1984] and Pearce [44, 1984] gave essentially this answer, via their concept of rationalizability.[1] Pearce [44, 1984] also defined the concept of a **best-response set** (**BRS**), and gave this as a more complete answer.

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[1] Under the original definition, which makes an independence assumption, the rationalizable strategies can be a strict subset of the iteratively undominated strategies. Recent definitions (e.g., Osborne-Rubinstein [43, 1994]) allow for correlation; in this case, the two sets are equal.

In this paper we ask: What is the answer to the above question, when rationality of a player is taken to incorporate an admissibility requirement–i.e., the avoidance of weakly dominated strategies?

In particular, we want to identify conditions under which the answer is that the players will choose **iteratively admissible** (**IA**) strategies–i.e., strategies that survive iterated deletion of weakly dominated strategies. We also want to find a weak-dominance analog to Pearce's concept of a BRS–we identify one, which we call a **self-admissible set** (**SAS**).

The case of weak dominance is important. Weak-dominance concepts give sharp predictions in many games of applied interest. For example, in the Vickrey auction, the players have weakly dominant strategies to bid their values. IA is useful in analyzing other auctions (e.g., Harstad-Levin [33, 1985] and Chung-Ely [27, 2001]). Farquharson [30, 1969] suggested that IA could yield strong predictions in voting games, and this has since been shown in a number of cases (see Moulin [42, 1994]). Bertrand pricing games can be analyzed using IA (e.g., Börgers [16, 1992]).[2] IA also makes sharp predictions in extensive games. It yields the backward-induction outcome in perfect-information games,[3] and the Defect-always strategies in the Finitely Repeated Prisoner's Dilemma. In signalling games, Osborne-Rubinstein [43, 1994, pp.110-111] observe that IA gives the forward-induction outcome in the original example of Kohlberg-Mertens [36, 1986, Section 2.3]. Van Damme [57, 1989] and Ben Porath-Dekel [10, 1992] employ IA to generate the forward-induction outcome in their striking Burn-a-Dollar game. IA also gives the signalling outcome in the well-known Beer-Quiche game of Cho-Kreps [26, 1987].[4] Of course, IU does not yield such sharp results in these games.

Separate from its power in applications, admissibility is a prima facie reasonable criterion: It captures the idea that a player takes all strategies of the other players into consideration; none is entirely ruled out. It also has a long heritage in decision and game theory. (See the discussion in Kohlberg-Mertens [36, 1986, Section 2.7].)

The paper is organized as follows. The next section is an informal discussion of the issues and results to follow. The formal treatment is in Sections 3-8. Section 9 discusses some conceptual aspects, and Section 10 covers the literature. The heuristic treatment of the next section can be read either before or in parallel with the formal treatment.

2 Heuristic Treatment

We begin with the standard equivalence: Strategy s is admissible if and only if there is a strictly positive probability measure on the strategy profiles of the other players, under which s is optimal. In an influential paper, Samuelson [47, 1992] pointed out that this poses a basic challenge for an analysis of admissibility in games. Consider the game in Figure 2.1, which is essentially Example 8 in Samuelson [47, 1992].

		Bob	
		L	*R*
Ann	*U*	1, 1	0, 1
	D	0, 2	1, 0

Figure 2.1

[2] We have borrowed from the list of applications in Ewerhart [29, 2002]. We come back to this paper in Section 10.

[3] Under a "no-ties" condition on the payoffs. See Appendix A.

[4] When IA is performed on the two-player strategic form, where one player is the Sender and the other is the Receiver, and the Sender's payoffs are calculated as expected payoffs before Nature chooses the Sender's type.

Suppose rationality incorporates admissibility. Then, if Ann is rational, she should assign positive probability to both L and R. If Bob is rational, he should assign positive probability to both U and D, and so will play L. But then, if Ann thinks Bob is rational, presumably she should assign probability 1 to L. (We deliberately use the loose term "thinks." We will be more precise below.) The condition that Ann is rational appears to conflict with the condition that she thinks Bob is rational. Moreover, there could be different implications for behavior. If Ann assigns positive probability to both L and R, then perhaps, depending on the exact assessment, she will play D. If Ann assigns probability 1 to L, she will play U.

2.1 Lexicographic Probabilities

Our method for overcoming this hurdle will be to allow Ann at the same time both to include and to exclude a strategy of Bob's. (In Section 10, we return to Samuelson's line of argument.) Ann will consider some of Bob's strategies infinitely less likely than others, but still possible. The strategies that get infinitesimal weight can be viewed as both included (because they don't get zero weight) and excluded (because they get only infinitesimal weight).

		1	[1]
		L	R
1	U	1, 1	0, 1
[1]	D	0, 2	1, 0

Figure 2.2

In Figure 2.2, each player has a **lexicographic probability system** (**LPS**) on the other player's strategies. (See Blume-Brandenburger-Dekel [13, 1991].) Ann's primary measure ("hypothesis") assigns probability 1 to L. Her secondary measure (depicted in square parentheses) assigns probability 1 to R. Ann considers it infinitely more likely that Bob is rational (plays L) than that he is irrational (plays R)–but doesn't entirely exclude R from consideration. We will say that Ann **assumes** Bob is rational.

Which strategy will Ann choose? In our lexicographic decision theory, Ann will choose strategy s over strategy s' if s yields a sequence of expected payoffs lexicographically greater than the sequence s' yields. So, she'll choose U (not D).

Note we can't say that Ann believes Bob is rational, because that would require the event that Bob is irrational to be Savage-null. That is, preference conditional on this event would have to be trivial–which is precluded if Ann follows the admissibility requirement, and so rules no event out. We settle for the weaker condition that Ann assumes Bob is rational. (We give assumption a preference basis later.) This is our resolution of the tension between requiring Ann to be rational–in the sense of admissibility–and requiring her to 'think' Bob is rational.

Back to Figure 2.2: Under Bob's primary hypothesis, L and R yield Bob an equal expected payoff of 1. But under his secondary hypothesis, L yields a higher expected payoff than R. If rational, Bob will choose L.

In this game, the conditions that Bob is rational, and that Ann is rational and assumes Bob is rational, imply a unique strategy for each player.

2.2 Rationality and Common Assumption of Rationality

In general, we can formulate a sequence of conditions:

(a1) Ann is rational;
(a2) Ann is rational and assumes (b1);
(a3) Ann is rational, assumes (b1), assumes (b2);
...

(b1) Bob is rational;
(b2) Bob is rational and assumes (a1);
(b3) Bob is rational, assumes (a1), assumes (a2);
...

There is **rationality and common assumption of rationality** (**RCAR**) if this sequence holds. RCAR is a natural 'baseline' epistemic condition on a game, when rationality incorporates admissibility. We want to know what strategies can be played under RCAR.

To answer, we need some more epistemic apparatus. Let T^a, T^b be spaces of **types** for Ann and Bob respectively. Each type t^a for Ann is associated with an LPS on the product of Bob's strategy and type spaces. Likewise for Bob. A state of the world is a 4-tuple (s^a, t^a, s^b, t^b), where s^a and t^a are Ann's actual strategy and type, and likewise for Bob. This is a standard **type structure** in the epistemic literature, with the difference that types are associated with LPS's, not single probability measures.

In these structures, rationality is a property of a strategy-type pair. A pair (s^a, t^a) is **rational** it satisfies the following **admissibility** requirement: The LPS σ associated with t^a has full support (rules nothing out), and s^a lexicographically maximizes Ann's expected payoff under σ (in particular, s^a is not weakly dominated). Otherwise the pair is irrational. Likewise for Bob.

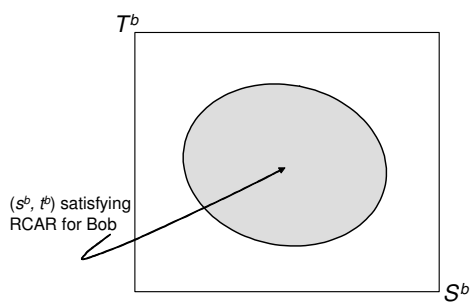

Figure 2.3

Starting with a game and an associated type structure, we get a picture like Figure 2.3. Here, S^b, T^b are the strategy and type spaces for Bob, and the outer rectangle is $S^b \times T^b$. The shaded area consists of strategy-type pairs (s^b, t^b) that satisfy RCAR for Bob. The set of rational pairs for Bob (which is not shown in the picture) is between the shaded area and the rectangle.

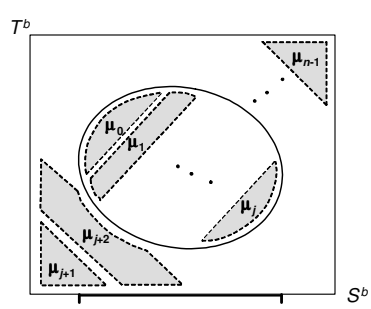

Figure 2.4

Now fix a strategy-type pair (s^a, t^a) that satisfies RCAR for Ann. Then Ann assumes (b1), assumes (b2), ... By a conjunction property of assumption, it follows that Ann assumes the joint

event (b1) and (b2) and ..., i.e., Ann assumes "RCAR for Bob." This gives a picture like Figure 2.4, where the sequence of measures $(\mu_0, \ldots, \mu_{n-1})$ is the LPS associated with t^a. There is an initial segment $(\mu_0, \ldots, \mu_j)$ of this sequence which concentrates exactly on the event "RCAR for Bob." This is because Ann considers pairs (s^b, t^b) inside this event infinitely more likely than pairs outside the event.

Consider the marginals on S^b of the measures $(\mu_0, \ldots, \mu_{n-1})$. Since (s^a, t^a) is rational, strategy s^a lexicographically maximizes Ann's expected payoff, under this sequence of marginals. This establishes (by taking a convex combination of the marginals) that there is a strictly positive measure on S^b under which s^a is optimal. That is, s^a must be admissible. Strategy s^a must also lexicographically maximize Ann's expected payoff, under the marginals of the initial segment $(\mu_0, \ldots, \mu_j)$. It follows (again taking a convex combination) that there is a strictly positive measure on the projection of the event "RCAR for Bob" under which s^a is optimal. That is, s^a must be admissible with respect to the projection.

Take the set of all states (s^a, t^a, s^b, t^b) satisfying RCAR, and let $Q^a \times Q^b$ be its projection into $S^a \times S^b$. By the discussion above, the product $Q^a \times Q^b$ has the following two properties:

(i) each $s^a \in Q^a$ is admissible (i.e., is admissible with respect to S^b);

(ii) each $s^a \in Q^a$ is admissible with respect to Q^b;

and likewise with a and b interchanged.

(Note the similarity of these properties to the definition of a best-response set (Pearce [44, 1984])– a concept based, of course, on strong dominance.) But these two properties are not yet enough to characterize RCAR, as the next example shows.

2.3 Convex Combinations

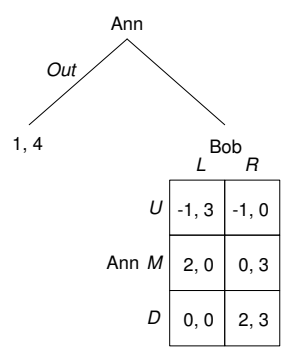

Figure 2.5

Consider the game in Figure 2.5. The set $\{Out\} \times \{L, R\}$ has properties (i) and (ii). But Out cannot be played under RCAR. Indeed, fix a type structure, and suppose (Out, t^a) is rational. Then it must be that (M, t^a) and (D, t^a) are also rational. Now consider a strategy-type pair (s^b, t^b) for Bob, which is rational and assumes Ann is rational (i.e., Bob assumes the event (a1) defined in Section 2.2). Consider the marginals on S^a of the measures in the LPS associated with t^b. Since Bob assumes Ann is rational, there is an initial segment of this sequence of marginals which concentrates exactly on the set $\{Out, M, D\}$. (Intuitively, Bob considers rational pairs infinitely more likely than irrational pairs.) From this, $s^b = R$. Now consider a strategy-type pair (s^a, t^a) for Ann, which is rational and such that Ann assumes "Bob is rational" and assumes "Bob is rational and assumes Ann is rational" (i.e., Ann assumes the events (b1) and (b2)). Consider the marginals on S^b of the

measures in the LPS associated with t^a. Using the conclusion $s^b = R$ above, we see there must be an initial segment of this sequence of marginals which concentrates on $\{R\}$. But now $s^a = D$, not $s^a = Out$.

The key to the example is that Out is a convex combination for Ann of M and D, so that (M, t^a) and (D, t^a) are rational whenever (Out, t^a) is. Also, our basic requirement is that a player–in order to assume another player is rational–must consider all of the rational strategy-type pairs for that other player infinitely more likely than the irrational pairs. That is, Bob must put (Out, t^a), (M, t^a), and (D, t^a) ahead of any pairs involving U. Then, L can't be rational (i.e., part of a rational strategy-type pair) for Bob.

This suggests that the projection of the RCAR set should have the following property:

> *(iii) if $s^a \in Q^a$, and r^a is part of a convex combination of strategies for Ann that is equivalent for her to s^a, then $r^a \in Q^a$;*
>
> *and likewise for Bob.*

We define a **self-admissible set (SAS)** to be a set $Q^a \times Q^b \subseteq S^a \times S^b$ of strategy pairs which has properties (i), (ii), and (iii). The strategies played under RCAR always constitute an SAS (Theorem 6.1(i)). But there is a remaining issue, before we get a characterization of RCAR.

2.4 Irrationality

Consider the game in Figure 2.6 (kindly provided by Pierpaolo Battigalli). The set $\{U, M, D\} \times \{C, R\}$ is an SAS. (It is also the IA set–i.e., the set of strategies that survive iterated deletion of weakly dominated strategies.)

		Bob		
		L	*C*	*R*
	U	4, 0	4, 1	0, 1
Ann	*M*	0, 0	0, 1	4, 1
	D	3, 0	2, 1	2, 1

Figure 2.6

Fix a type structure. Note that the set of Bob's rational strategy-type pairs looks as in Figure 2.7: Any strategy-pair (L, t^b) is irrational, since L is (strongly) dominated. Since C and R are dominant, strategy-type pairs (C, t^b) and (R, t^b) are rational for any type t^b–provided t^b has full support (i.e., the LPS associated with t^b has full support). Graphically, Bob's rational strategy-type pairs are represented by the solid vertical lines–and the issue is whether or not these lines extend all the way to the axis.

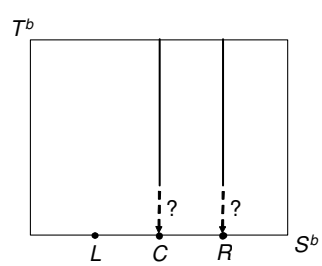

Figure 2.7

Our goal is to characterize the RCAR states; both players in these states have full-support types. Nevertheless, we shall allow types in our structures that do not have full support. So, in Figure 2.7, there may, in fact, be gaps between the solid vertical lines and the axis, reflecting the presence of non-full-support types u^b for Bob. Such pairs (C, u^b) and (R, u^b) will be irrational.

Conceptually, since Ann considers everything possible, she should, in particular, take into account the possibility that Bob doesn't consider everything possible.[5] (Cf. von Neumann-Morgenstern [58, 1944, p.32]: "The rules of rational behavior must provide definitely for the possibility of irrational conduct of the part of others.")

Now refer to Figure 2.6. Strategy D belongs to an SAS, and we want to ask whether it can be played under RCAR–i.e., whether there can be a type t^a such that (D, t^a) is consistent with RCAR. To answer, first note the following two facts: If D is optimal under a measure, then the measure either assigns probability $\frac{1}{2}$ to C and $\frac{1}{2}$ to R, or assigns positive probability to both L and R. Moreover, in the first case, U and M are also optimal.

Suppose that (D, t^a) is rational and assumes Bob is rational. Let $\left(\mu_0, \ldots, \mu_{n-1}\right)$ be the LPS associated with t^a. Since (D, t^a) is rational, the LPS associated with t^a must have full support (we are characterizing such states). So, there is some first measure μ_i that gives $\{L\} \times T^b$ positive probability. Given that type t^a assumes Bob is rational, it must consider the set of rational strategy-type pairs for Bob infinitely more likely than the irrational pairs. In particular, we have $i \neq 0$, because each pair (L, t^b) is irrational for Bob. Since (D, t^a) is rational, D must be optimal under the first measure μ_0. An easy induction then shows that for each measure μ_k that comes earlier than μ_i (i.e. $k < i$): (i) μ_k assigns probability $\frac{1}{2}$ to $\{C\} \times T^b$ and probability $\frac{1}{2}$ to $\{R\} \times T^b$; and (ii) U, M, and D are each optimal under μ_k. It follows that D is optimal under μ_i, and so μ_i must assign positive probability to both $\{L\} \times T^b$ and $\{R\} \times T^b$. Therefore, since t^a assumes Bob is rational, and each point in $\left(L, t^b\right)$ is irrational, μ_i must give positive probability to the set of irrational pairs in $\{R\} \times T^b$. But every pair in $\{R\} \times T^b$ with a full-support type is rational. We conclude that there must be non-full-support types for Bob.

In sum, we want the play of D to be allowed under RCAR–a fortiori, under the assumption that Ann is rational and assumes Bob is rational. This is possible only if we include types that do not satisfy the full support requirement. We think this is natural because a full analysis of the admissibility requirement should include the idea that other players do not conform to the requirement.

2.5 Characterization of RCAR

We now state the desired characterization of RCAR in games (Theorem 6.1(i) and (ii)). As usual, we start with a game and an associated type structure.

> *Let $Q^a \times Q^b$ be the projection into $S^a \times S^b$ of the states (s^a, t^a, s^b, t^b) satisfying RCAR. Then $Q^a \times Q^b$ is an SAS of the game.*

We also have:

> *Start with a game and an SAS $Q^a \times Q^b$. There is a type structure (with non-full-support types) such that $Q^a \times Q^b$ is the projection into $S^a \times S^b$ of the states (s^a, t^a, s^b, t^b) satisfying RCAR.*

[5] Naturally, our structures will often also contain strategy-type pairs that are irrational simply because a player isn't optimizing given his—even full-support—LPS. In Figure 2.7, pairs (L, t^b), where t^b has full support, are of this kind.

Appendix A examines the usefulness of this characterization, by showing how SAS's behave in various games of interest–Centipede, the Finitely Repeated Prisoner's Dilemma, and the Chain Store Game.

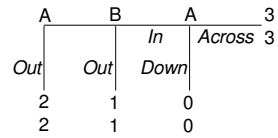

Figure 2.8

It is easy to check that the IA strategies constitute an SAS of a game. So, in particular, every game possesses an SAS (and RCAR is possible in every game). But a game may possess other SAS's too. In the game in Figure 2.8, there are three SAS's: $\{(Out, Out)\}$, $\{Out\} \times \{Out, In\}$, and $\{(Across, In)\}$. (The third is the IA set. Note that the other two SAS's aren't contained in the IA set. This is different from the case of strong dominance: It is well known that any Pearce best-response set is contained in the set of strategies that survives iterated strong dominance.)

2.6 Iterated Admissibility

What epistemic conditions select the IA set in a game, from among the family of SAS's? To investigate this, consider Figure 2.9, which gives a type structure for the game in Figure 2.8. Ann and Bob each have a single type. Ann's LPS assigns primary probability 1 to (Out, t^b), and secondary probability 1 (in square parentheses) to (In, t^b). Bob's LPS assigns primary probability 1 to (Out, t^a), secondary probability 1 (in square parentheses) to $(Down, t^a)$, and tertiary probability 1 (in double square parentheses) to $(Across, t^a)$. Ann (resp. Bob) has just one rational strategy-type pair, namely (Out, t^a) (resp. (Out, t^b)). Ann's unique type t^a assumes Bob is rational (the rational pair (Out, t^b) is considered infinitely more likely than the irrational pair (In, t^b)). Likewise, Bob's unique type t^b assumes Ann is rational (the rational pair (Out, t^a) is considered infinitely more likely than the irrational pairs $(Down, t^a)$ and $(Across, t^a)$). By induction, the RCAR set is then the singleton $\{(Out, t^a, Out, t^b)\}$. This is an instance of Theorem 6.1(i): The projection into $S^a \times S^b$ of $\{(Out, t^a, Out, t^b)\}$ is an SAS, viz. $\{(Out, Out)\}$.

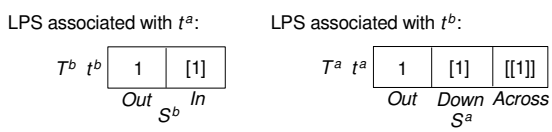

Figure 2.9

In this structure, Ann assumes Bob plays *Out*, making *Out* her unique rational choice. Both *Down* and *Across* are irrational for her. In fact, Bob considers it infinitely more likely that Ann plays *Down* than *Across*–which is why he plays *Out*. Bob is free to assign the probabilities this way. Given her type, the choice *Across* is irrational for Ann. So, to assume Ann is rational, it is enough that Bob considers *Out* infinitely more likely than both *Down* and *Across*, as he does. But if Bob considered *Across* infinitely more likely than *Down*, he'd rationally play *In* not *Out*. Presumably, Ann would then play *Across*. The IA set would result.

Figure 2.10 gives a scenario under which Bob will, in fact, consider *Across* infinitely more likely than *Down*. Add to the structure a type u^a for Ann that assumes Bob plays *In*. Now, there is a second rational pair for Ann, viz. $(Across, u^a)$. (Note there is no type v^a for Ann which we could add to the structure to make $(Down, v^a)$ rational for Ann, since *Down* is dominated.) If Bob

assumes Ann is rational, then he must consider the shaded pairs in Figure 2.10 infinitely more likely than the unshaded pairs. A rational Bob must then play In, as desired.

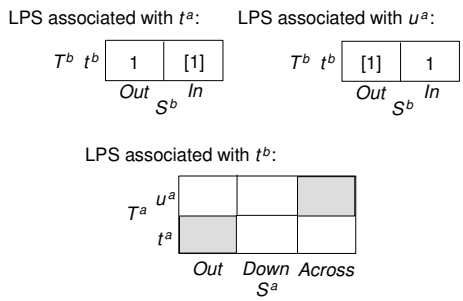

Figure 2.10

Call a type structure **complete** if the range of the map from T^a (Ann's type space) to the space of LPS's on $S^b \times T^b$ (Bob's strategy space cross Bob's type space) properly contains the set of full-support LPS's on $S^b \times T^b$, and similarly with Ann and Bob interchanged. More loosely, a type structure is complete if it contains 'all' possible full-support types, and at least some non-full-support type (as per Section 2.4 above). Complete type structures exist for every finite game (Proposition 5.2). As Figure 2.10 suggests, with this set-up, we can now identify the IA strategies.

For $m \geq 0$, say there is **rationality and mth-order assumption of rationality** (**RmAR**) if the sequence of conditions (a1), (b1), ..., (a$(m+1)$), (b$(m+1)$) of Section 2.2 holds. We have (Theorem 7.1):

> *Start with a game and an associated complete type structure. Let $Q^a \times Q^b$ be the projection into $S^a \times S^b$ of the states (s^a, t^a, s^b, t^b) satisfying RmAR. Then $Q^a \times Q^b$ is the set of strategies that survive $(m+1)$ rounds of iterated admissibility.*

2.7 A Negative Result

Note that our Theorem 7.1 actually identifies, for any m, the $(m+1)$-iteratively admissible strategies, not the IA strategies. Of course, for a given (finite) game, there is a number M such that for all $m \geq M$, the m-iteratively admissible strategies coincide with the IA strategies. Nevertheless, our result is not quite an epistemic condition for IA in all finite games. That would be one common condition–across all games–that yields IA. For example, one might hope to characterize the IA set as the projection of a set of states which is constructed in a uniform way in all complete type structures.

One would expect the RCAR set to be a natural candidate for this set of states. But the following negative result (Theorem 8.1) shows that RCAR will not work, and is the reason for our limited statement of Theorem 7.1. (The complete type structure we get from Proposition 5.2 is continuous–see Definition 5.7–and this result also uses continuity).

> *Start with a game in which Ann has more than one "strategically distinct" strategy and an associated continuous complete type structure. Then no state satisfies RCAR.*

Here is an intuition for the result. Suppose the RCAR set is nonempty. Then there must be a type t^a for Ann that assumes each of the decreasing sequence of events (b1), (b2), ... (these events were defined in Section 2.2). That is, strategy-type pairs not in (b1) must be considered infinitely less likely than pairs in (b1). Pairs not in (b2) must be considered infinitely less likely than pairs in (b2). And so on. Let $(\mu_0, \ldots, \mu_{n-1})$ be the LPS associated with t^a. Figure 2.11 shows the most

'parsimonious' way to arrange the measures μ_i, so that Ann indeed assumes each of (b1), (b2), ... But even in this case, we'll run out of measures, and Ann won't be able to assume any of the events (bn), (b$(n+1)$),.... More loosely, at some point Ann will 'hit' her primary hypothesis μ_0, at which point there is no next (more likely) order of likelihood.

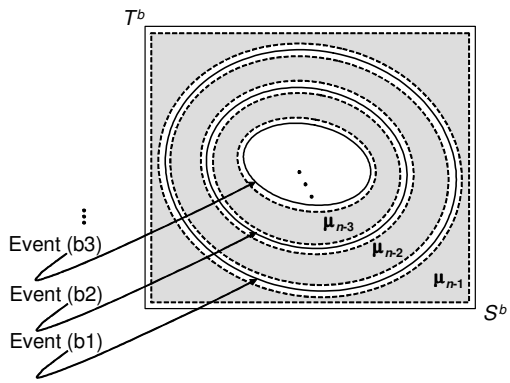

Figure 2.11

In the complete type structure we get from Proposition 5.2, each event (b$(m+1)$) is 'significantly' smaller than event (bm). This is because Bob has many types that assume the event (a$(m-1)$) but not the event (am). So the measures μ_i do indeed have to be arranged as shown. This wasn't true in the incomplete structure of Figure 2.9.

In Section 9f, we discuss further the meaning of this negative result.

3 SAS's and the IA Set

We now begin the formal treatment. Fix a two-player finite strategic-form game $\langle S^a, S^b, \pi^a, \pi^b \rangle$, where S^a, S^b are the (finite) strategy sets and π^a, π^b are payoff functions for Ann and Bob, respectively.[6] Given a finite set X, let $\mathcal{M}(X)$ denote the set of all probability measures on Ω. The definitions to come all have counterparts with a and b reversed. We extend π^a to $\mathcal{M}(S^a) \times \mathcal{M}(S^b)$ in the usual way, i.e. $\pi^a(\sigma^a, \sigma^b) = \sum_{(s^a,s^b)\in S^a\times S^b} \sigma^a(s^a)\sigma^b(s^b)\pi^a(s^a, s^b)$. Throughout, we adopt the convention that in a product $X \times Y$, if $X = \emptyset$ then $Y = \emptyset$ (and vice versa).

Definition 3.1 *Fix $X \times Y \subseteq S^a \times S^b$. A strategy $s^a \in X$ is* ***weakly dominated with respect to*** *$X \times Y$ if there exists $\sigma^a \in \mathcal{M}(S^a)$, with $\sigma^a(X) = 1$, such that $\pi^a(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$ for every $s^b \in Y$, and $\pi^a(\sigma^a, s^b) > \pi^a(s^a, s^b)$ for some $s^b \in Y$. Otherwise, say s^a is* ***admissible with respect to*** *$X \times Y$. If s^a is admissible with respect to $S^a \times S^b$, simply say that s^a is* ***admissible****.*

We have the usual equivalence:

Lemma 3.1 *A strategy $s^a \in X$ is admissible with respect to $X \times Y$ if and only if there exists $\sigma^b \in \mathcal{M}(S^b)$, with $\operatorname{Supp}\sigma^b = Y$, such that $\pi^a(s^a, \sigma^b) \geq \pi^a(r^a, \sigma^b)$ for every $r^a \in X$.*

Definition 3.2 *Say r^a* ***supports*** *s^a if there exists some $\sigma^a \in \mathcal{M}(S^a)$ with $r^a \in \operatorname{Supp}\sigma^a$ and $\pi^a(\sigma^a, s^b) = \pi^a(s^a, s^b)$ for all $s^b \in S^b$. Write $\operatorname{su}(s^a)$ for the set of $r^a \in S^a$ that support s^a.*

[6] For notational simplicity, we restrict attention throughout to two-player games. But the analysis can be extended without change to games with three or more players.

In words, the strategy r^a is contained in $\mathrm{su}\,(s^a)$ if it is part of a convex combination of Ann's strategies that is equivalent for her to s^a.

We can now define SAS's and the IA set:

Definition 3.3 *Fix $Q^a \times Q^b \subseteq S^a \times S^b$. The set $Q^a \times Q^b$ is a* ***self-admissible set (SAS)*** *if:*

(i) each $s^a \in Q^a$ is admissible,

(ii) each $s^a \in Q^a$ is admissible with respect to $S^a \times Q^b$,

(iii) for any $s^a \in Q^a$, if $r^a \in \mathrm{su}\,(s^a)$ then $r^a \in Q^a$,

and likewise for each $s^b \in Q^b$.

Definition 3.4 *Set $S^i_0 = S^i$ for $i = a, b$, and define inductively*

$$S^i_{m+1} = \{s^i \in S^i_m : s^i \text{ is admissible with respect to } S^a_m \times S^b_m\}.$$

*A strategy $s^i \in S^i_m$ is called m-****admissible****. A strategy $s^i \in \bigcap_{m=0}^{\infty} S^i_m$ is called* ***iteratively admissible (IA)****.*

Note that there is an M such that $\bigcap_{m=0}^{\infty} S^i_m = S^i_M$ for $i = a, b$. Moreover, each set S^i_m is nonempty, and hence IA is nonempty.

4 Lexicographic Probability Systems

Given a Polish space Ω, it will be helpful to fix a metric. (So "Polish" will mean complete separable metric.) Let $\mathcal{M}(\Omega)$ be the space of Borel probability measures on Ω with the Prohorov metric. Recall that $\mathcal{M}(\Omega)$ is again a Polish space, and has the topology of weak convergence (Billingsley [12, 1968, Appendix III]). Let $\mathcal{N}(\Omega)$ be the set of all finite sequences of Borel probability measures on Ω. That is, if $\sigma \in \mathcal{N}(\Omega)$, then there is some integer n with $\sigma = (\mu_0, \ldots, \mu_{n-1})$.

Define a metric on $\mathcal{N}(\Omega)$ as follows. The distance between two sequences of measures $(\mu_0, \ldots, \mu_{n-1})$ and $(\nu_0, \ldots, \nu_{n-1})$ of the same length is the maximum of the Prohorov distances between μ_i and ν_i for $i < n$. The distance between two sequences of measures of different lengths is 1. For each fixed n, this metric on the set of sequences in $\mathcal{N}(\Omega)$ of length n is easily seen to be separable and complete, and thus Polish (this is the usual finite product metric). The whole space $\mathcal{N}(\Omega)$ is thus a countable union of Polish spaces at uniform distance 1 from each other. This shows that $\mathcal{N}(\Omega)$ itself is a Polish space.

Definition 4.1 *Fix $\sigma = (\mu_0, \ldots, \mu_{n-1}) \in \mathcal{N}(\Omega)$, for some integer n. Say σ is a* ***lexicographic probability system (LPS)*** *if σ is* ***mutually singular****—that is, for each $i = 0, \ldots, n-1$, there are Borel sets U_i in Ω with $\mu_i(U_i) = 1$ and $\mu_i(U_j) = 0$ for $i \neq j$. Write $\mathcal{L}(\Omega)$ for the set of LPS's, and write $\overline{\mathcal{L}}(\Omega)$ for the closure of $\mathcal{L}(\Omega)$ in $\mathcal{N}(\Omega)$.*

An LPS is a finite measure sequence where the measures are non-overlapping (mutually singular). This has the usual interpretation: the player's primary hypothesis, secondary hypothesis, ..., and so on, until an nth hypothesis μ_{n-1}.

The space $\overline{\mathcal{L}}(\Omega)$ is Polish, since it is a closed subspace of the Polish space $\mathcal{N}(\Omega)$. The next proposition identifies $\overline{\mathcal{L}}(\Omega)$ in two natural cases.[7]

[7] The proofs not found in the main text, can be found in the appendices.

Proposition 4.1

(i) If Ω is a finite set, then $\mathcal{L}(\Omega)$ is closed, and thus $\overline{\mathcal{L}}(\Omega) = \mathcal{L}(\Omega)$.

(ii) If Ω has no isolated points, then $\overline{\mathcal{L}}(\Omega) = \mathcal{N}(\Omega)$.

In general, an LPS may have some null states which remain outside the support of each of its measures. We are also interested in the case that there are no such null states:

Definition 4.2 *A **full-support sequence** is a sequence $\sigma = (\mu_0, \ldots, \mu_{n-1}) \in \mathcal{N}(\Omega)$ such that $\Omega = \bigcup_{i<n} \operatorname{Supp} \mu_i$. We write $\mathcal{N}^+(\Omega)$ for the set of full-support sequences, and $\mathcal{L}^+(\Omega)$ for the set of **full-support LPS's**.*

The sets $\mathcal{N}^+(\Omega)$, $\mathcal{L}(\Omega)$, and $\mathcal{L}^+(\Omega)$ are each Borel (Corollary E.1).

Our definition of an LPS is an infinite version of the definition for finite spaces introduced in Blume-Brandenburger-Dekel (henceforth BBD) [13, 1991], who also gave an axiomatic derivation in terms of preferences over acts. Infinite spaces play a crucial role in this paper–complete type structures (recall the discussion in Section 2.6) are infinite. (A note on terminology: BBD [13, 1991] use the term LPS even if mutual singularity doesn't hold. We use the term LPS throughout.)

The next step is the formal treatment of our concept of "assumption" (Section 2.1). In the usual setting (subjective expected utility), Ann has one probability measure μ, and she believes an event E if $\mu(E) = 1$. With a sequence of measures $\sigma = (\mu_0, \ldots, \mu_{n-1})$, the obvious analog is to say that Ann believes E if $\mu_i(E) = 1$ for all i. (Appendix B gives a preference-based treatment. Say E is believed if $\Omega \backslash E$ is Savage-null under Ann's preference relation. This holds if and only if $\mu(E) = 1$, in the subjective expected utility setting, and if and only if $\mu_i(E) = 1$ for all i, in the lexicographic setting.)

But as we noted in Section 2.1, belief is too strong in the lexicographic setting. If Ann's LPS σ has full support, and E is the event that Bob is rational, then it may be impossible to have Ann believe E, that is, to have $\mu_i(E) = 1$ for all i. For any of Bob's strategies s^b, the event $\{s^b\} \times T^b$ is open, and so must get positive probability under some μ_i. But if s^b is inadmissible, then $\{s^b\} \times T^b$ is disjoint from the event that Bob is rational. While it may be impossible to have Ann believe E, it is possible to have Ann consider the event that Bob is irrational to be infinitely less likely than the event that he is rational. This is the motivation for our assumption concept:

Definition 4.3 *Say a set E is **assumed under** $\sigma \in \mathcal{L}(\Omega)$ (at **level** j) if E is Borel and there is a j such that:*

(i) $\mu_i(E) = 1$ for all $i \leq j$,

(ii) $\mu_i(E) = 0$ for all $j < i$,

(iii) $E \subseteq \bigcup_{i \leq j} \operatorname{Supp} \mu_i$.

The best way to understand this definition is via some examples. To begin, let $\Omega = \{a, b, c\}$ and σ be a two-level LPS with $\mu_0(a) = 1$ and $\mu_1(b) = 1$. Consider the events $E = \{a\}$ and $F = \{a, c\}$. Intuitively, we'd want to say that E is assumed (at level 0), because a is considered infinitely more likely than either b or c. But, we wouldn't want to say F is assumed, because c is considered infinitely less likely than b. Note that E satisfies (i)-(iii) (at level 0), while F satisfies (i) and (ii) but fails (iii). In particular then, condition (iii) is to ensure that the formal definition of assumption captures the intuitive idea. It guarantees that all of F is considered infinitely more likely than the complement of F.

If Ω is finite, condition (iii) implies condition (ii), so we could require conditions (i) and (iii) alone. But this is not true in the infinite case. To see this, let $\Omega = [0,1] \cup \{2\}$ and σ be a two-level LPS where μ_0 is uniform on $[0,1]$ and $\mu_1(\{1\}) = \mu_1(\{2\}) = \frac{1}{2}$. Consider the event $E = [0,1]$. Condition (iii) is satisfied (for $j = 0$), but condition (ii) is not. Moreover, intuitively, we wouldn't want to say E is assumed, because, under μ_1, the complement of E is considered as likely as E. This is why in the general–i.e., infinite–case, we require all three conditions (i)-(iii).

Like belief, assumption can also be defined axiomatically. (See Appendix B for the formal treatment.) Say E is assumed in the axiomatic sense if whenever Ann strictly prefers one act to another, conditional on E, she has the same preference unconditionally. More loosely, if Ann assumes E, she is prepared to make a decision based only on the states in E.

Here is the relationship between belief and assumption: If E is believed, then it is assumed (Proposition B.1). If Ann has one probability measure μ, then if she assumes E, she believes E (Proposition B.2). But, with an LPS, assumption is weaker than belief. In Appendix B, we show that Ann assumes E in the axiomatic sense if and only if conditions (i)-(ii) of Definition 4.3 hold. In Figure 2.2, for example, Ann assumes $\{L\}$ (in the axiomatic sense) but doesn't believe it.

(There are other examples of concepts that coincide under ordinary probabilities but are distinguished under lexicographic probabilities. For instance, BBD [13, 1991] give three definitions of stochastic independence that are equivalent under ordinary probabilities, but successively weaker under lexicographic probabilities. See also Hammond [32, 1994], Kohlberg-Reny [37, 1997], and Streufert [56, 2003].)

Condition (iii) can also be axiomatized (again see Appendix B) so that, in the lexicographic case, assumption plus an additional axiom is equivalent to Definition 4.3. (If Ann has one probability measure, the same axiom implies $\operatorname{Supp} \mu = E$ when E is closed.)

Next are some properties of assumption:

Property 4.1 (Convexity) *If E and F are assumed under σ at level j, then any Borel set G lying between $E \cap F$ and $E \cup F$ is also assumed under σ at level j.*

Property 4.2 (Closure) *If E and F are assumed under σ at level j then $\operatorname{cl}(E) = \operatorname{cl}(F)$ (where $\operatorname{cl}(\cdot)$ denotes closure).*

The Convexity property refers to convexity in the sense of orderings (where the order is set inclusion), and is a two-sided monotonicity. The Closure property implies that if E and F are each assumed then either $\operatorname{cl}(E) \subseteq \operatorname{cl}(F)$ or $\operatorname{cl}(F) \subseteq \operatorname{cl}(E)$. Consequently, for a finite space, there is only one set that is assumed at each level, and $E \subseteq F$ or $F \subseteq E$. But this is not true for an infinite space.

Overall, the mental picture we suggest for assumption is of rungs of a ladder, separated by gaps, where each rung is a convex family of sets with the same closure. (Each rung corresponds to the events assumed at the particular level.)

Next, notice that assumption is not monotonic. Here is an example: Set $\Omega = [0,1] \cup \{2,3\}$, and let $\sigma = (\mu_0, \mu_1)$ be a full-support LPS where μ_0 is uniform on $[0,1]$ and $\mu_1(\{2\}) = \mu_1(\{3\}) = \frac{1}{2}$. Then σ assumes $(0,1]$ but not $(0,1] \cup \{2\}$.

The best way to understand this nonmonotonicity is in terms of our axiomatic treatment.[8] Suppose Ann assumes $(0,1]$–i.e., she is willing to make a decision based solely on $(0,1]$ (when she has a strict preference). Now suppose Ann considers the possibility that 2 obtains. It doesn't seem natural to require that she should now be willing to make a decision based only on $(0,1] \cup \{2\}$. After all, once she considers the possibility that 2 obtains, presumably she should also consider the

[8] We thank a referee for this line of argument.

possibility that 3 obtains. (To give 2 positive probability, she must look to her secondary hypothesis, which also gives 3 positive probability.) Of course, the state 3 may well matter for her preferences.

On the other hand, if Ann assumes $(0,1]$ then certainly she should assume $[0,1]$. Admitting the possibility of 0 doesn't force her to look to her secondary hypothesis–it doesn't force her to consider 2 or 3 possible. Formally, Ann assumes $[0,1)$ and $(0,1]$ at the same level. Convexity then requires her to assume $[0,1]$ (at the same level).

While assumption is not monotonic in the lexicographic case, it is monotonic under ordinary probabilities. Now, Ann assumes $(0,1]$ if and only if she assigns probability 1 to this event. Monotonicity is immediately satisfied. Conceptually, Ann never considers the possibility that state 2 or state 3 obtains. But we use LPS's precisely to be able both to include and to exclude a given state (Section 2.1). This is why assumption is not monotonic in the lexicographic case.

Finally, note that because it is not monotonic (in the lexicographic case), assumption fails one direction of conjunction. Returning to the example, Ann assumes $(0,1] \cap ((0,1] \cup \{2\})$ even though she does not assume $(0,1] \cup \{2\}$. But the other direction of conjunction, and the analog for disjunction, are satisfied:

Property 4.3 (Conjunction and Disjunction) *Fix Borel sets $E_1, E_2, ...$ in Ω, and suppose, for each m, that E_m is assumed under σ. Then $\bigcap_m E_m$ and $\bigcup_m E$ are assumed under σ.*

In Section 10, we relate assumption to some other epistemic concepts in the literature, including "belief at level 0" (Brandenburger [19, 1992]) and strong belief (Battigalli-Siniscalchi [8, 2002]).

5 Type Structures

Fix again a two-player finite strategic-form game $\langle S^a, S^b, \pi^a, \pi^b \rangle$.

Definition 5.1 *An $\left(S^a, S^b\right)$**-based type structure** is a structure*

$$\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle,$$

*where T^a and T^b are nonempty Polish spaces, and $\lambda^a : T^a \to \overline{\mathcal{L}}\left(S^b \times T^b\right)$ and $\lambda^b : T^b \to \overline{\mathcal{L}}\left(S^a \times T^a\right)$ are Borel measurable. Members of T^a, T^b are called **types**. Members of $S^a \times T^a \times S^b \times T^b$ are called **states (of the world)**. A type structure is called **lexicographic** if $\lambda^a : T^a \to \mathcal{L}\left(S^b \times T^b\right)$ and $\lambda^b : T^b \to \mathcal{L}\left(S^a \times T^a\right)$.*

This is based on a standard epistemic definition: A type structure enriches the basic description of a game by appending spaces of epistemic types for both players, where a type of one player is associated with an LPS on the strategies and types of the other player. The difference from the standard definition is the use of LPS's rather than (ordinary) probability measures.

Our primary focus will be on lexicographic type structures, which have a natural interpretation in a game setting. Non-lexicographic type structures do not have such an interpretation, but, as we will see, they are useful as a step in the construction of lexicographic type structures. Note that lexicographic type structures can contain two different kinds of types–those associated with full-support LPS's and those associated with non-full-support LPS's. The reason for this was discussed in Section 2.4.

The following definitions apply to a given finite game and type structure. As before, they also have counterparts with a and b reversed.

Definition 5.2 *A strategy s^a is* ***optimal*** *under $\sigma = (\mu_0, \ldots, \mu_{n-1})$ if $\sigma \in \mathcal{L}(S^b \times T^b)$ and*

$$\left(\pi^a(s^a, \operatorname{marg}_{S^b} \mu_i(s^b))\right)_{i=0}^{n-1} \geq^L \left(\pi^a(r^a, \operatorname{marg}_{S^b} \mu_i(s^b))\right)_{i=0}^{n-1}$$

for all $r^a \in S^a$.[9]

Here, $\operatorname{marg}_{S^b} \mu_i$ denotes the marginal on S^b of the measure μ_i. In words, Ann will prefer strategy s^a to strategy r^a if the associated sequence of expected payoffs under s^a is lexicographically greater than the sequence under r^a. (If σ is a length-one LPS (μ_0), we will sometimes say that s^a is optimal under the measure μ_0 if it is optimal under (μ_0).)

Definition 5.3 *A strategy-type pair $(s^a, t^a) \in S^a \times T^a$ is* ***rational*** *if $\lambda^a(t^a)$ is a full-support LPS and s^a is optimal under $\lambda^a(t^a) \in \mathcal{L}(S^b \times T^b)$.*

In words, a strategy-type pair (s^a, t^a) will be rational if $\lambda^a(t^a)$ is a full-support LPS and s^a is optimal under $\lambda^a(t^a)$. This is the usual definition of rationality, plus the condition that Ann rules out no states. The full-support requirement is to capture our basic admissibility requirement. The following two lemmas say this formally:

Lemma 5.1 (BBD [14, 1991]) *Suppose s^a is optimal under a full-support LPS $(\mu_0, \ldots, \mu_{n-1}) \in \mathcal{L}^+(S^b \times T^b)$. Then there is a length-one full-support LPS $(\nu_0) \in \mathcal{L}^+\left(S^b \times T^b\right)$, under which s^a is optimal.*

Together with Lemma 3.1, this gives:

Lemma 5.2 *If (s^a, t^a) is rational, then s^a is admissible.*

Next, for $E \subseteq S^b \times T^b$, set

$$A^a(E) = \{t^a \in T^a : E \text{ is assumed under } \lambda^a(t^a)\}.$$

In words, $A^a(E)$ is the set of types $t^a \in T^a$ with associated LPS's $\lambda^a(t^a)$ that assume the event E (in $S^b \times T^b$). Of course, this set depends on the specific type structure.

For finite m, define R^a_m inductively by

$$R^a_{m+1} = R^a_m \cap [S^a \times A^a(R^b_m)].$$

The sets R^a_m are Borel (Lemma E.6).

Definition 5.4 *If $(s^a, t^a, s^b, t^b) \in R^a_{m+1} \times R^b_{m+1}$, say there is* ***rationality and mth-order assumption of rationality (RmAR)*** *at this state. If $(s^a, t^a, s^b, t^b) \in \bigcap_{m=1}^{\infty} R^a_m \times \bigcap_{m=1}^{\infty} R^b_m$, say there is* ***rationality and common assumption of rationality (RCAR)*** *at this state.*

In words, there is RCAR at a state if Ann is rational, Ann assumes the event "Bob is rational," Ann assumes the event "Bob is rational and assumes Ann is rational," and so on, and similarly starting with Bob.

Note, we cannot replace this definition with $\widehat{R}^a_1 = R^a_1$ and $\widehat{R}^a_{m+1} = \widehat{R}^a_1 \cap [S^a \times A^a(\widehat{R}^b_m)]$. To clarify, suppose $(s^a, t^a) \in R^a_3$. Then $(s^a, t^a) \in R^a_1 \cap [S^a \times A^a(R^b_1)] \cap [S^a \times A^a(R^b_1 \cap [S^b \times A^b(R^a_1)])]$. In words, Ann is rational, she assumes the event "Bob is rational," and she assumes the event "Bob is rational and assumes Ann is rational." Now suppose $(s^a, t^a) \in \widehat{R}^a_3$. Then $(s^a, t^a) \in$

[9] If $x = (x_0, \ldots, x_{n-1})$ and $y = (y_0, \ldots, y_{n-1})$, then $x \geq^L y$ if and only if $y_j > x_j$ implies $x_k > y_k$ for some $k < j$.

$R_1^a \cap [S^a \times A^a(R_1^b \cap [S^b \times A^b(R_1^a)])]$. In words, Ann is rational, and she assumes the event "Bob is rational and assumes Ann is rational." But, because assumption is not monotonic, she might not assume the event "Bob is rational." We think that under a good definition of R2AR, Ann should assume this event.

We now introduce a notion of equivalence between type structures.

Definition 5.5 *Two type structures $\langle S^a, S^b, T^a, T^b, \kappa^a, \kappa^b \rangle$ and $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ are* ***equivalent*** *if:*

(i) they have the same strategy and type spaces;

(ii) for each $t^a \in T^a$, if either $\kappa^a(t^a)$ or $\lambda^a(t^a)$ belongs to $\mathcal{L}^+(S^b \times T^b)$ then $\kappa^a(t^a) = \lambda^a(t^a)$ (and likewise with a and b reversed).

Proposition 5.1

(i) For every type structure there is an equivalent lexicographic type structure.

(ii) If two type structures are equivalent, then for each m they have the same R_m^a and R_m^b sets.

This proposition shows that any statement about rationality and assumptions of rationality that is true for every lexicographic type structure is true for every type structure. Conceptually, we are interested in type structures which satisfy the hypothesis of being lexicographic, but the proposition tells us that we will never need this hypothesis in our theorems. In practice, we will state and prove theorems for arbitrary type structures. By Proposition 5.1, in these proofs we can always assume without loss of generality that the type structure is lexicographic.

We conclude this section with complete type structures.

Definition 5.6 *A type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ is* ***complete*** *if $\mathcal{L}^+\left(S^b \times T^b\right) \subsetneq \operatorname{range} \lambda^a$ and $\mathcal{L}^+\left(S^a \times T^a\right) \subsetneq \operatorname{range} \lambda^b$.*

In words, a complete structure contains all full-support LPS's for Ann and Bob, and (at least) one non-full-support LPS. (Refer back to Sections 2.4 and 2.6.) We see at once from the definition that any type structure which is equivalent to a complete type structure is complete.

Proposition 5.2 *For any finite sets S^a, S^b, there is a complete type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ such that the maps λ^a and λ^b are continuous.*

Definition 5.7 *A type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ is* ***continuous*** *if it is equivalent to a type structure where the λ^a and λ^b maps are continuous.*

Thus, in a continuous type structure, players associate neighboring full-support LPS's with neighboring full-support types. Propositions 5.1 and 5.2 immediately give:

Corollary 5.1 *For any finite sets S^a, S^b, there exists a complete continuous lexicographic (S^a, S^b)-based type structure.*

6 Characterization of RCAR

Theorem 6.1

(i) Fix a type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b\rangle$. Then $\mathrm{proj}_{S^a} \bigcap_{m=1}^{\infty} R_m^a \times \mathrm{proj}_{S^b} \bigcap_{m=1}^{\infty} R_m^b$ is an SAS. That is, the projection of the set of RCAR states into $S^a \times S^b$ is an SAS.

(ii) Fix an SAS $Q^a \times Q^b$. There is a lexicographic type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b\rangle$ with $Q^a \times Q^b = \mathrm{proj}_{S^a} \bigcap_{m=1}^{\infty} R_m^a \times \mathrm{proj}_{S^b} \bigcap_{m=1}^{\infty} R_m^b$. That is, $Q^a \times Q^b$ is the projection of the set of RCAR states into $S^a \times S^b$.

Proof. For part (i), if $\bigcap_m R_m^a \times \bigcap_m R_m^b = \emptyset$, then the conditions of an SAS are automatically satisfied. So we'll suppose this set is nonempty.

Fix $s^a \in \mathrm{proj}_{S^a} \bigcap_m R_m^a$. Then $(s^a, t^a) \in \bigcap_m R_m^a$ for some $t^a \in T^a$. Certainly $(s^a, t^a) \in R_1^a$. Using Lemma 5.2, s^a is admissible, establishing condition (i) of an SAS. By Property 4.3, $t^a \in A^a\left(\bigcap_m R_m^b\right)$. We therefore get a picture like Figure 6.1 (for some $j < n$), and, as illustrated,

$$\bigcup_{i \le j} \mathrm{Supp}\,\mathrm{marg}_{S^b}\, \mu_i = \mathrm{proj}_{S^b} \bigcap_m R_m^b.$$

(This is formally established as Lemma D.1 in the appendix to this section, and uses condition (iii) of the definition of assumption.) As in Lemma 5.1, there is a length-one LPS (ν_0) on S^b, with $\mathrm{Supp}\,\nu_0 = \mathrm{proj}_{S^b} \bigcap_m R_m^b$, under which s^a is optimal. Thus s^a is admissible with respect to $S^a \times \mathrm{proj}_{S^b} \bigcap_m R_m^b$, establishing condition (ii) of an SAS. Next suppose $r^a \in \mathrm{su}\,(s^a)$. Then, for any t^a, $(s^a, t^a) \in R_1^a$ implies $(r^a, t^a) \in R_1^a$ (Lemma D.2), and so we have for all m, $(s^a, t^a) \in R_m^a$ implies $(r^a, t^a) \in R_m^a$. This establishes condition (iii) of an SAS.

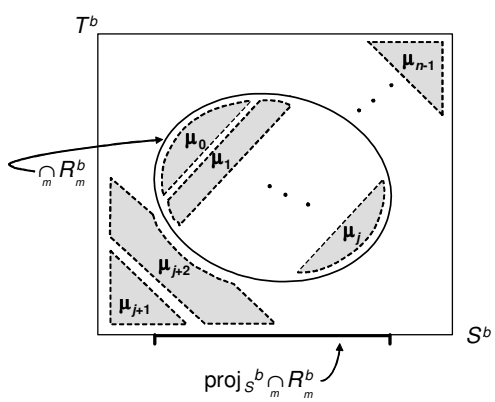

Figure 6.1

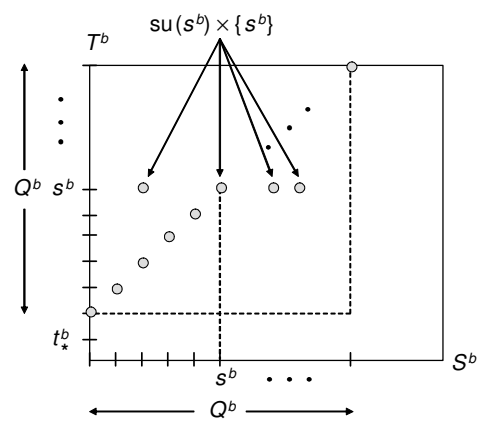

Figure 6.2

For part (ii), fix an SAS $Q^a \times Q^b$. (Recall the convention that if $Q^a = \emptyset$ then $Q^b = \emptyset$, and vice versa.) By conditions (i) and (ii) of an SAS, for each $s^a \in Q^a$ there are measures $\nu_0, \nu_1 \in \mathcal{M}\left(S^b\right)$, with $\mathrm{Supp}\,\nu_0 = S^b$ and $\mathrm{Supp}\,\nu_1 = Q^b$, under which s^a is optimal. We can choose ν_0 so that r^a is optimal under ν_0 if and only if $r^a \in \mathrm{su}\,(s^a)$. (This is Lemma D.4 in the appendix to this section.)

Define type spaces $T^a = Q^a \cup \{t_*^a\}$ and $T^b = Q^b \cup \{t_*^b\}$, where t_*^a and t_*^b are arbitrary labels. For $t^a = s^a \in Q^a$, the associated $\lambda^a(t^a) \in \mathcal{L}^+(S^b \times T^b)$ will be a two-level full-support LPS (μ_0, μ_1) where $\mathrm{marg}_{S^b}\, \mu_0 = \nu_1$ and $\mathrm{marg}_{S^b}\, \mu_1 = \nu_0$.[10] (Further conditions are specified below.) Let $\lambda^a(t_*^a)$ be an element of $\mathcal{L}(S^b \times T^b) \backslash \mathcal{L}^+(S^b \times T^b)$. Define the map λ^b similarly.

[10] We reverse the indices for consistency with the proof of Theorem 7.1 below.

Figure 6.2 shows the construction of $\lambda^a(t^a)$: Under the above specifications, points (s^b, s^b) on the diagonal are rational, i.e., lie in R_1^b. Other points (r^b, s^b) are rational if and only if $r^b \in \mathrm{su}(s^b)$. Since $\mathrm{su}(s^b) \subseteq Q^b$, by condition (iii) of an SAS, the set R_1^b contains the diagonal and is contained in the rectangle $Q^b \times Q^b$. Moreover, for each $s^b \in S^b$, $\left(s^b, t_*^b\right) \in (S^b \times T^b) \backslash R_1^b$. Thus we can take the measures μ_0 and μ_1 to satisfy:

$$\begin{array}{ll} \mathrm{marg}_{S^b}\, \mu_0 = \nu_1, & \mathrm{Supp}\, \mu_0 = R_1^b, \\ \mathrm{marg}_{S^b}\, \mu_1 = \nu_0, & \mathrm{Supp}\, \mu_1 = (S^b \times T^b) \backslash R_1^b. \end{array}$$

Likewise for the map λ^b.

We now show that $\mathrm{proj}_{S^a} \bigcap_m R_m^a = Q^a$, and likewise for b. By the same argument as in the previous paragraph, $\mathrm{proj}_{S^a} R_1^a = Q^a$. Moreover, the construction yields $A^a(R_1^b) = Q^a$, so that $R_2^a = R_1^a$. Likewise for b. Thus $R_m^a = R_1^a$ and $R_m^b = R_1^b$ for all m, by induction. Certainly $\mathrm{proj}_{S^a} R_1^a \times \mathrm{proj}_{S^b} R_1^b = Q^a \times Q^a$. It follows that $\mathrm{proj}_{S^a} \bigcap_m R_m^a \times \mathrm{proj}_{S^b} \bigcap_m R_m^b = Q^a \times Q^b$, as required. ■

7 Characterization of RmAR in a Complete Structure

Theorem 7.1 *Fix a complete type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. Then, for each m,*

$$\mathrm{proj}_{S^a} R_m^a \times \mathrm{proj}_{S^b} R_m^b = S_m^a \times S_m^b.$$

That is, the projection of the set of RmAR states into $S^a \times S^b$ is the set of strategy pairs that survive $(m+1)$ rounds of iterated admissibility.

Proof. We may assume that the type structure is lexicographic. The proof is by induction on m. Begin by fixing some $(s^a, t^a) \in R_1^a$. By Lemma 5.2, $s^a \in S_1^a$. This shows that $\mathrm{proj}_{S^a} R_1^a \times \mathrm{proj}_{S^b} R_1^b \subseteq S_1^a \times S_1^b$.

Next fix some $s^a \in S_1^a$. By Lemma 3.1, there is an LPS $(\nu_0) \in \mathcal{L}^+\left(S^b\right)$ under which s^a is optimal. We want to construct an LPS $(\mu_0) \in \mathcal{L}^+\left(S^b \times T^b\right)$ with $\mathrm{marg}_{S^b}\, \mu_0 = \nu_0$. By completeness, there will then be a type t^a with $\lambda^a(t^a) = (\mu_0)$. By construction, the pair $(s^a, t^a) \in R_1^a$. This will establish that $\mathrm{proj}_{S^a} R_1^a \times \mathrm{proj}_{S^b} R_1^b = S_1^a \times S_1^b$.

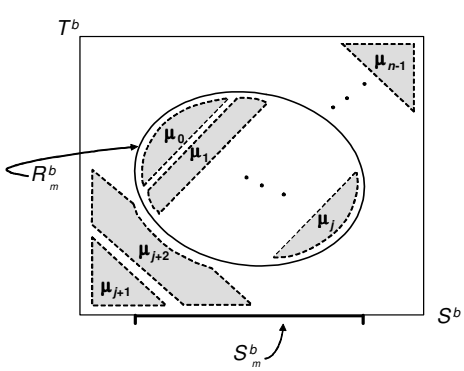

Figure 7.1

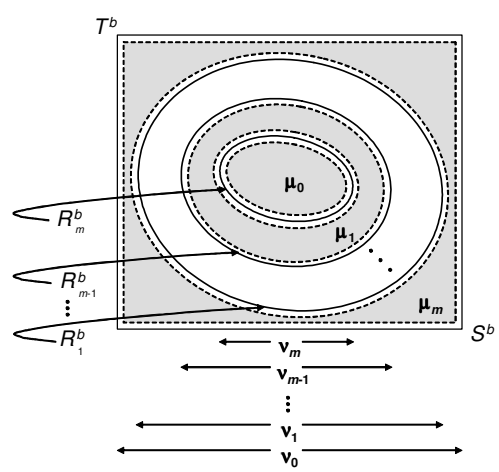

Figure 7.2

To construct (μ_0), fix some $s^b \in S^b$ and set $X = \{s^b\} \times T^b$. Note that $\nu_0(s^b) > 0$. By rescaling and combining measures over different s^b, it is enough to find $(\xi_0) \in \mathcal{L}^+(X)$. By separability, X has a countable dense subset Y. So, by assigning positive weight to each point in Y we get a measure ξ_0 where $\xi_0(Y) = 1$ and $\operatorname{Supp} \xi_0$ is the closure of Y, as required.

Now assume the result for all $1 \leq i \leq m$. We will show it is also true for $i = m+1$. Fix some $(s^a, t^a) \in R^a_{m+1}$, where $\lambda^a(t^a) = (\mu_0, \ldots, \mu_{n-1})$. Then $(s^a, t^a) \in R^a_m$ and so, by the induction hypothesis, $s^a \in S^a_m$. Also, $t^a \in A^a(R^b_m)$. Since $\operatorname{proj}_{S^b} R^b_m = S^b_m$, by the induction hypothesis, we get a picture like Figure 7.1 (for some $j < n$). By the same argument as in the proof of Theorem 6.1, we conclude that s^a is admissible with respect to $S^a \times S^b_m$ (so certainly with respect to $S^a_m \times S^b_m$). Thus $s^a \in S^a_{m+1}$.

Next fix some $s^a \in S^a_{m+1}$. It will be useful to set $S^b_0 = S^b$ and $R^b_0 = S^b \times T^b$. For each $0 \leq i \leq m$ there is a measure $\nu_i \in \mathcal{M}(S^b)$, with $\operatorname{Supp} \nu_i = S^b_i$, under which s^a is optimal among all strategies in S^a. (This is Lemma E.1 in the appendix to this section, which uses Lemma 3.1.) Thus s^a is (lexicographically) optimal under the sequence of measures $(\nu_0, \ldots, \nu_m)$. Also, using the induction hypothesis, $S^b_i = \operatorname{proj}_{S^a} R^b_i$ for all $0 \leq i \leq m$. We want to construct an LPS $(\mu_0, \ldots, \mu_m) \in \mathcal{L}^+\left(S^b \times T^b\right)$ where:

(i) $\operatorname{marg}_{S^b} \mu_i = \nu_{m-i}$,

(ii) R^b_i is assumed at level $m - i$.

It will then follow from completeness that there is a t^a with $\lambda^a(t^a) = (\mu_0, \ldots, \mu_m)$, and hence $(s^a, t^a) \in R^a_{m+1}$. (Refer to Figure 7.2.)

Now fix some $s^b \in S^b$ and set $X = \{s^b\} \times T^b$ as above. Let h be the greatest $i \leq m$ such that $s^b \in S^b_i$. Note that for each $i \leq h$ we have $s^b \in S^b_i = \operatorname{Supp} \nu_i$, and so $\nu_i(s^b) > 0$. By rescaling and combining the measures over different s^b, it is enough to find $(\xi_0, \ldots, \xi_h) \in \mathcal{L}^+(X)$ with:

(iii) $\xi_0(X \cap R^b_h) = 1$,

(iv) $\xi_i(X \cap (R^b_{h-i} \backslash R^b_{h-i+1})) = 1$ for each $1 \leq i \leq h$,

(v) $X \cap R^b_{h-i} \subseteq \bigcup_{j=0}^{i} \operatorname{Supp} \xi_j$ for each $0 \leq i \leq h$.

Each R^b_{h-i} is Borel (Lemma E.6). We also have $\operatorname{proj}_{S^b} R^b_{h-i} = \operatorname{proj}_{S^b}(R^b_{h-i} \backslash R^b_{h-i+1})$. (This is Lemma E.8. It is the place where we need the fact that a complete lexicographic type structure has a non-full-support LPS.) Since $s^b \in \operatorname{proj}_{S^b} R^b_{h-i}$, for each $1 \leq i \leq h$ the set $X_i = X \cap (R^b_{h-i} \backslash R^b_{h-i+1})$ is nonempty. The set $X_0 = X \cap R^b_h$ is also nonempty. The proof is finished by the same argument as in the base step above: By separability, each X_i has a countable dense subset Y_i. Assign positive probability to each point in Y_i to get a measure ξ_i where $\xi_i(Y_i) = 1$ and $\operatorname{Supp} \xi_i$ is the closure of Y_i. Then $(\xi_0, \ldots, \xi_h) \in \mathcal{L}^+(X)$ and satisfies (iii)-(v), completing the induction. ■

8 A Negative Result

Definition 8.1 *Say that player a is* ***indifferent*** *if $\pi^a(r^a, s^b) = \pi^a(s^a, s^b)$ for all r^a, s^a, s^b.*

In words, the condition is that a player has more than one "strategically distinct" strategy.

Theorem 8.1 *Fix a complete continuous type structure $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. If player a is not indifferent, then there is no state at which there is RCAR. In fact,*

$$\bigcap_{m=1}^{\infty} R^a_m = \bigcap_{m=1}^{\infty} R^b_m = \emptyset.$$

The result assumes continuity (see Section 9b below for a discussion of this assumption). We don't know if the result holds if λ^a, λ^b are required only to be measurable. Alternatively put, does there exist a complete type structure in which RCAR is nonempty? We leave this as a—we think very interesting—open question.

9 Discussion

Here we discuss some further conceptual and technical aspects of our analysis.

a. Definition of an LPS The original definition of LPS's in BBD [13, 1991] considers both the mutually singular and non-mutually singular cases. Our Definition 4.1 imposes mutual singularity. But, in fact, both types of LPS's are used in this paper, too. Fix a strategy-type pair (s^a, t^a) for Ann. Type t^a is associated with an LPS $\lambda^a(t^a) = (\mu_0, \ldots, \mu_{n-1})$ on $S^b \times T^b$, where the measures μ_i are mutually singular. But the optimality of s^a under $\lambda^a(t^a)$ (Definition 5.2) depends on the marginals on S^b of the measures μ_i. These marginals may not be mutually singular, of course. (The game of Figure 2.6 shows that mutual singularity of the marginals may be impossible.)

We define an LPS to be a finite sequence of probability measures, not an infinite sequence. The main reason is that finite sequences are general enough for this paper. But it would certainly be worth exploring extensions of our definition–e.g., with applications to infinite games in mind. (See Section 9g below.) For example, in Halpern [31, 2003], LPS's are sequences of measures indexed by an initial segment of the ordinals. An axiomatic treatment of LPS's on infinite spaces would also be valuable.

It is interesting to ask whether Theorem 8.1 would go through with infinite sequences of measures. Certainly, the intuition given in Section 2.7 appears to depend only on the condition that an LPS has a primary hypothesis, secondary hypothesis, etc. Given this, we'll eventually 'hit' the primary hypothesis, when trying to 'count on' smaller and smaller events. In other words, it seems that the well-foundedness of an LPS is really what is responsible for the impossibility.

The idea that a player has an initial hypothesis about a game seems very basic. That said, we don't know if Theorem 8.1 would be overturned if we used non-well-founded LPS's. This is another important open question.

b. Definition of a Complete Structure In the literature, the more common concept of a "model of all possible types" is the universal (or canonical) model. (See Armbruster-Böge [1, 1979], Böge-Eisele [15, 1979], Mertens-Zamir [40, 1985], Brandenburger-Dekel [21, 1993], Heifetz [34, 1993], and Battigalli-Siniscalchi [7, 1999], among others.) Why do we define and use completeness instead (Definition 5.6)?[11] Completeness is very easy to define–it is just "two-way surjectivity." Further, it is exactly the completeness property that is used in Theorems 7.1 and 8.1, so it seems right to isolate this property.[12]

Also, the constructions of universal models in the papers cited above, as well as the construction of complete type structures in this paper, give continuous maps from types to measures. For this reason, we don't view our continuity assumption in Theorem 8.1 as ad hoc–though, as we said, we are interested in knowing what happens without it.

c. Complete vs. Incomplete Structures The use of complete structures (in Theorems 7.1 and 8.1) is not uncontroversial. To see why, we need the concept of the "context" of a game. Formally,

[11] The concept is from Brandenburger [20, 2003].

[12] The universal models cited are also complete models. That said, we aren't aware of a general treatment of the relationship between universal and complete models (absent specific structure). On this point, see also Brandenburger-Keisler [25, 1999, Section 9].

this is just an associated type structure (Definition 5.1). Informally, the idea is that the types present in the structure reflect the players' possible characteristics, what they could have observed prior to the start of the game (their possible histories), etc. Viewed this way, a complete structure is the special "context-free" case, where all possible types are present. But, arguably, like the Savage [48, 1954, pp.82-91] "small-worlds" view of decision theory, there is always a history before the start of the game as described. This will likely narrow down, at least to some extent, the players' possible types. In short, if the game has a particular context, we won't be in a complete structure.

We don't want to overstate this point, but we do think this argument highlights the interest in analyzing RCAR in incomplete structures. Moreover, it is perhaps not obvious, a priori, that one would be able to characterize RCAR regardless of the context of the game. But Theorem 6.1 on SAS's says that one can.[13]

d. Order Dependence We defined IA (Definition 3.4) to be simultaneous maximal deletion (SMD). It is well known that different orders of deletion can give different answers. Theorem 7.1 delivers SMD. Of course, we don't rule out that some other epistemic treatment could yield a different order of deletion.

	L	*C*	*R*
U	0, 1	1, 2	1, 0
M	1, 1	0, 0	0, 2
D	-1, -2	-1, -1	-1, -1

Figure 9.1

Interestingly, if a different order of deletion from SMD is used, the result may not even be an SAS. In Figure 9.1, SMD gives $\{(U, C)\}$, an SAS. But if we delete D first, no further deletions are possible, and the remaining set $\{U, M\} \times \{L, C, R\}$ is not an SAS, since L is inadmissible.[14] It is also true that an SAS may not correspond to any order of deletion. In Figure 4.1, there were three SAS's, but there was a unique order of deletion.

e. Irrationality The presence of a non-full-support type (i.e., a type with associated LPS that doesn't have full support) is needed in the proofs of each of our three main theorems. To be precise, one such type is used in the converse direction of Theorem 6.1 (the characterization of RCAR), in Theorem 7.1 (the characterization of RmAR and completeness), and in Theorem 8.1 (our negative result). In each case, the key fact is that, for every strategy s^a, there is some type t^a such that (s^a, t^a) is irrational.

Let us repeat the conceptual meaning of this. The theorems concern states where players have full-support types. But since these types consider all possibilities, they don't rule out the event that another player is of a non-full-support type. The idea, as discussed in Section 2.4, is that a player who considers everything possible should, in particular, take into account the possibility that another player doesn't consider everything possible.

In Theorem 8.1 we also used the hypotheses that the maps λ^a, λ^b are continuous, and Ann is not indifferent. The key here is that under these hypotheses, $S^a \times T^a$ contains a nonempty open set of irrational pairs. This is used to get the first step of an induction (Lemma F.1). At each later step

[13] We are grateful to Dov Samet, who suggested this framing at a seminar presentation of this work.

[14] The same example indicates that it would be hard to give an epistemic treatment of certain orders of deletion. Provided we have the requirement that if a strategy-type pair is rational then the strategy is admissible, we can't get $\{U, M\} \times \{L, C, R\}$.

of the induction, continuity is again needed to guarantee that the pre-image of an open set is still open.

f. Interpretation of the Negative Result Conceptually, Theorem 8.1 appears to indicate a limit on players' ability to reason about all possibilities in a game. The underlying decision theory (incorporating admissibility) asks a player to take all states into consideration. RCAR asks players to assume "rationality and mth-order assumption of rationality" for all m. Completeness asks players to consider all possible types that are implied by the model. Continuity asks players to associate neighboring full-support LPS's with neighboring full-support types. It seems that not all of this is possible. (Our open question is whether all but the continuity requirement is possible.)

g. Infinite Games Finally, Theorem 8.1 may be suggestive of limitations to the analysis of infinite games.[15] For a fixed infinite game, it may be that one needs the full force of RCAR in a complete structure to characterize IA. Will this be possible? Of course, to answer this question, we have to rebuild all the ingredients of this paper for infinite games.[16] This seems an important direction for future work.

10 Related Papers

Our goal in this paper was to provide a unified analysis of admissibility in games. We've shown that RCAR is characterized by an SAS, an analog to a Pearce best-response set. Moreover, the IA set is one SAS, and, in the case of a complete type structure, the $(m+1)$-admissible strategies are the strategies consistent with RmAR.

There are a number of existing analyses of admissibility, to which we now relate our work. (We discuss other connections, including to the work by Asheim-Dufwenberg [3, 2003] and Basu-Weibull [5, 1991], in an online appendix.[17])

A pioneering paper is Samuelson [47, 1992], which–as we mentioned in Section 2–pointed out the fundamental inclusion-exclusion challenge in the area. To see how Samuelson tackles this, go back to the game of Figure 2.1. Suppose an analysis yields the answer that Ann plays (only) U. Then, Samuelson requires Ann to include all of Bob's strategies that are optimal with respect to U–so Ann must include both L and R. Turning to Bob, by the same principle he should then include D for Ann, since D could be optimal with respect to $\{L, R\}$. Contradiction. What if the answer includes Ann's playing D? But then Bob must play L (admissibility). From this, Ann will play U, so D is excluded, not included. Another contradiction.

More generally, Samuelson formalizes a condition of "common knowledge of admissibility" (CKA) and shows that a "consistent pair" (Börgers-Samuelson [18, 1992])–another weak dominance analog to a Pearce best-response set–is always consistent with CKA.[18] Consistent pairs may or may not exist. In particular, Figure 2.1 is a game where no consistent pair exists. By contrast, SAS's always exist (the IA set is an SAS). In Figure 2.1, $\{(U, L)\}$ is the (unique) SAS. The reason for the difference is that while we also require Ann to include R, she can consider R infinitely less likely than L, in which case only U (and not D) is optimal. A consistent pair (when it exists) may contain inadmissible strategies, and so may not be an SAS. If it contains only admissible strategies, it is an SAS.

[15] We are grateful to the editor for this observation.

[16] In particular, we have to revisit the relationship between admissibility, optimality under a full-support measure, and rationality under a LPS (Lemmas 3.1 and 5.1).

[17] "Admissibility in Games: Online Appendix" available on our webpages.

[18] He gives a mixed analog to consistent pairs as defined in [18, 1992]. We are stating his result for the case of pure strategies.

Samuelson's work led to various papers investigating ways for Ann both to include all of Bob's strategies and, at the same time, to exclude his irrational strategies. Börgers [17, 1994] and Brandenburger [19, 1992] took a step with "$(1-\varepsilon)$-belief" (Monderer-Samet [41, 1989], Stinchcombe [55, 1988]) and "belief at level 0," respectively. The second analysis is lexicographic–let's restate it in terms of the current paper. Types are associated with LPS's and rationality is as in Definition 5.3. If $\lambda^a(t^a) = (\mu_0, \ldots, \mu_{n-1})$ and $\mu_0(R_1^b) = 1$, then we say Ann "believes at level 0" that Bob is rational. This is one way of allowing Ann both to include and to exclude strategy-type pairs for Bob. Note that, unlike assumption, belief at level 0 is monotonic. But also notice that it needn't be true that all of Bob's rational strategy-type pairs are infinitely more likely than his irrational pairs. Of course, this is exactly what is ruled out under assumption. (Conditions (ii)-(iii) of assumption may be violated for $j = 0$.) In fact, for a full-support LPS, assumption lies between belief at level 0 and belief.

The condition of rationality and common belief at level 0 of rationality can be characterized as follows. Call a subset $Q^a \times Q^b$ of $S^a \times S^b$ a weak best-response set (WBRS) if: (i) each $s^a \in Q^a$ is admissible; (ii) each $s^a \in Q^a$ is not strongly dominated with respect to Q^b; and likewise with a and b interchanged. Recall that the $S^\infty W$ set (Dekel-Fudenberg [28, 1990]) consists of the strategies that survive one round of deletion of inadmissible strategies followed by iterated deletion of strongly dominated strategies. It is easy to check that $S^\infty W$ is a WBRS, and every WBRS is a subset of $S^\infty W$. We have the following (partial) analog to our Theorem 6.1: Let $Q^a \times Q^b$ be the projection into $S^a \times S^b$ of the states (s^a, t^a, s^b, t^b) satisfying rationality and common belief at level 0 of rationality. Then $Q^a \times Q^b$ is a WBRS. Conversely, given a WBRS $Q^a \times Q^b$ there is a type structure such that $Q^a \times Q^b$ is contained in the projection into $S^a \times S^b$ of the states (s^a, t^a, s^b, t^b) satisfying rationality and common belief at level 0 of rationality.[19]

There have been several attempts to refine $S^\infty W$. Asheim [2, 2001] provides epistemic conditions for proper rationalizability (Schuhmacher, [49, 1999]), a non-equilibrium analog to proper equilibrium and a refinement of $S^\infty W$. (But the proper rationalizable profiles may not be contained in the IA set.) Asheim considers the case where Ann has a full-support LPS and believes the relevant events. He further asks that Ann's LPS "respect preferences"–i.e., if Bob strictly (lexicographically) prefers (s^b, t^b) to (r^b, t^b), then Ann should consider (s^b, t^b) infinitely more likely than (r^b, t^b). Go back to the game of Figure 2.8 and the type structure of Figure 2.9. Bob's LPS does not respect preferences. Given her type t^a, Ann strictly prefers *Across* to *Down*, but Bob considers $(Across, t^a)$ infinitely less likely than $(Down, t^a)$. As we saw, Bob does assume Ann is rational. Both $(Across, t^a)$ and $(Down, t^a)$ are irrational, and considered infinitely less likely than the rational pair (Out, t^a). Our route to 'upsetting' the (Out, Out) answer was different–we did so by rationalizing *Across* (but not *Down*) by adding more types (à la completeness). Of course, the understanding that there are these two different routes goes back to Kohlberg-Mertens [36, 1986, p.1009].

Also related is Stahl [53, 1995], who gets the IA set. Stahl uses LPS's and supposes that Ann considers Bob's strategy s^b as infinitely less likely than his strategy r^b if s^b is eliminated on an earlier round of IA than r^b. We want this condition as an output, not an input, of our analysis. For us, the crucial ingredient, to get IA, is completeness. We saw that without this we get SAS, not the IA set.

Other epistemic conditions for IA were given by Ewerhart [29, 2002]. His conditions use provability (in the sense of mathematical logic). In his model, Ann assigns probability 0 to a strategy of Bob's if and only if it is not provable that it is possible that Bob plays that strategy. In effect, Ann eliminates a strategy of Bob's unless it is provable that it should not be eliminated. The philosophy in Ewerhart seems almost opposite to ours. Ewerhart's players are "aggressive" (his terminology)

[19] There may also be an analog–presumably involving a complete structure–to our Theorem 7.1, but we aren't aware of a treatment in the literature.

in eliminating strategies of the other player. Our players are, in a sense, cautious in eliminating strategies of the other player–since no strategy is ever entirely ruled out. Ewerhart shows that if the players follow his rule, and if they work in a self-referential system such as Peano Arithmetic, then they will choose IA strategies. Completeness does not appear to play a role in Ewerhart's analysis.

Next, we turn to analysis on the tree. The extensive-form rationalizable (EFR) strategies are an extensive-form analog to the iteratively undominated (IU) strategies, defined by Pearce [44, 1984] and later simplified by Battigalli [6, 1997].[20] In a generic tree, IA and EFR coincide (Shimoji [51, 2004] and Brandenburger-Friedenberg [22, 2003]). Of course, many games of interest are non-generic.[21] In a simultaneous-move game, EFR reduces to IU. IA and EFR will then differ whenever IA and IU do. (In general, SAS's and EFR are quite different.)

Nonetheless, there are some very interesting connections between IA and EFR at the epistemic level. Battigalli-Siniscalchi (henceforth B-S) [8, 2002] gave epistemic conditions for EFR. As will be clear, our paper owes a big debt to their work.

Let's begin by summarizing the B-S characterization of EFR. B-S use conditional probability systems (CPS's). (The concept goes back to Rényi [45, 1955].) A CPS is defined relative to a family $\mathcal{E}$ of conditioning events. (In B-S, the family $\mathcal{E}$ is associated with information sets of the tree.) A CPS then specifies a probability measure for each conditioning event, together with certain conditions tying the measures together. The next ingredient is "strong belief": Ann strongly believes an event E if she assigns it probability 1, given any conditioning event F with $E \cap F \neq \emptyset$. B-S show that in a CPS-based complete type structure, the strategies consistent with rationality (defined on the tree) and common strong belief of rationality are the EFR strategies.

Now the relationship between our analysis and B-S. First, the relationship between LPS's and CPS's. In Brandenburger-Friedenberg-Keisler [24, 2006], we show: Fix a finite subalgebra $\mathcal{E}$ of conditioning events where each $F \in \mathcal{E}$ is clopen. (The conditioning events in B-S are always clopen.) There is a natural surjective mapping from the set of full-support LPS's onto the CPS's defined relative to $\mathcal{E}$. Moreover, an event E is strongly believed under a given CPS if and only if it is assumed under some full-support LPS that maps to that CPS.

Next, the role of completeness. B-S assume completeness of their (CPS-based) type structure. We assume (LPS-based) completeness in two of our results–in Theorem 7.1, where we characterize RmAR under completeness, and in Theorem 8.1, where we show RCAR is impossible under completeness. Our third result (Theorem 6.1) characterizes RCAR alone, without completeness.

In sum, there is a close correspondence between the ingredients of our analysis and the B-S analysis, just as there is between the solution concepts of IA and EFR. A difference is that we also treat the case of incomplete type structures–which leads to our SAS solution concept. Refer back to Section 9c above, and also to the open question below.

Ben Porath [9, 1997] is an important predecessor on the tree. Like us, he works in an incomplete type structure. Instead of strong belief, he uses what we'll call "initial belief": Ann initially believes an event E if she assigns it probability 1, given the root of the tree. (Strong belief implies initial belief, but not vice versa.) Ben Porath considers perfect-information trees satisfying a no-ties condition. He shows that if a state satisfies rationality (defined on the tree) and common initial belief of rationality, the strategies played lie in the $S^{\infty}W$ set.

Asheim-Søvik [4, 2005] offer more results on the relationship between assumption, strong belief, and other related concepts in the literature. One such concept is Stalnaker's [54, 1998] "absolutely robust belief." Assumption (defined with probability measures) and absolutely robust belief (defined

[20] Shimoji-Watson [52, 1998] show that EFR is equivalent to iterated deletion of strategies that are "conditionally dominated" in the tree.

[21] Examples include auction games, voting games, Bertrand, and zero-sum games. See Mertens [39, 1989] and Marx-Swinkels [38, 1997] for the same observation on non-genericity, and lists of examples.

with possibility sets) are analogs in the case of finite spaces. (Unlike B-S and this paper, Stalnaker doesn't consider infinite spaces.)

We end with a question. B-S characterize rationality and common strong belief of rationality in a complete CPS-based type structure. But can we characterize rationality and common strong belief of rationality in incomplete structures? In other words, is there an extensive-form analog to our result (Theorem 6.1) that SAS characterizes RCAR on the matrix? This seems an important question to answer about the tree. In Sections 8 and 9, we noted various open questions on extending our analysis on the matrix.

Appendix A Self-Admissible Sets

Here we look at how the SAS concept works in various games of interest in the literature.

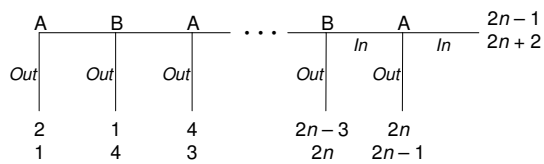

Figure A.1

Example A.1 (Centipede) *Consider n-legged Centipede (Rosenthal [46, 1981]), as in Figure A.1. We show that if $Q^a \times Q^b$ is an SAS for Centipede, and $(s^a, s^b) \in Q^a \times Q^b$, then s^a is Ann's strategy of playing Out at the first node.*[22]

Proof. Suppose, to the contrary, that there is an $(s^a, s^b) \in Q^a \times Q^b$ where s^a involves Ann's playing across at the first node. In particular, pick a profile (s^a, s^b) which yields the longest path of play (before Ann or Bob plays *Out*). Let h be the node on this path at which *Out* is played. Suppose Bob moves at h. (A similar argument applies if Ann moves at h.) Then, by condition (ii) of the definition of an SAS, and Proposition 3.1, Ann's strategy s^a must be optimal under a measure that assigns: (i) probability 1 to Bob's playing *Out* at node h or earlier; and (ii) positive probability to Bob's playing across until node h and *Out* at h. Now consider the strategy r^a for Ann that plays across until node h' (where h' is the immediate predecessor of h) and plays *Out* at h'. Then r^a does strictly better than s^a under any such measure—a contradiction. ■

This analysis of Centipede seems very intuitive. It starts at the beginning of the tree and works forwards–reaching a contradiction because if Bob is the player to end the game (playing *Out* at node h), then Ann should have ended the game earlier.

Interestingly, with SAS, we get the BI outcome, not necessarily the BI strategies. For instance, in the three-legged game, $\{Out\} \times \{Out, In\}$ is an SAS. IA makes a sharper prediction strategy-wise: The IA set is a singleton, consisting of the (reduced) backward-induction (BI) strategies.

For our next game, the Finitely Repeated Prisoner's Dilemma, we need the following projection property of SAS's: Fix a game tree Γ, with strategic form G, and a proper subtree Δ of Γ. Any SAS of G which allows Δ induces an SAS on the strategic form of Δ.[23]

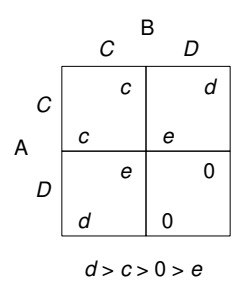

Figure A.2

[22] Of course, we are not prescribing play. The same is true in the examples below. Also, note that we consider the reduced strategic form of the game.

[23] Kohlberg-Mertens [36, 1986] introduced (essentially) this property of a solution concept. See Brandenburger-Friedenberg [23, 2004] for a proof of the projection property for SAS's.

Example A.2 (Finitely Repeated Prisoner's Dilemma) *Consider the Prisoner's Dilemma (Figure A.2) and the T-fold repetition, for some integer T. Fix an SAS of this game. We show that any strategy profile in the SAS yields the Defect-Defect path throughout.*

Proof. The proof is by induction on the number of rounds. For $T = 1$, the result is immediate from the fact that any strategy in an SAS is admissible. Now assume the result for T, and fix an SAS $Q^a \times Q^b$ of the $(T+1)$-fold game. Suppose $s^a \in Q^a$ involves Ann's playing C on the first round. Then for any $s^b \in Q^b$, Ann gets a first-round payoff of c if s^b involves Bob's playing C on the first round, and e if s^b involves Bob's playing D on the first round. These are also Ann's total payoffs from the game when (s^a, s^b) is played, since the projection property above and the induction hypothesis together imply that the profile (s^a, s^b) yields the Defect-Defect path on rounds $2, \ldots, T+1$. Suppose instead that Ann chooses the "Defect always" strategy. Then she gets a first-round payoff of d if s^b involves Bob's playing C on the first round, and 0 if s^b involves Bob's playing D on the first round. On subsequent rounds Ann gets at least 0. But then the "Defect always" strategy does strictly better than s^a against every $s^b \in Q^b$, contradicting the definition of an SAS. ■

As in Centipede, IA gives a sharper prediction strategy-wise in the Finitely Repeated Prisoner's Dilemma than SAS does: Each player has a unique IA strategy, namely "Defect always."

In the next example, SAS and IA give different answers outcome-wise.

Example A.3 (Twice Repeated Chain Store Game) *Figure A.3 gives the twice-repeated Chain Store game (Selten [50, 1978]). On each round, the payoffs are:* $(0, 3)$ *(to entrant and incumbent resp.) if the entrant chooses Out;* $(1, 1)$ *if the entrant chooses In and the incumbent chooses Cede; and* $(-2, -2)$ *if the entrant chooses In and the incumbent chooses Fight. There are two SAS's. In one, the entrant chooses In unconditionally and the incumbent chooses Cede unconditionally. This is also the IA set and BI strategy profile. In the other SAS, the entrant chooses Out at the first stage and In at the second stage. The incumbent chooses Fight at the first stage and Cede at the second stage.*

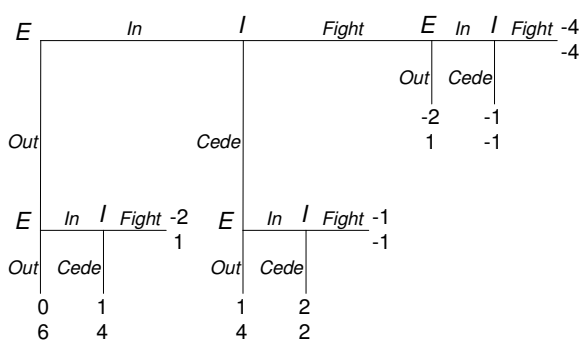

Figure A.3

Of course, the fact that SAS can deliver a non-BI outcome in this game is not a flaw of the concept. It is by now well accepted that non-BI outcomes are of interest in many games–including the repeated Chain Store game.

Still, the example prompts the question: What does SAS yield in perfect-information (PI) games? Here is the answer. Fix a PI game satisfying a "no-ties" condition: Any strategy profile that lies in

an SAS is outcome equivalent to a pure Nash equilibrium (and admissible). Conversely, any pure admissible Nash equilibrium lies in some SAS. (See Brandenburger-Friedenberg [23, 2004] for a precise statement and proof.) This is different from IA. In a PI game satisfying a no-ties condition, any strategy profile that lies in the IA set is outcome equivalent to the BI strategies. (See Battigalli [6, 1997] and Marx-Swinkels [38, 1997] for various versions of this relationship.)

Our result on SAS's in PI games resembles Theorem 2 in Ben Porath [9, 1997], which gives conditions for a Nash outcome in a generic PI game. We discuss the relationship further in our [23, 2004].

Appendix B Assumption

We begin with an axiomatic justification of assumption (Definition 4.3).

Let Ω be a Polish space and let $\mathcal{A}$ be the set of all measurable functions from Ω to $[0,1]$. A particular function $x \in \mathcal{A}$ is an act, where $x(\omega)$ is the payoff to the player of choosing the act x, if the true state is $\omega \in \Omega$. For $x, y \in \mathcal{A}$ and $0 \leq \alpha \leq 1$, write $\alpha x + (1-\alpha)y$ for the act that in state ω gives payoff $\alpha x(\omega) + (1-\alpha)y(\omega)$. For $c \in [0,1]$, write $\overrightarrow{c}$ for the constant act associated with c, i.e. $\overrightarrow{c}(\omega) = c$ for all $\omega \in \Omega$. Also, given acts $x, z \in \mathcal{A}$, and a Borel subset E in Ω, write $\left(x_E, z_{\Omega\setminus E}\right)$ for the act:

$$\left(x_E, z_{\Omega\setminus E}\right)(\omega) = \begin{cases} x(\omega) & \text{if } \omega \in E, \\ z(\omega) & \text{if } \omega \notin E. \end{cases}$$

Let $\succsim$ be a preference relation on $\mathcal{A}$, and write $\succ$ (resp. $\sim$) for strict preference (resp. indifference). We maintain three axioms throughout:

A1 (Order) $\succsim$ is a complete, transitive, reflexive binary relation on $\mathcal{A}$.

A2 (Independence) For all $x, y, z \in \mathcal{A}$ and $0 < \alpha \leq 1$,

$$x \succ y \text{ implies } \alpha x + (1-\alpha)z \succ \alpha y + (1-\alpha)z, \text{ and } x \sim y \text{ implies } \alpha x + (1-\alpha)z \sim \alpha y + (1-\alpha)z.$$

A3 (Nontriviality) There are $x, y \in \mathcal{A}$ with $x \succ y$.

Given a Borel set E, define conditional preference given E in the usual way:

Definition B.1 *$x \succsim_E y$ if for some $z \in \mathcal{A}$, $(x_E, z_{\Omega\setminus E}) \succsim (y_E, z_{\Omega\setminus E})$.*

(As is well known, under A1 and A2, B.1 for some z implies B.1 for all z.)

Definition B.2 *Say a set E is **believed under** $\succsim$ if E is Borel and, for all $x, y \in \mathcal{A}$, $x \sim_{\Omega\setminus E} y$.*

This is just the statement that the event $\Omega\setminus E$ is Savage-null.

Definition B.3 *Say a set E is **assumed** under $\succsim$ if E is Borel and:*

(i) there are $x, y \in \mathcal{A}$ with $x \succ_E y$,

(ii) for all $x, y \in \mathcal{A}$, $x \succ_E y$ implies $x \succ y$.

Condition (ii) says that in the comparison between any two acts x, y, the payoffs in E are determining for strict preference. More loosely, the player is prepared to make a decision based only on states in E. Condition (i) is a nontriviality requirement, stating that E is not Savage-null. (Without (i), the empty set, for example, would be assumed.)

Proposition B.1 *If E is believed under $\succsim$, it is assumed under $\succsim$.*

Proof. Fix a set E that is believed under $\succsim$ and note that E is Borel. We first establish condition (i). By A3, there are $x, y \in \mathcal{A}$ with $x \succ y$. Since E is believed under $\succsim$, we have $(x_E, x_{\Omega\backslash E}) \sim (x_E, z_{\Omega\backslash E})$. Similarly, $(y_E, y_{\Omega\backslash E}) \sim (y_E, z_{\Omega\backslash E})$. Using A1, $(x_E, z_{\Omega\backslash E}) \succ (y_E, z_{\Omega\backslash E})$, as required.

For condition (ii), consider any $x, y \in \mathcal{A}$ with $x \succ_E y$. Then certainly $(x_E, x_{\Omega\backslash E}) \succ (y_E, x_{\Omega\backslash E})$. Since E is believed under $\succsim$, we have $(y_E, x_{\Omega\backslash E}) \sim (y_E, y_{\Omega\backslash E})$. Using A1, $(x_E, x_{\Omega\backslash E}) \succ (y_E, y_{\Omega\backslash E})$, as required. ■

We now define two preference relations on $\mathcal{A}$. It is clear that both of them satisfy Axioms A1–A3. Given a probability measure $\mu \in \mathcal{M}(\Omega)$, define $\succsim^{\mu}$ by:

$$x \succsim^{\mu} y \iff \int_{\Omega} x(\omega)\, \mathrm{d}\mu(\omega) \geq \int_{\Omega} y(\omega)\, \mathrm{d}\mu(\omega).$$

Similarly, given an LPS $\sigma = (\mu_0, \ldots, \mu_{n-1}) \in \mathcal{L}(\Omega)$, define $\succsim^{\sigma}$ on $\mathcal{A}$ by:

$$x \succsim^{\sigma} y \iff \left(\int_{\Omega} x(\omega)\mathrm{d}\mu_i(\omega)\right)_{i=0}^{n-1} \geq^{L} \left(\int_{\Omega} y(\omega)\mathrm{d}\mu_i(\omega)\right)_{i=0}^{n-1}.$$

For ordinary probabilities we have:

Proposition B.2 *Fix $\mu \in \mathcal{M}(\Omega)$ and a Borel set E. The following are equivalent:*

(i) $\mu(E) = 1$,

(ii) *E is believed under $\succsim^{\mu}$,*

(iii) *E is assumed under $\succsim^{\mu}$.*

Proof. In light of Proposition B.1, it suffices to show that (i) implies (ii), and (iii) implies (i).

Suppose (i) holds. Then $\mu(\Omega\backslash E) = 0$, and so for any $x, y \in \mathcal{A}$, $x \sim^{\mu}_{\Omega\backslash E} y$. Thus (ii) holds.

Now suppose (iii) holds. By Definition B.3(i), there are $x, y \in \mathcal{A}$ with $x \succ^{\mu}_{E} y$. Thus $\mu(E) > 0$. Set $C = \operatorname{Supp}\mu$, and define $x, y \in \mathcal{A}$ as follows:

$$x(\omega) = \begin{cases} \mu(C\backslash E) & \text{if } \omega \in C \cap E, \\ 0 & \text{otherwise,} \end{cases}$$

$$y(\omega) = \begin{cases} 1 & \text{if } \omega \in C\backslash E, \\ 0 & \text{otherwise.} \end{cases}$$

Acts x and $(x_E, \overrightarrow{0}_{\Omega\backslash E})$ are evaluated as $\mu(C\backslash E)\,\mu(C \cap E)$. Act y is evaluated as $\mu(C\backslash E)$, while act $(y_E, \overrightarrow{0}_{\Omega\backslash E})$ is evaluated as 0. Note that if $0 < \mu(E) < 1$, then $\mu(C\backslash E) > 0$ and $0 < \mu(C \cap E) < 1$. Thus $x \succ^{\mu}_{E} y$ and $y \succ^{\mu} x$, contradicting Definition B.3(ii). ■

For lexicographic probabilities we have:

Proposition B.3 *Fix $\sigma = (\mu_0, \ldots, \mu_{n-1}) \in \mathcal{L}(\Omega)$ and a Borel set E in Ω. The following are equivalent:*

(i) $\mu_i(E) = 1$ *for all i,*

(ii) *E is believed under $\succsim^{\sigma}$.*

Proof. Suppose (i) holds. Then $\mu_i(\Omega\backslash E) = 0$ for all i, and so for any $x, y \in \mathcal{A}$, $x \sim^{\sigma}_{\Omega\backslash E} y$. Thus (ii) holds. Now suppose (ii) holds. Then $\overrightarrow{1} \sim^{\sigma}_{\Omega\backslash E} \overrightarrow{0}$. That is

$$\left(\mu_i(\Omega\backslash E) + \int_E z(\omega)\mathrm{d}\mu_i(\omega)\right)_{i=0}^{n-1} = \left(0 + \int_E z(\omega)\mathrm{d}\mu_i(\omega)\right)_{i=0}^{n-1},$$

or $\mu_i(\Omega\backslash E) = 0$ for all i, as required. ■

Proposition B.4 *Fix $\sigma = (\mu_0, \ldots, \mu_{n-1}) \in \mathcal{L}(\Omega)$ and a Borel set E in Ω. The following are equivalent:*

(i) there is a $j < n$ such that $\mu_i(E) = 1$ for all $i \leq j$ and $\mu_i(E) = 0$ for all $i > j$,

(ii) E is assumed under $\succsim^{\sigma}$.

Proof. Suppose (i) holds. The act $(\overrightarrow{1}_E, \overrightarrow{0}_{\Omega\backslash E})$ is evaluated as $(1, \ldots, 1, 0, \ldots, 0)$ where the last 1 corresponds to μ_j, and $(\overrightarrow{0}_E, \overrightarrow{0}_{\Omega\backslash E})$ is evaluated as $(0, \ldots, 0)$. Thus $\overrightarrow{1} \succ^{\sigma}_E \overrightarrow{0}$, establishing condition (i) of Definition B.3. To establish condition (ii), note that $x \succ^{\sigma}_E y$ implies

$$\left(\int_E x\mathrm{d}\mu_0, \ldots, \int_E x\mathrm{d}\mu_j, \int_{\Omega\backslash E} z\mathrm{d}\mu_{j+1}, \ldots, \int_{\Omega\backslash E} z\mathrm{d}\mu_{n-1}\right) >^L \left(\int_E y\mathrm{d}\mu_0, \ldots, \int_E y\mathrm{d}\mu_j, \int_{\Omega\backslash E} z\mathrm{d}\mu_{j+1}, \ldots, \int_{\Omega\backslash E} z\mathrm{d}\mu_{n-1}\right),$$

so that certainly

$$\left(\int_E x\mathrm{d}\mu_0, \ldots, \int_E x\mathrm{d}\mu_j, \int_{\Omega\backslash E} x\mathrm{d}\mu_{j+1}, \ldots, \int_{\Omega\backslash E} x\mathrm{d}\mu_{n-1}\right) >^L \left(\int_E y\mathrm{d}\mu_0, \ldots, \int_E y\mathrm{d}\mu_j(\omega), \int_{\Omega\backslash E} y\mathrm{d}\mu_{j+1}, \ldots, \int_{\Omega\backslash E} y\mathrm{d}\mu_{n-1}\right).$$

Thus $x \succ^{\sigma} y$, as required.

Now suppose (i) fails. There are three cases to consider.

Case B.1 *$\mu_i(E) = 0$ for all i.*

This contradicts Definition B.3(i).

Case B.2 *$\mu_i(E) = 0$ and $\mu_h(E) = 1$ where $h > i$.*

Let U_i and U_h be Borel sets as in Definition 4.1 (i.e. with $\mu_i(U_i) = 1$ and, for $i \neq k$, $\mu_i(U_k) = 0$ and similarly for h). Define:

$$x(\omega) = \begin{cases} 1 & \text{if } \omega \in E \cap U_h, \\ 0 & \text{otherwise,} \end{cases}$$
$$y(\omega) = \begin{cases} 1 & \text{if } \omega \in U_i\backslash E, \\ 0 & \text{otherwise.} \end{cases}$$

Acts x and $(x_E, \overrightarrow{0}_{\Omega\backslash E})$ are evaluated as $(0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 corresponds to μ_h. (Here, we use $\mu_k(U_h) = 0$ for all $k \neq h$.) Act y is evaluated as $(0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 corresponds to μ_i, while act $(y_E, \overrightarrow{0}_{\Omega\backslash E})$ is evaluated as $(0, \ldots, 0)$. Thus $x \succ^{\sigma}_E y$. But, since $h > i$, $y \succ^{\sigma} x$, contradicting Definition B.3(ii).

Case B.3 $0 < \mu_i(E) < 1$ *for some* i.

Let U_i be a Borel set as in Definition 4.1, and define:

$$x(\omega) = \begin{cases} \mu_i(U_i \backslash E) & \text{if } \omega \in E \cap U_i, \\ 0 & \text{otherwise,} \end{cases}$$

$$y(\omega) = \begin{cases} 1 & \text{if } \omega \in U_i \backslash E, \\ 0 & \text{otherwise.} \end{cases}$$

Acts x and $(x_E, \overrightarrow{0}_{\Omega \backslash E})$ are evaluated as

$$(0, \ldots, 0, \mu_i(U_i \backslash E)\, \mu_i(E \cap U_i), 0, \ldots, 0),$$

where the non-zero entry corresponds to μ_i. This entry is indeed non-zero, as $1 > \mu_i(E) > 0$ implies $\mu_i(U_i \backslash E) > 0$ and $\mu_i(E \cap U_i) > 0$. Act y is evaluated as

$$(0, \ldots, 0, \mu_i(U_i \backslash E), 0, \ldots, 0),$$

where the non-zero entry corresponds to μ_i. This entry is indeed non-zero, since $1 > \mu_i(E)$. The act $(y_E, \overrightarrow{0}_{\Omega \backslash E})$ is evaluated as $(0, \ldots, 0)$. Thus $x \succ^{\sigma}_{E} y$. But since $1 > \mu_i(E \cap U_i)$, $y \succ^{\sigma} x$, contradicting Definition B.3(ii). ■

The next definition says that "all of E should matter" in terms of preference. For completeness, we characterize the condition for both ordinary and lexicographic probabilities.

Definition B.4 *A set E is* ***whole under*** $\succsim$ *if E is Borel and, for each $e \in E$ and open neighborhood U of e, there are $x, y \in \mathcal{A}$ such that $x \succ_{E \cap U} y$.*

Proposition B.5 *Suppose that E is assumed under $\succsim^{\mu}$. Then E is whole under $\succsim^{\mu}$ if and only if $E \subseteq \operatorname{Supp} \mu$.*

Proof. The proof is the same as the proof of Lemma B.1 below (adapted for a one-level LPS). ■

Lemma B.1 *Suppose that E is assumed under $\succsim^{\sigma}$ at level j (as per Definition B.3). Then E is whole under $\succsim^{\sigma}$ if and only if $E \subseteq \bigcup_{i \leq j} \operatorname{Supp} \mu_i$.*

Proof. First suppose $E \subseteq \bigcup_{i \leq j} \operatorname{Supp} \mu_i$, and fix some $e \in E$ and an open neighborhood U of e. Then $U \cap \operatorname{Supp} \mu_i \neq \emptyset$ for some $i \leq j$, and so $\mu_i(U) > 0$. But $\mu_i(E) = 1$ by Proposition B.4, and so $\mu_i(E \cap U) > 0$. Thus $\overrightarrow{1} \succ^{\sigma}_{E \cap U} \overrightarrow{0}$.

Next suppose $E \nsubseteq \bigcup_{i \leq j} \operatorname{Supp} \mu_i$. Then there is an $e \in E$ such that $U = \Omega \backslash \bigcup_{i \leq j} \operatorname{Supp} \mu_i$ is an open neighborhood of e. But $\mu_i(U) = 0$ for all $i \leq j$. Also, $\mu_i(E) = 0$ for all $i > j$, using Proposition B.4. Thus $\mu_i(E \cap U) = 0$ for all i, and so $x \sim^{\mu}_{E \cap U} y$ for all $x, y \in \mathcal{A}$. ■

Proposition B.6 *A set E is assumed and whole under $\succsim^{\sigma}$ if and only if it is assumed under σ in the sense of Definition 4.3.*

We now turn to some properties of assumption.

Proof of Property 4.1 (Convexity). Let $\sigma = (\mu_0, \ldots, \mu_{n-1})$ and fix events E and F that are assumed at level j. Fix also a Borel set G with $E \cap F \subseteq G \subseteq E \cup F$. We will show that G is also assumed at level j.

First fix $i \leq j$ and note that $\mu_i(E) = \mu_i(F) = 1$. So certainly $\mu_i(E \cap F) = 1$. Since $E \cap F \subseteq G$, $\mu_i(G) = 1$, establishing property (i) of assumption. Next fix $i > j$. Note that $\mu_i(E) = \mu_i(F) = 0$, and so $\mu_i(E \cup F) = 0$. Since $G \subseteq E \cup F$, $\mu_i(G) = 0$, establishing property (ii) of assumption. Finally, note that since E and F are assumed at level j and $G \subseteq E \cup F$,

$$G \subseteq E \cup F \subseteq \bigcup_{i \leq j} \operatorname{Supp} \mu_i.$$

This establishes property (iii) of assumption. ■

Proof of Property 4.2 (Closure). Let $\sigma = (\mu_0, \ldots, \mu_{n-1})$. It suffices to show that if E is assumed at level j then $\operatorname{cl}(E) = \bigcup_{i \leq j} \operatorname{Supp} \mu_i$. By condition (iii) of Definition 4.3, $E \subseteq \bigcup_{i \leq j} \operatorname{Supp} \mu_i$. Since $\bigcup_{i \leq j} \operatorname{Supp} \mu_i$ is closed, $\operatorname{cl}(E) \subseteq \bigcup_{i \leq j} \operatorname{Supp} \mu_i$. Moreover, for all $i \leq j$, $\mu_i(\operatorname{cl}(E)) = 1$ so that $\bigcup_{i \leq j} \operatorname{Supp} \mu_i \subseteq \operatorname{cl}(E)$. ■

Proof of Property 4.3 (Conjunction and Disjunction). We will only prove the Conjunction property. The proof of the Disjunction property is similar.

Let $\sigma = (\mu_0, \ldots, \mu_{n-1})$. For each m, E_m is assumed under σ at some level j_m. Let $j_M = \min\{j_m : m = 1, 2, \ldots\}$. Then, for each m, $\mu_i(E_m) = 1$ for all $i \leq j_M$. Thus $\mu_i(\bigcap_m E_m) = 1$ for all $i \leq j_M$. Also, $\mu_i(E_M) = 0$ for all $i > j_M$. Then certainly $\mu_i(\bigcap_m E_m) = 0$ for all $i > j_M$. This establishes conditions (i) and (ii) of Definition 4.3 (for $j = j_M$). Finally, $\bigcap_m E_m \subseteq E_M \subseteq \bigcup_{i \leq j_M} \operatorname{Supp} \mu_i$, where the second inclusion follows from the fact that E_M is assumed at level j_M. This establishes condition (iii) of Definition 4.3. ■

Finally in this section, we prove Proposition 4.1. Let $\mathcal{L}_n(\Omega)$ be the set of all σ in $\mathcal{L}(\Omega)$ of length n, and define $\mathcal{N}_n(\Omega)$ analogously.

Proof of Proposition 4.1. (i) It suffices to prove that for each n, $\mathcal{L}_n(\Omega)$ is closed in $\mathcal{N}_n(\Omega)$. Suppose Ω has at least two elements. (The result is trivial otherwise.) Take a point $\sigma = (\mu_0, \ldots, \mu_{n-1})$ in $\mathcal{N}(\Omega) \backslash \mathcal{L}(\Omega)$. Since Ω is finite, there is a point $\omega \in \Omega$ and a pair of indices $i < j < n$ such that $\mu_i(\omega) > 0$ and $\mu_j(\omega) > 0$. The set of $\tau = (\nu_0, \ldots, \nu_{n-1})$ such that $\nu_i(\omega) > 0$ and $\nu_j(\omega) > 0$ is an open neighborhood of σ in $\mathcal{N}_n(\Omega)$ which is disjoint from $\mathcal{L}_n(\Omega)$, so $\mathcal{L}_n(\Omega)$ is closed.

(ii) It suffices to prove that for each n, $\mathcal{L}_n(\Omega)$ is dense in $\mathcal{N}_n(\Omega)$. We begin with the fact that for any Polish space Ω, the set of probability measures with finite support is dense in $\mathcal{M}(\Omega)$ (Billingsley [12, 1968, Appendix III Theorem 4]). Therefore the set F of n-tuples of probability measures with finite support is dense in $\mathcal{N}_n(\Omega)$. To complete the proof it is enough to show that every element $(\mu_0, \ldots, \mu_{n-1}) \in F$ is the limit of a sequence of mutually singular elements of F. The support of μ_i is a finite set $\{x_{ij} : j < m_j\}$. Since Ω has no isolated points, each x_{ij} is a limit of a sequence $y_{ij}^k, k = 0, 1, \ldots$. By taking subsequences, we can get all the points y_{ij}^k to be distinct from each other. For each k, let $(\nu_0^k, \ldots, \nu_{n-1}^k)$ be the n-tuple of measures such that $\nu_i^k(x_{ij}) = \mu_i(x_{ij})$. That is, ν_i^k is the measure μ_i shifted over to the kth points in the sequence converging to the support of μ_i. Then $(\nu_0^k, \ldots, \nu_{n-1}^k)$ is in F and converges to $(\mu_0, \ldots, \mu_{n-1})$. Since the points y_{ij}^k are distinct, each $(\nu_0^k, \ldots, \nu_{n-1}^k)$ is mutually singular. ■

Appendix C Proofs for Section 5

Proof of Proposition 5.1. (i) Start with a type structure $\langle S^a, S^b, T^a, T^b, \kappa^a, \kappa^b \rangle$. The case that $S^b \times T^b$ is a singleton is trivial, so we may assume that it is not. Pick any $\sigma \in \mathcal{L}(S^b \times T^b)$ which does not have full support. Define $\lambda^a(t^a) = \kappa^a(t^a)$ if $\kappa^a(t^a) \in \mathcal{L}^+(S^b \times T^b)$ and $\lambda^a(t^a) = \sigma$ otherwise. Since $\mathcal{L}^+(S^b \times T^b)$ is Borel, λ^a is a Borel map. Define λ^b similarly.

(ii) It is clear from the definitions that they have the same rationality sets R_1^a and R_1^b. The result for $m > 1$ then follows at once by induction, since only assumption by full-support LPS's is involved. ■

Proof of Proposition 5.2. Let T^a and T^b be the Baire space, i.e., the metric space $\mathbb{N}^{\mathbb{N}}$ with the product metric, where $\mathbb{N}$ has the discrete metric. There is a continuous surjection λ^a (resp. λ^b) from T^a (resp. T^b) onto any Polish space, in particular onto $\overline{\mathcal{L}}\left(S^b \times T^b\right)$ (respectively $\overline{\mathcal{L}}\left(S^b \times T^b\right)$). (See Kechris [35, 1995, p.13 and Theorem 7.9].) These maps give us a complete type structure. ■

Appendix D Proofs for Section 6

Lemma D.1 *Suppose t^a assumes $E \subseteq S^b \times T^b$ at level j, where $\lambda^a(t^a) = \left(\mu_0, \ldots, \mu_{n-1}\right)$. Then $\bigcup_{i \leq j} \operatorname{Supp} \operatorname{marg}_{S^b} \mu_i = \operatorname{proj}_{S^b} E$.*

Proof. Fix $s^b \in \operatorname{proj}_{S^b} E$, i.e. $\left(s^b, t^b\right) \in E$ for some t^b. Then, by condition (c) of Definition 4.3, $(s^b, t^b) \in \operatorname{Supp} \mu_i$ for some $i \leq j$. Therefore $0 < \mu_i(\{s^b\} \times T^b) = \operatorname{marg}_{S^b} \mu_i(s^b)$ and hence $s^b \in \operatorname{Supp} \operatorname{marg}_{S^b} \mu_i$. Next fix $s^b \notin \operatorname{proj}_{S^b} E$. Then $\{s^b\} \times T^b$ is disjoint from E. But for each $i \leq j$ we have $\mu_i(E) = 1$, so $\mu_i(\{s^b\} \times T^b) = \operatorname{marg}_{S^b} \mu_i(s^b) = 0$ and hence $s^b \notin \operatorname{Supp} \operatorname{marg}_{S^b} \mu_i$. ■

The next series of lemmas concerns the geometry of polytopes. We will first review some notions from geometry, then state the lemmas, then explain the connection between the geometric notions and games, then present some intuitive examples, and finally give the formal proofs of the lemmas.

Throughout this section, we will fix a finite set $X = \{x_1, \ldots, x_n\} \subseteq \mathbb{R}^d$. The **polytope** generated by X, denoted by P, is the closed convex hull of X, i.e., the set of all sums $\sum_{i=1}^n \lambda_i x_i$, where $\lambda_i \geq 0$ for each i, and $\sum_{i=1}^n \lambda_i = 1$. The **affine hull** of P, denoted by $\operatorname{aff}(P)$, is the set of all affine combinations of finitely many points in P, i.e., the set of all sums $\sum_{i=1}^k \lambda_i y_i$, where $y_1, \ldots, y_k \in P$ and $\sum_{i=1}^k \lambda_i = 1$. The **relative interior** of P, denoted by $\operatorname{relint}(P)$, is the set of all $x \in \operatorname{aff}(P)$ such that there is an open ball $B(x)$ centered around x, with $\operatorname{aff}(P) \cap B(x) \subseteq P$.

A **hyperplane** in $\mathbb{R}^d$ is a set of the form

$$H(u, \alpha) = \left\{x \in \mathbb{R}^d : \langle x, u \rangle = \alpha\right\}$$

for some nonzero $u \in \mathbb{R}^d$. A hyperplane $H(u, \alpha)$ **supports** a polytope P if

$$\alpha = \sup\left\{\langle x, u \rangle : x \in P\right\}.$$

A **face** of P is either P itself or a set of the form $H \cap P$ where H is a hyperplane that supports P. If $F \neq P$ is a face of P, we say F is a **proper face**. A face $H \cap P$ is **strictly positive** if $H = H(u, \alpha)$ for some (u, α) such that each coordinate of u is strictly positive.

Given a point x in a polytope P, say the points $x_1, \ldots, x_k \in P$ each **support** $x \in P$ if there are $\lambda_1, \ldots, \lambda_k$, with $0 < \lambda_i \leq 1$ for each i, $\sum_{i=1}^k \lambda_i = 1$, and $x = \sum_{i=1}^k \lambda_i x_i$. Write $\operatorname{su}(x)$ for the set of points that support $x \in P$. (Note the slight abuse of notation relative to that introduced before Definition 3.3.)

Here are the lemmas we will need:

Lemma D.2 *If F is a face of a polytope P and $x \in F$ then $\operatorname{su}(x) \subseteq F$.*

Lemma D.3 *For each point x in a polytope P, $\operatorname{su}(x)$ is a face of P.*

Lemma D.4 *If x belongs to a strictly positive face of a polytope P, then* $\mathrm{su}(x)$ *is a strictly positive face of P.*

We now give the interpretation of the geometric notions in game theory. Let d be the cardinality of the finite strategy set S^b. Each strategy $s^a \in S^a$ corresponds to the point

$$\overrightarrow{\pi}^a(s^a) = \left(\pi^a(s^a, s^b) : s^b \in S^b\right) \in \mathbb{R}^d.$$

For any probability measure $\mu \in \mathcal{M}(S^a)$, $\overrightarrow{\pi}^a(\mu)$ is the point

$$\overrightarrow{\pi}^a(\mu) = \textstyle\sum_{s^a \in S^a} \mu\left(s^a\right) \overrightarrow{\pi}^a(s^a).$$

Notice that $\overrightarrow{\pi}^a(\mu)$ is in the polytope P generated by the finite set $\{\overrightarrow{\pi}^a(s^a) : s^a \in S^a\}$.

Let us identify each probability measure $\nu \in \mathcal{M}(S^b)$ with the point $(\nu(s^b) : s^b \in S^b) \in \mathbb{R}^d$. Then for each pair $(\mu, \nu) \in \mathcal{M}(S^a) \times \mathcal{M}(S^b)$, $\langle \overrightarrow{\pi}^a(\mu), \nu \rangle$ is the expected payoff to Ann. Thus, a pair (μ, ν) gives expected payoff α to Ann if and only if $\overrightarrow{\pi}^a(\mu)$ belongs to the hyperplane $H(\nu, \alpha)$. It follows that a set F is a strictly positive face of P if and only if there is a probability measure ν with support S^b such that

$$F = \{\overrightarrow{\pi}^a(\mu) : \mu \in \mathcal{M}(S^a) \text{ is optimal under } \nu\}.$$

Consider an admissible strategy s^a. By Lemma 3.1, $\overrightarrow{\pi}^a\left(s^a\right)$ is optimal under some measure ν with support S^b. That is, $\overrightarrow{\pi}^a\left(s^a\right)$ belongs to some strictly positive face of P. Lemma D.4 shows that $\mathrm{su}(\overrightarrow{\pi}^a\left(s^a\right))$ is a strictly positive face of P. So, we can pick ν so that, for every $r^a \in S^a$, $\overrightarrow{\pi}^a\left(r^a\right)$ is optimal under ν if and only if $\overrightarrow{\pi}^a\left(r^a\right) \in \mathrm{su}(\overrightarrow{\pi}^a\left(s^a\right))$. This is the fact we use in the proof of Theorem 6.1(ii).

We next give some intuition for Lemmas D.2-D.4. Let P be a tetrahedron, as in Figure D.1. The point x^* is supported by the hyperplane H, and the corresponding face $H \cap P$ is the shaded region shown. The set of points that support x^*, i.e., the set $\mathrm{su}\left(x^*\right)$, is the line segment from x_2 to x_4. Note that these points are also contained in the face $H \cap P$. The general counterpart of this is Lemma D.2.

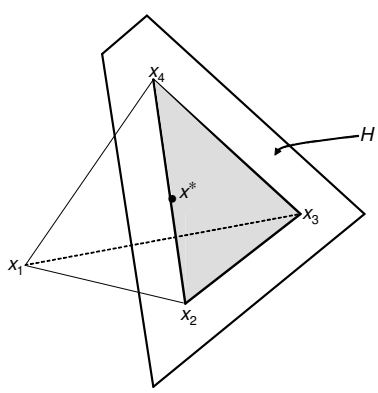

Figure D1

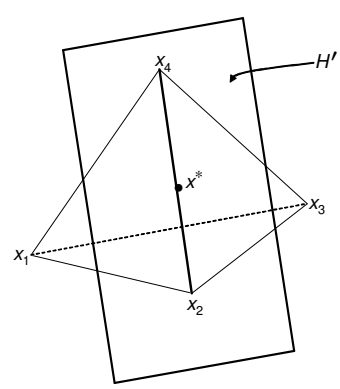

Figure D2

Now a converse. In Figure D.1, the point x_3 lies in $H \cap P$ but does not support x^*. However, we can 'tilt' the hyperplane H to get a new supporting hyperplane H' as in Figure D.2. Here, $H' \cap P$ is the line segment from x_2 to x_4, i.e., exactly the set $\mathrm{su}\left(x^*\right)$. The general counterpart is Lemma D.3.

Consider another example, in Figure D.3. Here P is the line segment from $(1,0)$ to $(1,1)$. Note that $\mathrm{su}\,((1,0)) = \{(1,0)\}$. The hyperplane H supports $(1,0)$, and $H \cap P = P$. We can tilt the hyperplane to get H' where $H' \cap P = \{(1,0)\}$ (in accordance with Lemma D.3). But note that we can't do this if we require the hyperplane to be nonnegative. (Indeed, H is the unique nonnegative hyperplane supporting $(1,0)$.) Intuitively, though, we will have room to tilt the hyperplane and maintain nonnegativity–in fact, strict positivity–if the original hyperplane is strictly positive. This is Lemma D.4.

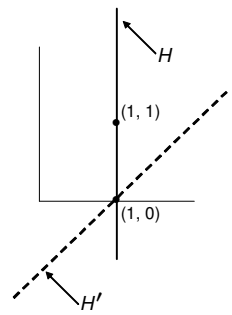

Figure D3

We now turn to the proofs of Lemmas D.2-D.4.

Proof of Lemma D.2. Fix a face F that contains x. If $F = P$ then certainly $\mathrm{su}\,(x) \subseteq F$. If $F \neq P$, there is a hyperplane $H = H\,(u, \alpha)$ that supports P, with $F = H \cap P$. Fix $y \in \mathrm{su}\,(x)$. Then there are $x_1, \ldots, x_k \in P$ and $\lambda_1, \ldots, \lambda_k$, with $0 < \lambda_i \leq 1$ for each i, $\sum_{i=1}^{k} \lambda_i = 1$, $y = x_1$, and $x = \sum_{i=1}^{k} \lambda_i x_i$. Let $z = \sum_{i=2}^{k} \frac{\lambda_i}{1-\lambda_1} x_i$, and note that $z \in P$, since P is convex. Also note that $x = \lambda_1 y + (1 - \lambda_1)\, z$; that is, x lies on the line segment from y to z.

Since $x \in H$ and $y, z \in P$,

$$\langle x, u \rangle = \alpha, \quad \langle y, u \rangle \leq \alpha, \quad \langle z, u \rangle \leq \alpha.$$

Moreover, since x lies on the line segment from y to z,

$$\langle y, u \rangle \leq \langle x, u \rangle \leq \langle z, u \rangle.$$

It follows that $\langle y, u \rangle = \alpha$, so $y \in F$. ■

For the next proofs we need the following basic facts about a general polytope P (see Ziegler [59, 1998, Chapter 2]):

P1 Every face of P is a polytope.

P2 Every face of a face of P is a face of P.

P3 If $x \in P$, either $x \in \mathrm{relint}\,(P)$ or x belongs to a proper face of P.

P4 P has finitely many faces.

We record an immediate consequence of P1-P4.

Lemma D.5 *If $x \in P$ then there exists a face F of P with $x \in \mathrm{relint}\,(F)$.*

Proof. If $x \in \text{relint}\,(P)$, the result holds trivially. So suppose $x \notin \text{relint}\,(P)$. By P3, x is contained in some proper face F of P. By P1, the face F is a polytope. Using P2 and P4, we can choose F so that there does not exist a proper face of F that contains x. P3 then implies $x \in \text{relint}\,(F)$. ■

The next lemma establishes a fact about points in the relative interior of a face F of P.

Lemma D.6 *Let F be a face of P. If $x \in \text{relint}(F)$ then $F \subseteq \text{su}(x)$.*

Proof. Fix $x \in \text{relint}(F)$ and some $x' \in F$. If $x' = x$ then certainly $x' \in \text{su}\,(x)$. If not, consider the line going through both x and x', to be denoted by $L\,(x, x')$. Since $x \in \text{relint}\,(F)$, there is some open ball $B\,(x)$ centered around x, with $\text{aff}\,(F) \cap B\,(x) \subseteq F$. Then $\text{aff}\,(F) \cap B\,(x)$ must meet $L\,(x, x')$. Certainly, we can find a point x'' both on $L\,(x, x')$ and in $\text{aff}\,(F) \cap B\,(x)$, with $d\,(x', x) < d\,(x', x'')$ for the Euclidean metric d. Then there must exist $0 < \lambda < 1$ with $x = \lambda x' + (1 - \lambda)\,x''$. Since $x', x'' \in P$, this establishes $x' \in \text{su}\,(x)$. ■

We now turn to the proofs of lemmas D.3 and D.4.

Proof of Lemma D.3. Fix $x \in P$. By Lemma D.5, there exists a face F of P with $x \in \text{relint}\,(F)$. We then have $\text{su}\,(x) \subseteq F$ by Lemma D.2, and $F \subseteq \text{su}\,(x)$ by Lemma D.6. ■

Proof of Lemma D.4. Let $H(u, \alpha) \cap P$ be a strictly positive face of P containing x. By Lemma D.3, $\text{su}(x) = H(u', \alpha') \cap P$ is a face of P. Set

$$u'' = u' + \beta u, \quad \alpha'' = \alpha' + \beta\alpha,$$

for some $\beta > 0$. If $y \in H(u', \alpha') \cap P$, we get

$$\langle y, u'' \rangle = \langle y, u' \rangle + \beta \langle y, u \rangle = \alpha' + \beta\alpha = \alpha'',$$

using $\text{su}(x) \subseteq H(u, \alpha) \cap P$. If $y \in P \backslash H(u', \alpha')$, we get

$$\langle y, u'' \rangle = \langle y, u' \rangle + \beta \langle y, u \rangle < \alpha' + \beta \langle y, u \rangle \leq \alpha' + \beta\alpha = \alpha''.$$

Thus $H(u'', \alpha'')$ is a supporting hyperplane with $\text{su}\,(x) = H(u'', \alpha'') \cap P$. Moreover, since $\beta > 0$, $u'' \gg 0$ as required. ■

Appendix E Proofs for Section 7

Lemma E.1 *If $s^a \in S^a_m$ then there exists $\mu \in \mathcal{M}\left(S^b\right)$, with $\text{Supp}\,\mu = S^b_{m-1}$, such that $\pi^a\,(s^a, \mu) \geq \pi^a\,(r^a, \mu)$, for each $r^a \in S^a$.*

Proof. By Lemma 3.1, there exists $\mu \in \mathcal{M}\left(S^b\right)$, with $\text{Supp}\,\mu = S^b_{m-1}$, such that $\pi^a\,(s^a, \mu) \geq \pi^a\,(r^a, \mu)$ for all $r^a \in S^a_{m-1}$. Suppose there is an $r^a \in S^a \backslash S^a_{m-1}$ with

$$\pi^a\,(s^a, \mu) < \pi^a\,(r^a, \mu)\,. \tag{E.1}$$

We have $r^a \in S^a_l \backslash S^a_{l+1}$, for some $l < m - 1$. Choose r^a (and l) so that there does not exist $q^a \in S^a_{l+1}$ with $\pi^a\,(s^a, \mu) < \pi^a\,(q^a, \mu)$.

Fix some $\nu \in \mathcal{M}(S^b)$, with $\text{Supp}\,\nu = S^b_l$, and define a sequence of measures $\mu^n \in \mathcal{M}(S^b)$, for each $n \in \mathbb{N}$, by $\mu^n = (1 - \frac{1}{n})\mu + \frac{1}{n}\nu$. Note that $\text{Supp}\,\mu^n = S^b_l$ for each n. Using $r^a \notin S^a_{l+1}$, and

Lemma 3.1 applied to the $(l+1)$-admissible strategies, it follows that for each n there is a $q^a \in S_l^a$ with

$$\pi^a(q^a, \mu^n) > \pi^a(r^a, \mu^n). \tag{E.2}$$

We can assume that $q^a \in S_{l+1}^a$. (Choose $q^a \in S_l^a$ to maximize the left-hand side of equation E.2 among all strategies in S_l^a.) Also, since S_{l+1}^a is finite, there is a $q^a \in S_{l+1}^a$ such that E.2 holds for infinitely many n. Letting $n \to \infty$ yields

$$\pi^a(q^a, \mu) \geq \pi^a(r^a, \mu). \tag{E.3}$$

From E.1 and E.3 we get $\pi^a(q^a, \mu) > \pi^a(s^a, \mu)$, contradicting our choice of r^a. ■

In what follows, we will need to make use of the following characterizations of full support.

Lemma E.2 *A sequence $\sigma = \left(\mu_0, \ldots, \mu_{n-1}\right) \in \mathcal{N}(\Omega)$ has full support if and only if, for each nonempty open set U, there is an i with $\mu_i(U) > 0$.*

Proof. Fix a sequence $\sigma = \left(\mu_0, \ldots, \mu_{n-1}\right) \in \mathcal{N}(\Omega)$ which does not have full support. Then $U = \Omega \backslash \bigcup_{i<n} \operatorname{Supp} \mu_i$ is nonempty. The set U is open and $\mu_i(U) = 0$ for all i. For the converse, fix a full-support sequence $\sigma = \left(\mu_0, \ldots, \mu_{n-1}\right) \in \mathcal{N}(\Omega)$ and a nonempty open set U. Since σ has full support, $U \cap \operatorname{Supp} \mu_i \neq \emptyset$ for some i. Then $(\Omega \backslash U) \cap \operatorname{Supp} \mu_i$ is closed and strictly contained in $\operatorname{Supp} \mu_i$, so that $\mu_i\left((\Omega \backslash U) \cap \operatorname{Supp} \mu_i\right) < 1$. From this, $\mu_i(U) > 0$, as required. ■

Lemma E.3 *Fix a sequence $\sigma = (\mu_0, \ldots, \mu_{n-1}) \in \mathcal{N}(\Omega)$ which has full support. Then $\sigma \in \mathcal{L}^+(\Omega)$ if and only if there are Borel sets U_i that form a partition of Ω with $\mu_i(U_i) = 1$ and $U_i \subseteq \operatorname{Supp} \mu_i$.*

Proof. Certainly, if there are Borel sets U_i that form a partition of Ω with $\mu_i(U_i) = 1$ and $U_i \subseteq \operatorname{Supp} \mu_i$, then σ is mutually singular.

Now suppose that, for $i = 0, \ldots, n-1$, there are Borel sets U_i in Ω with $\mu_i(U_i) = 1$ and $\mu_i(U_j) = 0$ for $i \neq j$. For $i > 0$, define $V_i = U_i \backslash \bigcup_{j<i} U_j$. Also define $V_0 = \Omega \backslash \bigcup_{i>0} V_i$. Then V_i is a partition of Ω with $\mu_i(V_i) = 1$ and $\mu_i(V_j) = 0$ for $i \neq j$.

Next, let $W = \bigcup_j (V_j \backslash \operatorname{Supp} \mu_j)$. We have $\mu_i(W) = 0$ for all i. Let $W_0 = W \cap \operatorname{Supp} \mu_0$ and

$$W_i = [W \backslash (W_0 \cup \cdots \cup W_{i-1})] \cap \operatorname{Supp} \mu_i$$

for $i > 0$. Using $\bigcup_j \operatorname{Supp} \mu_j = \Omega$, we see that the sets W_i partition W. Also $\mu_i(W_j) = 0$ for all i and j, since $W_j \subseteq W$. Finally, set $X_i = (V_i \cap \operatorname{Supp} \mu_i) \cup W_i$. Then the X_i partition Ω and have the required properties: $\mu_i(X_i) = 1$, $\mu_i(X_j) = 0$ for $j \neq i$, and $X_i \subseteq \operatorname{Supp} \mu_i$. ■

In the next three lemmas, Borel without qualification means Borel in $\mathcal{N}(\Omega)$, and $\sigma = (\mu_0, \ldots, \mu_{n-1})$ varies over $\mathcal{N}(\Omega)$. We make repeated use of the following facts:

(i) There is a countable open basis $E_1, E_2, \ldots$ for Ω.

(ii) For each Borel set B in Ω and $r \in [0,1]$, the set of μ such that $\mu(B) > r$ is Borel in $\mathcal{M}(\Omega)$.

(iii) For each Borel set Y in $\mathcal{M}(\Omega)$ and each k, the set of σ such that $\mu_k \in Y$ is Borel.

Fact (i) follows from the assumption that Ω is separable. Fact (ii) says that the function $\mu \mapsto \mu(B)$ is Borel, which follows from Kechris [35, 1995, Theorem 17.24]. Fact (iii) follows from the continuity of the projection function $\sigma \mapsto \mu_k$ from $\mathcal{N}(\Omega)$ to $\mathcal{M}(\Omega)$.

Let $\mathcal{L}_n^+(\Omega)$ be the set of all σ in $\mathcal{L}^+(\Omega)$ of length n, and define $\mathcal{N}_n^+(\Omega)$ analogously.

Lemma E.4 *Fix $n \in \mathbb{N}$. For any Polish space Ω, the set $\mathcal{L}_n^+(\Omega)$ is Borel.*

Proof. Fix $n \in \mathbb{N}$. It follows from (iii) that $\mathcal{N}_n(\Omega)$ is Borel. By Lemma E.2 and (i), a sequence $\sigma \in \mathcal{N}_n(\Omega)$ has full support if and only if for each basic open set E_i there exists $j < n$ such that $\mu_j(E_i) > 0$. By (ii) and (iii), for each i and j the set of σ such that $\mu_j(E_i) > 0$ is Borel. Therefore $\mathcal{N}_n^+(\Omega)$ is Borel.

Write $\mu \perp \nu$ if there is a Borel set $U \subseteq \Omega$ such that $\mu(U) = 1$ and $\nu(U) = 0$. It is easy to see that mutual singularity holds for an element $\sigma \in \mathcal{N}_n^+$ if and only if $\mu_i \perp \mu_j$ for all $i < j$. To complete the proof it suffices to prove that for each $i < j$, the set of σ such that $\mu_i \perp \mu_j$ is Borel. Note that $\mu_i \perp \mu_j$ if and only if for each m, there is an open set V such that $\mu_i(V) = 1$ and $\mu_j(V) < \frac{1}{m}$. By (i), this in turn holds if and only if for each m there exists k such that $\mu_i(E_k) > 1 - \frac{1}{m}$ and $\mu_j(E_k) < \frac{1}{m}$. By (ii) and (iii), the set of σ such that $\mu_i(E_k) > 1 - \frac{1}{m}$ is Borel, and the set of σ such that $\mu_j(E_k) < \frac{1}{m}$ is Borel. The set of σ such that $\mu_i \perp \mu_j$ is a Borel combination of these sets, and hence is Borel as required. ■

Corollary E.1 *For any Polish space Ω, the set $\mathcal{L}^+(\Omega)$ is Borel.*

Proof. Each $\mathcal{L}_n^+(\Omega)$ is Borel, and $\mathcal{L}^+(\Omega) = \bigcup_n \mathcal{L}_n^+(\Omega)$. ■

Lemma E.5 *For each Polish space Ω and Borel set E in Ω, the set of $\sigma \in \mathcal{L}^+(\Omega)$ such that E is assumed under σ is Borel.*

Proof. Fix n and $j < n$. By Fact (ii), the sets of μ such that $\mu(E) = 1$ and such that $\mu(E) = 0$ are Borel in $\mathcal{M}(\Omega)$. Therefore the set of $\sigma = (\mu_0, \ldots, \mu_{n-1}) \in \mathcal{L}_n^+(\Omega)$ such that conditions (i) and (ii) in Definition 4.3 hold is Borel. Let $\{d_0, d_1, \ldots\}$ be a countable dense subset of E. For each k and $\mu \in \mathcal{M}(\Omega)$, we have $d_k \in \operatorname{Supp} \mu$ if and only if $\mu(B) > 0$ for every open ball B with center d_k and rational radius. Then by Fact (ii), the set of μ such that $d_k \in \operatorname{Supp} \mu$ is Borel in $\mathcal{M}(\Omega)$. We have $E \subseteq \bigcup_{i \leq j} \operatorname{Supp} \mu_i$ if and only if $d_k \in \bigcup_{i \leq j} \operatorname{Supp} \mu_i$ for all $k \in \mathbb{N}$. Therefore, the set of $\sigma \in \mathcal{L}_n^+(\Omega)$ satisfying condition (iii) in Definition 4.3 is Borel. Thus the set of $\sigma \in \mathcal{L}^+(\Omega)$ such that E is assumed under σ is Borel, as required. ■

Lemma E.6 *For each m,*

(i) $R_m^a = R_1^a \cap [S^a \times \bigcap_{i<m} A^a(R_i^b)]$,

(ii) R_m^a *is Borel in* $S^a \times T^a$.

Proof. Part (i) is immediate.

Part (ii) is by induction. For $m = 1$, first note that since λ^a is Borel measurable, Lemma E.4 says that for each n the set $(\lambda^a)^{-1}(\mathcal{L}_n^+(S^b \times T^b))$ is Borel in T^a. From Definition 5.3, for each $s^a \in S^a$ there is a finite Boolean combination C of linear equations in $n \cdot \mid S^b \mid$ variables such that whenever $\lambda^a(t^a) = (\mu_0, \ldots, \mu_{n-1}) \in \mathcal{L}_n^+(S^b \times T^b)$, the pair (s^a, t^a) is rational if and only if C holds for $\{\operatorname{marg}_{S^b} \mu_i(s^b) : i < n, s^b \in S^b\}$. Since S^a and S^b are finite, this shows that R_1^a is Borel in $S^a \times T^a$.

Now turn to $m > 1$. Assume the result holds for all $i \leq m$. It suffices to show that each $A^a(R_i^b)$ is Borel in T^a. By the induction hypothesis and Lemma E.5, the set of σ such that R_i^b is assumed under σ is Borel. Since λ^a is Borel measurable, $A^a(R_i^b)$ is Borel in T^a. ■

The next lemma will guarantee that we will have enough room to build the measures we need to establish Lemma E.8. For $t^a, u^a \in T^a$, write $t^a \approx u^a$ if for each i the component measures $(\lambda^a(t^a))_i$ and $(\lambda^a(u^a))_i$ have the same marginals on S^b and are mutually absolutely continuous (have the same null sets).

Lemma E.7 *In a complete type structure:*

(i) If $\lambda^a(t^a) \in \mathcal{L}^+\left(S^b \times T^b\right)$ and $u^a \approx t^a$, then $\lambda^a(u^a) \in \mathcal{L}^+\left(S^b \times T^b\right)$.

(ii) If $\lambda^a(t^a) \in \mathcal{L}^+\left(S^b \times T^b\right)$, then there are continuum many u^a such that $u^a \approx t^a$.

(iii) For each set $E \subseteq S^b \times T^b$, the set $A^a(E)$ is closed under the relation $\approx$. In fact, for each j, if $t^a \approx u^a$ and E is assumed under $\lambda^a(t^a)$ at level j, then E is assumed under $\lambda^a(u^a))$ at level j.

(iv) If $t^a \approx u^a$ then for each m and $s^a \in S^a$, $(s^a, t^a) \in R^a_m$ if and only if $(s^a, u^a) \in R^a_m$.

Proof. Part (i) follows from the fact that $\lambda^a(t^a) \in \mathcal{L}(S^b \times T^b)$ and the mutual absolute continuity of the measures $\lambda^a(t^a)$ and $\lambda^a(u^a)$. For part (ii), note that full support implies that $\mu_i = (\lambda^a(t^a))_i$ has infinite support for some i. Therefore, there are continuum many different measures ν_i with the same null sets and marginal on S^b as μ_i. The sequence of measures obtained by replacing μ_i by ν_i belongs to $\mathcal{L}^+(S^b \times T^b)$, and by completeness this sequence is equal to $\lambda^a(u^a)$ for some u^a. It follows that $u^a \approx t^a$. For part (iii), we need to show that if $\lambda^a(t^a) = \left(\mu_0, \ldots, \mu_{n-1}\right)$ assumes E (at level j) and $u^a \approx t^a$ then $\lambda^a(u^a) = (\nu_0, \ldots, \nu_{n-1})$ assumes E (at level j). Conditions (i)-(ii) follow immediately from the fact that $u^a \approx t^a$. For condition (iii), fix $i \leq j$ and note that $\operatorname{Supp}\mu_i \subseteq E$ and $U_i = S^b \times T^b \backslash \operatorname{Supp}\mu_i$ is open. It follows that $\operatorname{Supp}\nu_i \backslash E \subseteq U_i$. If there exists $\left(s^b, t^b\right) \in \operatorname{Supp}\nu_i \backslash E$ then $\nu_i(U_i) > 0$ and, using the fact that $u^a \approx t^a$, $\mu_i(U_i) > 0$, a contradiction. With this, $\operatorname{Supp}\nu_i \subseteq E$ as required. For part (iv), the case of $m = 1$ follows immediately from part (i). The case of $m > 1$, it is proved by induction and makes use of part (iii). ■

Set $R^a_0 = S^a \times T^a$ and $R^b_0 = S^b \times T^b$.

Lemma E.8 *In a complete type structure,* $\operatorname{proj}_{S^a} R^a_m = \operatorname{proj}_{S^a}\left(R^a_m \backslash R^a_{m+1}\right)$ *for each $m \geq 0$.*

Proof. The proof is by induction on m.

$m = 0$: Choose t^a so that $\lambda^a(t^a) \notin \mathcal{L}^+(S^b \times T^b)$ and note that $S^a \times \{t^a\}$ is disjoint from R^a_1. So, $\operatorname{proj}_{S^a}(R^a_0 \backslash R^a_1) = S^a$.

$m = 1$: Fix $(s^a, t^a) \in R^a_1$. It suffices to show that there is a type $u^a \in T^a$ with $(s^a, u^a) \in R^a_1 \backslash R^a_2$. To see this, first notice that there is a full-support LPS (μ) of length one such that s^a is optimal under (μ). (This is by Lemma 5.1.) By completeness, there is a type u^b such that $\lambda^b\left(u^b\right) \notin \mathcal{L}^+(S^a \times T^a)$. Construct a probability measure $\nu \in \mathcal{M}\left(S^b \times T^b\right)$ with $\operatorname{marg}_{S^b}\mu = \operatorname{marg}_{S^b}\nu$ and $\nu\left(S^b \times \left\{u^b\right\}\right) = 1$. Let ρ be the measure $(\mu + \nu)/2$. Then ρ is a full-support LPS, so by completeness there is a type $u^a \in T^a$ with $\lambda^a(u^a) = (\rho)$. Note, s^a is optimal under (ρ), so $(s^a, u^a) \in R^a_1$. But $\rho\left(R^b_1\right) \leq \frac{1}{2}$ because $\lambda^b\left(u^b\right) \notin \mathcal{L}^+(S^a \times T^a)$. So R^b_1 is not assumed under (ρ) and therefore $(s^a, u^a) \notin R^a_2$.

$m \geq 2$: Assume the result holds for $m - 1$. Let $(s^a, t^a) \in R^a_m$ and $\lambda^a(t^a) = \sigma = (\mu_0, \ldots, \mu_{n-1})$. Then $t^a \in A^a(R^b_i)$ for each $i < m$. We will find a type u^a such that $(s^a, u^a) \in R^a_m \backslash R^a_{m+1}$.

By the induction hypothesis and the fact that S^b is finite, there is a finite set $U \subseteq R^b_{m-1} \backslash R^b_m$ with $\operatorname{proj}_{S^b} U = \operatorname{proj}_{S^b} R^b_{m-1}$. Since $m \geq 2$, $U \subseteq R^b_1$, so $\lambda^b\left(t^b\right) \in \mathcal{L}^+\left(S^a \times T^a\right)$ for each $\left(s^b, t^b\right) \in U$. Therefore, by Lemma E.7, the set U can be chosen so that $\mu_i(U) = 0$ for all i.

We will get a point $(s^a, u^a) \in R^a_m \backslash R^a_{m+1}$ by adding a measure to the beginning of the sequence σ. Since U is finite, $\operatorname{proj}_{S^b} U = \operatorname{proj}_{S^b} R^b_{m-1}$, and $\mu_0(R^b_{m-1}) = 1$, there is a probability measure ν such that $\nu(U) = 1$ and $\operatorname{marg}_{S^b}\nu = \operatorname{marg}_{S^b}\mu_0$. Let τ be the sequence $(\nu, \mu_0, \ldots, \mu_{n-1})$. By completeness there is a $u^a \in T^a$ with $\lambda^a(u^a) = \tau$. Since $\sigma \in \mathcal{L}^+(S^b \times T^b)$ and $\mu_i(U) = 0$ for each i, we see that $\tau \in \mathcal{L}^+(S^b \times T^b)$. Since ν has the same marginal on S^b as μ_0, and $(s^a, t^a) \in R^a_1$, we have $(s^a, u^a) \in R^a_1$. Since $U \subseteq R^b_{m-1}$ and $t^a \in A^a(R^b_k)$ for each $k < m$, it follows that $u^a \in A^a(R^b_k)$

for each $k < m$. Then, by Lemma E.6(i), we have $(s^a, u^a) \in R^a_m$. However, since U is disjoint from R^b_m we have $\nu(R^b_m) = 0$, so $u^a \notin A^a(R^b_m)$ and hence $(s^a, u^a) \notin R^a_{m+1}$. This completes the induction. ■

Appendix F Proofs for Section 8

For the following two lemmas we assume that $\langle S^a, S^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ is a complete type structure in which the maps λ^a, λ^b are continuous.

Lemma F.1 *If player a is not indifferent, then $R^a_0 \setminus \mathrm{cl}(R^a_1)$ is uncountable.*

Proof. We have that $\pi^a(r^a, s^b) < \pi^a(s^a, s^b)$ for some r^a, s^a, s^b. Then S^a has more than one element, and by completeness, T^b has more than one element. Therefore, using completeness again, there is a type $t^a \in T^a$ such that $\lambda^a(t^a) = (\mu_0, \mu_1)$ is a full-support LPS of length 2, and $\mu_0(\{s^b\} \times T^b) = 1$. Let U be the set of all $u^a \in T^a$ such that r^a is not optimal for $(\lambda^a(u^a))_0$, that is, for some $q^a \in S^a$,

$$\textstyle\sum_{s^b \in S^b} \pi^a(r^a, s^b) \operatorname{marg}_{S^b}(\lambda^a(u^a))_0(s^b) < \sum_{s^b \in S^b} \pi^a(q^a, s^b) \operatorname{marg}_{S^b}(\lambda^a(u^a))_0(s^b).$$

We now show that $t^a \in U$. Note first that since $\mu_0(\{s^b\} \times T^b) = 1$, the function $\operatorname{marg}_{S^b}(\lambda^a(t^a))_0$ has value 1 at s^b and 0 everywhere else in S^b. Therefore for each $q^a \in S^a$,

$$\textstyle\sum_{s^b \in S^b} \pi^a(r^a, s^b) \operatorname{marg}_{S^b}(\lambda^a(t^a))_0(s^b) = \pi^a(q^a, s^b).$$

Since $\pi^a(r^a, s^b) < \pi^a(s^a, s^b)$, the inequality defining U holds with $(q^a, u^a) = (s^a, t^a)$, and hence $t^a \in U$.

We next show that U is open. Since λ^a is continuous, the function $u^a \mapsto (\lambda^a(u^a))_0$ is continuous. Convergence in the Prohorov metric is equivalent to weak convergence, so the function

$$u^a \mapsto \operatorname{marg}_{S^b}(\lambda^a(u^a))_0(s^b) = \textstyle\int \mathbf{1}(\{s^b\} \times T^b) \mathrm{d}(\lambda^a(u^a))_0$$

is continuous. Thus U is defined by a strict inequality between two continuous real functions of u^a, and hence U is open.

Since $\{r^a\}$ is open in S^a, the set $\{r^a\} \times U$ is open in $S^a \times T^a$. By definition, the set $\{r^a\} \times U$ is disjoint from R^a_1. Now suppose $u^a \approx t^a$. Then $(\lambda^a(u^a))_0$ has the same marginals as $\lambda^a(t^a))_0$, so $u^a \in U$ and hence $(r^a, u^a) \in \{r^a\} \times U$. Since $\{r^a\} \times U$ is open and disjoint from R^a_1, we have $(r^a, u^a) \notin \mathrm{cl}(R^a_1)$. By Lemma E.7, there are uncountably many u^a such that $u^a \approx t^a$, so $R^a_0 \setminus \mathrm{cl}(R^a_1)$ is uncountable. ■

Lemma F.2 *Suppose that $m \geq 1$ and $R^b_{m-1} \setminus \mathrm{cl}\left(R^b_m\right)$ is uncountable. Then $R^a_m \setminus \mathrm{cl}\left(R^a_{m+1}\right)$ is uncountable.*

Proof. The proof is similar to the proof of Lemma E.8. Fix $(s^a, t^a) \in R^a_m$. By the proof of Theorem 7.1, we can choose t^a so that $\lambda^a(t^a) = \sigma = \left(\mu_0, \ldots, \mu_{m-1}\right)$ and R^b_{m-1} is assumed at level 0. We will get uncountably many points $(s^a, u^a) \in R^a_m \setminus \mathrm{cl}(R^a_{m+1})$ by adding one more measure to the beginning of the sequence σ and using Lemma E.7.

We claim that there is a finite set $U \subseteq R^b_{m-1} \setminus R^b_m$ such that $\operatorname{proj}_{S^b} U = \operatorname{proj}_{S^b} R^b_{m-1}$ and $\mu_i(U) = 0$ for all $i < m$.

$m = 1$: Recall that, for each $(s^a, t^a) \in R^a_1$, there is a u^a such that $\lambda^a(u^a)$ is a full-support LPS and $(s^a, u^a) \in R^a_1 \setminus R^a_2$. (This was shown in the proof of Lemma E.8.) The claim for $m = 1$ now follows from Lemma E.7 and the fact that S^a is finite.

$m \geq 2$: The claim was already established in the induction step of Lemma E.8.

Now, since $R^b_{m-1} \setminus \mathrm{cl}(R^b_m)$ is uncountable, there is a point $(s^b, t^b) \in R^b_{m-1} \setminus \mathrm{cl}(R^b_m)$ such that $\mu_i(s^b, t^b) = 0$ for all $i < m$. Therefore we may also take U to contain such a point (s^b, t^b). Let ν be a probability measure such that $\nu(U) = 1, \mathrm{marg}_{S^b} \nu = \mathrm{marg}_{S^b} \mu_0$, and $\nu(s^b, t^b) = \mathrm{marg}_{S^b} \mu_0(s^b)$. Since R^b_{m-1} is assumed under σ at level 0, we have $(s^b, t^b) \in \mathrm{Supp}\, \mu_0$, and thus $\mu_0(\{s^b\} \times T^b) = \mathrm{marg}_{S^b} \mu_0(s^b) > 0$. Therefore $\nu(s^b, t^b) > 0$.

Let τ be the sequence $(\nu, \mu_0, \ldots, \mu_{m-1})$. By completeness there is a $v^a \in T^a$ with $\lambda^a(v^a) = \tau$. Then $(\lambda^a(v^a))_0 = \nu$. As in Lemma E.8, we have $(s^a, v^a) \in R^a_m$. Since (s^a, t^a) is rational, the tail $(\mu_0, \ldots, \mu_{m-1})$ already has full support. Because of this, the proof of Lemma E.7(ii) shows that there are uncountably many $u^a \approx v^a$ such that $(\lambda^a(u^a))_0 = \nu$.

Suppose $u^a \approx v^a$ and $(\lambda^a(u^a))_0 = \nu$. Then $\lambda^a(u^a)$ has length $m+1$. By Lemma E.7, we have $(s^a, u^a) \in R^a_m$. However, since $(s^b, t^b) \notin \mathrm{cl}(R^b_m)$, the measure ν has an open neighborhood W where, for each $\nu' \in W$, $\nu'\left(R^b_m\right) < 1$. (An example of such a neighborhood is the set $\{\nu' : \nu'(V) > \nu(s^b, t^b)/2\}$ where V is an open neighborhood of (s^b, t^b) which is disjoint from R^b_m.) Then the set

$$X = \{\tau \in \mathcal{N}_{m+1}(S^b \times T^b) : \tau_0 \in W\}$$

is an open neighborhood of $\lambda^a(u^a)$, and no LPS $\xi \in X$ can assume R^b_m at level 0. It follows that an LPS $\xi \in X$ cannot assume all of the $m+1$ sets $R^b_k, k \leq m$, because by the inductive hypothesis all these sets have different closures, and hence by Property 4.2 at most one can be assumed at each level. By continuity of λ^a, the set $Y = (\lambda^a)^{-1}(X)$ is an open neighborhood of u^a. Then $\{s^a\} \times Y$ is an open neighborhood of (s^a, u^a) which is disjoint from R^a_{m+1}, so (s^a, u^a) is not in the closure of R^a_{m+1}. By Lemma E.7, there are uncountably many $u^a \approx v^a$, and therefore $R^a_m \setminus \mathrm{cl}(R^a_{m+1})$ is uncountable. ■

Proof of Theorem 8.1. By Proposition 5.1(ii), it suffices to assume that λ^a, λ^b are continuous. As such, Lemma F.1 gives that the set $R^a_0 \setminus \mathrm{cl}(R^a_1)$ is uncountable. Suppose that $(s^b, t^b) \in \bigcap_m R^b_m$. Then, for each m, we have that R^a_m is assumed under $\lambda^b(t^b)$ at some level $j(m)$. Moreover, the sequence $j(m)$ is non-increasing. Using Lemma F.2, we see by induction that for each m, the set $R^a_{2m} \setminus \mathrm{cl}(R^a_{2m+1})$ is uncountable and the set $R^b_{2m+1} \setminus \mathrm{cl}(R^b_{2m+2})$ is uncountable. Then by Property 4.2, for each m, we have $j(2m+1) < j(2m)$. But this contradicts the fact that $\lambda^b(t^b)$ has finite length. ■

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