Generalized Laws of Large Numbers for the Simulation of Dynamic Economies

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In this paper we are concerned with the simulation of stationary dynamic economies that go beyond the convex representative agent framework. We allow for non-continuous Markovian laws of motion that may occur in models with discrete choices, multiple equilibria, bargaining power, optimal monetary and fiscal policies, heterogeneous agents, and market distortions. We prove the existence of an invariant distribution for the equilibrium correspondence, and establish some convergence and accuracy properties for the simulated moments. We obtain these results without imposing arbitrary convexity assumptions on the equilibrium correspondence.

KEY WORDS: Markov equilibrium, invariant distribution, computed solution, simulated moments.

1 Introduction

In this paper we are concerned with the simulation of dynamic economic models. Simulations are mechanically implemented in macroeconomics and other disciplines for assessing model's predictions, estimation, and testing [Cooley and Prescott (1965) and Peralta-Alva and Santos (2010)]. But an obvious question is how these models should be simulated. For many models there is no clear answer, since we lack a general theory that can justify the validity of these simulations.

Available laws of large numbers for non-linear dynamical systems rely on the ergodic theorem [e.g., Arnold (1998) and Kifer (1986)]. The ergodic theorem is not practical because the starting point may not belong to the ergodic set, and this set is usually hard to locate in economic applications. As discussed in Stokey, Lucas and Prescott (1989, Ch. 11), a certain technical condition – known as Hypotesis D in Doob (1953) – insures that for every starting point the dynamical system will enter one its ergodic points almost surely. Condition H seems hard to check in many applications. In our earlier work [Santos and Peralta-Alva (2005)] we provide some generalized laws of large numbers for continuous policy functions. There functions are encountered in representative agent economies or convex dynamic optimization. Here we go a step further, and consider a simple Markovian law of motion as defined by an upper semicontinuous correspondence on a compact set. We show existence of an invariant distribution, and establish some laws of large numbers that guarantee convergence of the simulated moments to the population moments of these invariant distributions. These results apply to several classes of models whose equilibrium solutions cannot be characterized by the techniques of convex dynamic programming. We have in mind models with non-continuous solutions stemming from discrete choices and non-convexities, multiple equilibria, bargaining power and strategic behavior, heterogeneous agents, various market frictions, numerical approximations, learning, and approximate equilibria. Under standard conditions the equilibrium set of these economies is an upper semicontinuous correspondence but may not contain a continuous Markovian selection. Therefore, the current paper opens the door for the simulation, estimation, and testing of more general models widely used in macroeconomics, finance, and several other disciplines.

The lack of continuity of the equilibrium selection is formally addressed in Blume (1982) and Duffie *et al*. (1994) who work directly with equilibrium correspondences. These authors substantiate existence of an invariant distribution by imposing a convexity condition on the set of equilibria. Convexity is not guaranteed by primitive assumptions, and amounts to a randomization over the set of exact equilibria by lotteries or added noise. We dispense with the convexity condition. We therefore focus on the simulation of exact equilibria. Of course, in practise exact equilibria may not be available, since a researcher may only have access to numerical solutions. Consequently, we also address the approximation of equilibrium correspondences. Our results extend naturally to these numerical solutions so that we also establish the convergence of the simulated moments to those of some invariant distribution of the numerical approximation. Then, we show some accuracy properties for the simulated moments as the numerical error of the correspondence goes to zero.

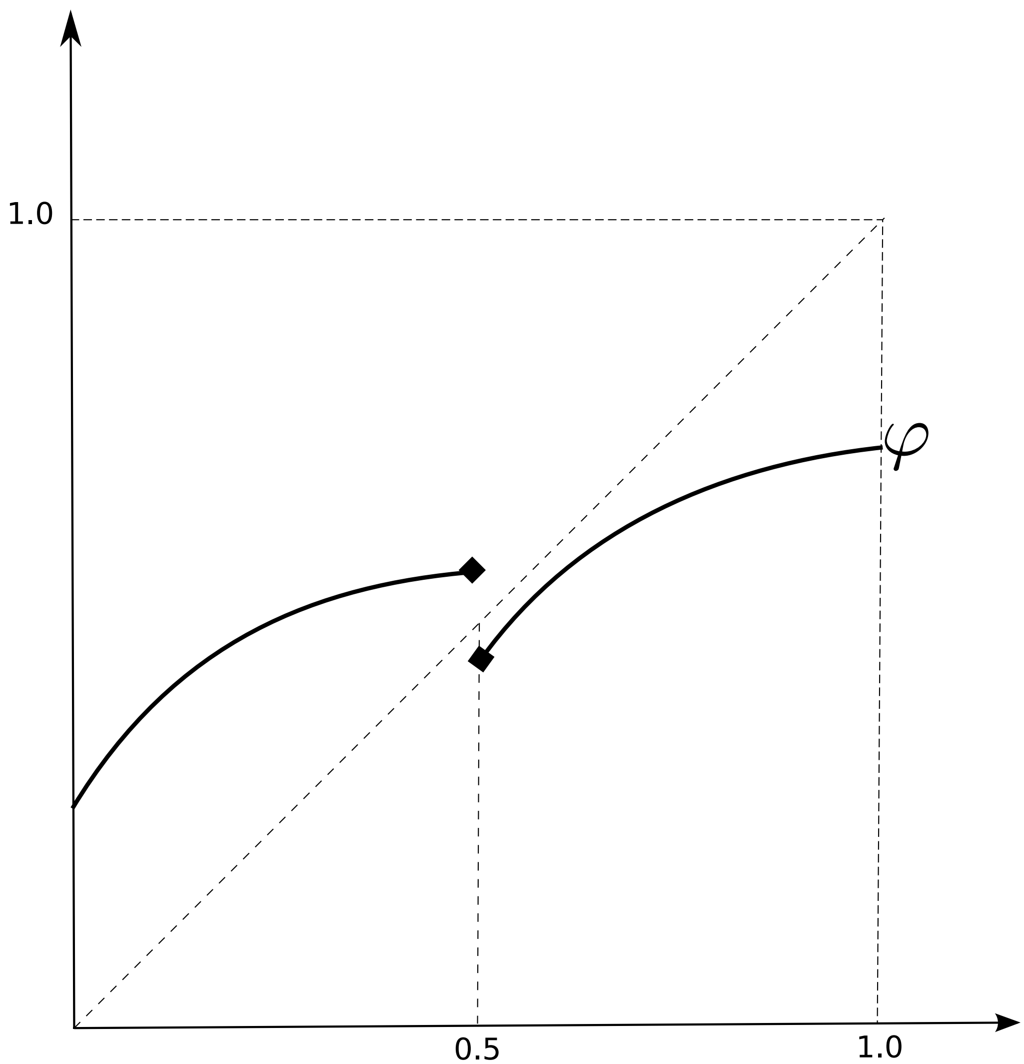
There are two basic ways to establish existence of an invariant distribution [e.g., Crauel (2002)]: (*i*) Via a Markov-Kakutani fixed-point theorem argument, and (*ii*) Via a Krylov-Bolgolyubov type argument, which iterates over some selected initial distribution, and finds the invariant distribution as a limit of these iterations. Blume (1982) and Duffie *et al.* (1994) follow (*i*), and are required to randomize over the equilibrium correspondence. We follow (*ii*), and build on the upper semicontinuity of this correspondence. Therefore, an important contribution of our analysis is to free rigorous simulation exercises from arbitrary randomization devices.

Besides existence of an invariant distribution, a second important technical step is to show equality of the range of variation between the moments computed from simulations and those computed from the invariant distributions of the model. For this result, we rely on two basic contributions in probability theory: (i) *Kingman's subadditive ergodic theorem* [Kingman (1968)] which selects upper and lower bounds for the moments of the simulated paths (almost surely), and (ii) *The ergodic decomposition theorem* [Kifer (1986, Ch. 1)] which provides an integral representation of an invariant measure via ergodic measures.

To make further progress in this discussion, in Section 2 we illustrate our method of analysis with a very simple example. Section 3 lays down a simplified framework that encompasses several dynamic economic models. Our main analytical results are presented in Section 4. A few examples are then discussed in Section 5 to highlight some of the pitfalls that may occur in the simulation of macroeconomic models. We conclude in Section 6 with a further evaluation of our findings as well as their application to economic models.

2 An Example

Let $\varphi : [0,1] \to [0,1]$ be an upper semicontinuous, convex-valued correspondence. As is well known, by the Kakutani fixed-point theorem the correspondence φ has a fixed point, i.e., there is some x^* such that $x^* \in \varphi(x^*)$. In other words, a measurable selection $\hat{\varphi} \subset \varphi$ contains a fixed point $x^* = \hat{\varphi}(x^*)$. The convexity of the correspondence is obtained in some regular economic optimization programs, but convexity may be lost in many other economic situations.

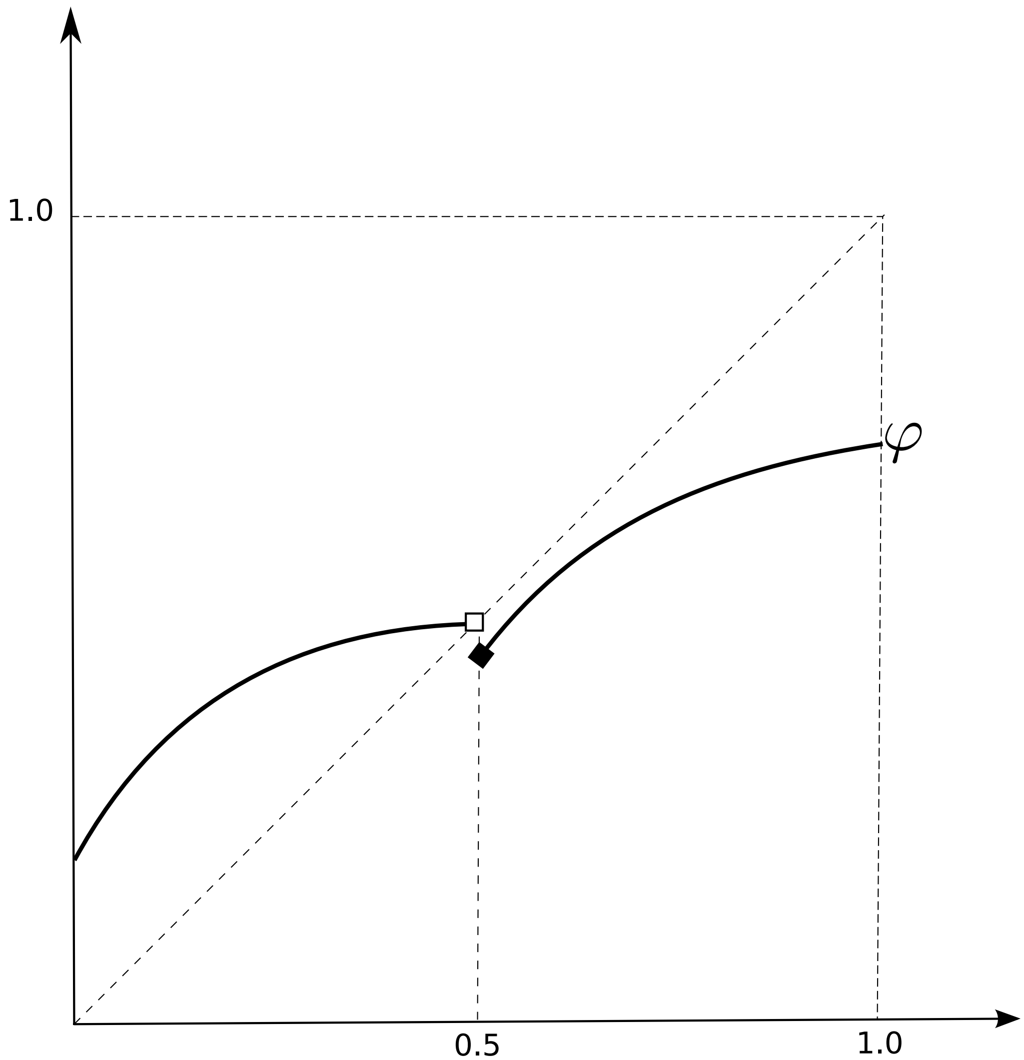


As in Figure 1, we now drop the convexity assumption, and still assume that φ is an upper semicontinuous correspondence. Then, a fixed-point solution x^* may not longer exist. Our analysis will nevertheless show that this correspondence may generate finite cycles or some other invariant set. To locate such stationary solutions, we are going to use a Krylov-Bolgolyubov type argument in which an invariant distribution is obtained as the limit of a convergent subsequence of empirical distributions. The idea is to construct T-period paths. For each path we associate a distribution μ^T that places the same weight to each point in the path. As we extend this path by one more period, the distribution μ^T hardly changes. Then, as T goes to ∞, by the upper semicontinuity of the correspondence we show that a well chosen limiting distribution is indeed a stationary solution.

Our proof is actually a bit more refined as we try to ascertain the range of variation of the moments. We want to select upper and lower bounds for the moments of both the simulated paths and those of the invariant solutions of the model. Assume that f is a function of interest that may represent a certain moment or some other statistic. Again, let $\hat{\varphi} \subset \varphi$ be a measurable selection. Then, to pick up an upper bound for the sample moments we let $\sup_{x_0 \in [0,1]} \frac{1}{T} \sum_{t=1}^{T} f(x_t)$ where $x_1, x_2, \cdots, x_T$ is a sequence generated under function $\hat{\varphi}$. For simplicity, assume that the sup is always attained. Let μ^T be the empirical measure that places the same weight in each point of the path. A limit point of $\{\mu^T\}_{T \geq 0}$ may not be an invariant distribution μ^* for $\hat{\varphi}$. But we show that every limit point of the sequence $\{\mu^T\}_{T \geq 0}$ is an invariant distribution μ^* under the action of some other selection $\hat{\varphi}'$ of our correspondence φ.

Let $I(\varphi)$ be the set of invariant distributions under φ. It follows from the previous argument that $\lim_{t \to \infty} \sup_{x_0 \in [0,1]} \frac{1}{T} \sum_{t=1}^{T} f(x_t) \leq \max_{\mu^* \in I(\varphi)} \int f(x) \mu^*(dx)$, where by construction the limit on the left-hand side does exist. (For stochastic models existence (almost surely) follows from Kingman's subadditive ergodic theorem.) To prove the reverse equality, we will invoke the ergodic decomposition theorem, which partitions the domain of an invariant distribution into ergodic subsets with measures Q. Then, at each ergodic component the expected value $\int f(x) Q(dx)$ attains a maximum, and t population moments are equal to sample moments almost surely.

The upper semicontinuity of the correspondence cannot be dropped. For instance, the correspondence in Figure 2 has no stationary solution and all paths converge to point $x = 1/2$. Hence, all simulations $\frac{1}{T} \sum_{t=1^T f(x_t)}$ converge to $f(1/2)$. Since our focus is on the simulated moments, we could actually consider models in which an invariant distribution μ^* may not exist. The existence of an invariant distribution is mainly needed in our accuracy results: It becomes easier to study changes in the population moments (rather than in the simulated moments) under perturbations of the model.



Therefore, our primary objectives are to show existence of a stationary solution and the convergence of the simulated moments. As in these figures we do not require our correspondence to be convex valued. Hence, we cannot show existence of a simple fixed point x^*.

But we show existence of a more general invariant set, which could be a finite cycle, or a limiting recurrent set. Our constructive algorithm may not pick some general stochastic solutions such as sunspots, but our arguments could be extended to arbitrary randomizations of correspondence φ.

3 The Analytical Framework

Time is discrete, $n = 0, 1, 2, \cdots$. As in most of the quantitative economics literature we assume that the equilibrium law of motion of the state variables can be specified by a time-invariant dynamical system of the following form

$$s_{n+1} \in \varphi(s_n, \varepsilon_{n+1}), \qquad\qquad n = 0, 1, 2, \cdots , \tag{3.1}$$

where $\varphi : S \times E \to S$ is a correspondence. S is the state space, and is endowed with σ-albgebra $\mathbb{S}$. This space is conformed by a subset of economic variables defining a Markovian law of motion. More specifically, vector s may include: (i) *Exogenous state variables* such as some indices of factor productivity or international market prices, (ii) *Predetermined state variables* such as physical capital and financial assets, and (iii) *Endogenous state variables* such as consumption, investment, asset prices and interest rates. In models with multiple equilibria, this latter set of variables may be needed to pin down a Markov equilibrium. We consider that ε *iid* shock that follows a probability law ν on a measurable space $(E, \mathbb{E})$.

Assumption 3.1 *S is compact subset of some finite-dimensional space R^m, and $\mathbb{S}$ is its Borel σ-algebra. E is a compact metric space.*

Assumption 3.2 *Correspondence $\varphi : S \times E \to S$ is upper semicontinuous.*

These assumptions are completely standard. We can think of two ways in which our results could be estended: (i) The domain $S \times E$ may be an unbounded set, and (ii) Correspondence φ may be defined for ν-almost all ε.

To study the dynamics generated by φ, we consider the set of its measurable functions $\hat{\varphi} \in \varphi$. These selections exist by the measurable selection theorem [e.g., Crauel (2002) and

Hildenbrand (1974)]. For each measurable selection $\hat{\varphi}$, it is useful to define the transition probability

$$P_{\hat{\varphi}}(s, A) = \nu(\{\varepsilon | \hat{\varphi}(s, \varepsilon) \in A\}). \tag{2.2}$$

Then, given an initial probability μ_0 on $\mathbb{S}$, the evolution of future probabilities, $\{\mu_n\}$, can be specified by the following operator $T^*_{\hat{\varphi}}$ that takes the space of probabilities on $\mathbb{S}$ into itself

$$\mu_{n+1}(A) = (T^*_{\hat{\varphi}}\mu_n)(A) = \int P_{\hat{\varphi}}(s, A)\mu_n(ds), \tag{2.3}$$

for all A in $\mathbb{S}$ and $n \geq 0$. An invariant probability measure or invariant distribution μ^* is a fixed point of operator $T^*_{\hat{\varphi}}$, i.e., $\mu^* = T^*_{\hat{\varphi}}\mu^*$. This invariant distribution is therefore a stochastic stationary solution of the original system (2.1).

Let $f : S \to R$ be a function of interest defining a moment or some other characteristic. Let $C(S)$ be the space of all continuous real-valued functions f on S. The integral $\int f(s)\mu(ds)$ or expected value of f over μ will be denoted by $E(f)$ whenever distribution μ is clear from the context. The weak topology is the coarsest topology such that every linear functional in the set $\{\mu \to \int f(s)\mu(ds), f \in C(S)\}$ is continuous. A sequence $\{\mu_j\}$ of probability measures on $\mathbb{S}$ is said to converge weakly to a probability measure μ if $\int f(s)\mu_j(ds) \to_j \int f(s)\mu(ds)$ for every $f \in C(S)$. The weak topology is metrizable [e.g., see Billingsley (1968)].
Let $P_{\varphi}(s, \cdot) = \{P_{\hat{\varphi}}(s, \cdot) : \hat{\varphi} \in \varphi\}$ and $T^*_{\varphi} = \{T^*_{\hat{\varphi}} : \hat{\varphi} \in \varphi\}$. Then, $s \to P_{\varphi}(s, \cdot)$ is a correspondence of stochastic probabilities, and often called a multivalued stochastic kernel. We say that $s \to P_{\varphi}(s, \cdot)$ is an upper semicontinuous correspondence in the weak topology of measures if for every fixed $f \in C(S)$ the associated functional $s \to \int f(s')P_{\hat{\varphi}}(s, ds')$, all $P_{\hat{\varphi}}(s, \cdot) \in P_{\varphi}(s, \cdot)$, is an upper semicontinuous correspondence. The following results follow from Blume (1982, Th. 3.1 and Prop. 2.3). These results can be viewed as an extension of the Feller property for correspondences [cf. Stokey, Lucas and Prescott (1989, Ch. 8)].

Theorem 3.1 (Feller property for correspondences) *Under Assumptions 3.1-3.2 the correspondence $P_{\varphi}(s, \cdot)$ is upper semicontinuous in s in the weak topology of measures μ on $\mathbb{S}$.*

A similar result concerns operator T^*_{φ}.

Theorem 3.2 (Feller property for correspondences) *Under the above assumptions the correspondence T_{φ}^* is upper semicontinuous in the the weak topology of measures μ on $\mathbb{S}$.*

4 Results

In preparation for our analysis, we define a new probability space comprising all infinite sequences $\omega = (\varepsilon_1, \varepsilon_2, \cdots)$. Let $\Omega = E^{\infty}$ be the countably infinite cartesian product of copies of E. Let $\mathbb{F}$ be the σ-field in E^{∞} generated by the collection of all cylinders $A_1 \times A_2 \times \cdots \times A_n \times E \times E \times E \times \cdots$ where $A_i \in \mathbb{E}$ for $i = 1, \cdots, n$. A probability measure λ can be constructed over these finite-dimensional sets as

$$\lambda\{\omega : \varepsilon_1 \in A_1, \varepsilon_2 \in A_2, \cdots, \varepsilon_n \in A_n\} = \prod_{i=1}^{n} Q(A_i). \tag{4.1}$$

This measure λ has a unique extension on $\mathbb{F}$. Hence, the triple $(\Omega, \mathbb{F}, \lambda)$ denotes a probability space. Finally, for every initial value s_0 and sequence of shocks $\omega = \{\varepsilon_n\}$, let $\{s_n(s_0, \omega, \hat{\varphi})\}$ be the sample path generated by function $\hat{\varphi}$; that is, $s_{n+1}(s_0, \omega, \hat{\varphi}) = \hat{\varphi}(s_n(s_0, \omega, \hat{\varphi}), \varepsilon_{n+1})$ for all $n \geq 1$ and $s_1(s_0, \omega, \hat{\varphi}) = \hat{\varphi}(s_0, \varepsilon_1)$.

We start with a version of the ergodic subadditive theorem of [Kingman (1968)]. A main advantage of this theorem is that we do not need to know existence of an invariant distribution for function $\hat{\varphi}$. Indeed, function $\hat{\varphi}$ may not even have an invariant probability measure μ^*.

Theorem 4.1 (The subadditive ergodic theorem) *Consider a measurable selection $\hat{\varphi} \in \varphi$. Let f belong to $C(S)$. Then, under Assumptions 3.1-3.2 there are constants $L(f, \hat{\varphi})$ and $U(f, \hat{\varphi})$ such that for λ-almost all ω,*

$$(i) \quad \lim_{N\to\infty} (inf_{s_0 \in S}[\frac{1}{N}\sum_{n=1}^{N} f(s_n(s_0, \omega, \hat{\varphi}))]) = L(f, \hat{\varphi}) \tag{4.2a}$$

$$(ii) \quad \lim_{N\to\infty} (sup_{s_0 \in S}[\frac{1}{N}\sum_{n=1}^{N} f(s_n(s_0, \omega, \hat{\varphi}))]) = U(f, \hat{\varphi}). \tag{4.2b}$$

The importance of Theorem 4.1 is that it establishes both upper and lower bounds for the simulated moments, and these bounds are the same for λ-almost all ω. The equality of the bounds follows from the *iid* process $\{\varepsilon_t\}$. This result could be extended to general ergodic stochastic systems.

We now prove our central result on the existence of an invariant probability measure. As before, we say that $P_\varphi(s,\cdot)$ has an invariant probability μ^* if there is $P_{\hat{\varphi}}(s,\cdot) \in P(s,\cdot)$ such that $\mu^*(A) = (T^*_{\hat{\varphi}}\mu^*)(A) = \int P_{\hat{\varphi}}(s,A)\mu^*(ds)$, for all A in $\mathbb{S}$.

Theorem 4.2 (Existence of an invariant probability measure) *The transition correspondence $P_\varphi(s,\cdot)$ has an invariant probability μ^*.*

As already pointed out the proof of this theorem builds on a Krylov-Bogolyubov argument [Crauel (2002)]. This result illustrates that convexity is not needed for the existence of invariant measures. It is well understood however that convexity may be needed for the existence of special fixed-point solutions such as point $x^* \in \varphi(x^*)$.

An ergodic decomposition of an invariant measure μ^*, or more precisely an ergodic decomposition of of a $\hat{\varphi}$-invariant measure μ^* into its ergodic components Q, is a measure ρ satisfying $\rho(I(\hat{\varphi})) = 1$, and $\mu^* = \int_S Q\rho(dQ)$. For every $f \in C(S)$ we then have $\int f(s)\mu(ds) = \int_{I(\hat{\varphi})}(\int(f(s)Q(ds))\rho(dQ)$.

Theorem 4.3 (The ergodic decomposition theorem) *Assume that μ^* is an invariant probability measure for $P(s,\cdot)$. Then, μ^* has an invariant decomposition $\mu^* = \int_S Q\rho(dQ)$.*

See Kifer (1968) and Klunger (1998) for various general conditions that guarantee the existence of an ergodic decomposition.

Let $E^{inf}(f) = \inf_{\mu^* \in I(\varphi)} \int f(s)\mu^*(ds)$, and $E^{sup}(f) = \sup_{\mu^* \in I(\varphi)} \int f(s)\mu^*(ds)$, where as before $I(\varphi)$ is the set of invariant probabilities for φ. Our next result shows that the range of variation of the sample moments is equal to the range of variation of the population moments. This is like a generalized law of large numbers for stochastic systems with multiple invariant probabilities. A similar result is proved in Santos and Peralta-alva (2005) under stronger assumptions.

Theorem 4.4 (A generalized law of large numbers) *Consider a measurable selection $\hat{\varphi} \in \varphi$. Let f belong to $C(S)$. Then, under Assumptions 3.1-3.2 for λ-almost all ω,*

$$(i) \quad \inf_{\hat{\varphi}\in\varphi} \lim_{N\to\infty} (inf_{s_0\in S}[\frac{1}{N}\sum_{n=1}^{N} f(s_n(s_0,\omega,\hat{\varphi}))]) = E^{inf}(f) \tag{3.2a}$$

$$(ii) \quad \sup_{\hat{\varphi}\in\varphi} \lim_{N\to\infty} (sup_{s_0\in S}[\frac{1}{N}\sum_{n=1}^{N} f(s_n(s_0,\omega,\hat{\varphi}))]) = E^{sup}(f). \tag{3.2b}$$

Let $E(\varphi)$ be the set of ergodic measures for φ. Then, by the ergodic decomposition theorem we can define $E^{inf}(f) = \inf_{\mu^*\in E(\varphi)} \int f(s)\mu^*(ds)$, and $E^{sup}(f) = \sup_{\mu^*\in E(\varphi)} \int f(s)\mu^*(ds)$. The above theorem does not follow from the ergodic theorem as it applies to any arbitrary conditions $s_0 \in S$ that may not be contained in any ergodic set.

We now apply the above results to the numerical simulation of stochastic dynamic models. A researcher is concerned with the predictions of a stochastic dynamic model whose equilibrium law of motion can be specified by a correspondence φ. Usually, this Markovian solution φ does not have an analytical representation, and so it is approximated by numerical methods. Moreover, the invariant distributions or stationary solutions of the numerical approximation cannot be calculated analytically. Hence, numerical methods are again brought up into the analysis. This time in connection with some law of large numbers.

As is typical in the simulation of stochastic models we suppose that the researcher can draw sequences $\{\widehat{\varepsilon}_n\}$ from a generating process that can mimic the distribution of the shock process $\{\varepsilon_n\}$. A probability measure λ is defined over all sequences $\omega = (\varepsilon_1, \varepsilon_2, ...)$. Once a numerical approximation $\hat{\varphi}_j$ is available, it is generally not so costly to generate sample paths $\{s_n(s_0,\omega,\hat{\varphi}_j)\}$ defined recursively as $s_{n+1}(s_0,\omega,\hat{\varphi}_j) = \hat{\varphi}_j(s_n(s_0,\omega,\hat{\varphi}_j), \varepsilon_{n+1})$ for every $n \geq 0$ for fixed s_0 and ω. Averaging over these sample paths we get sequences of simulated moments or characteristics $\{\frac{1}{N}\sum_{n=1}^{N} f(s_n(s_0,\omega,\hat{\varphi}_j))\}$ as defined by some functions of interest f. We suppose that a sequence $\{\hat{\varphi}_j\}$ may approach a selection $\hat{\varphi} \in \varphi$ in the sup norm [cf. Santos (1999)]. We next show that for a sufficiently good numerical approximation $\hat{\varphi}_j$ and for a sufficiently large N the series $\{\frac{1}{N}\sum_{n=1}^{N} f(s_n(s_0,\omega,\hat{\varphi}_j))\}$ is close (almost surely) to the range of variation of the simulated moments over correspondence φ.

Theorem 4.5 (Accuracy of the simulated moments) *Consider a sequence of measurable functions $\{\hat{\varphi}_j\}$ that approach correspondence φ. Let f belong to $C(S)$. Then, under*

Assumptions 3.1-3.2 for every $\eta > 0$ there are functions $N_j(w)$ and an integer J such that for all $j \geq J$ and $N \geq N_j(\omega)$,

$$\inf_{\hat{\varphi} \in \varphi} L(f, \hat{\varphi}) - \eta < \frac{1}{N} \sum_{n=1}^{N} f(s_n(s_0, \omega, \hat{\varphi}_j)) < \sup_{\hat{\varphi} \in \varphi} U(f, \hat{\varphi}) + \eta \tag{3.3}$$

for all s_0 and λ-almost all ω.

Note that $L(f, \hat{\varphi})$ and $U(f, \hat{\varphi})$ are defined in Theorem 4.1. If φ has a unique invariant probability we have that $\inf_{\hat{\varphi} \in \varphi} L(f, \hat{\varphi}) = \sup_{\hat{\varphi} \in \varphi} U(f, \hat{\varphi})$. Hence, by Theorem 4.5 there is convergence of the simulated moments from numerical approximations to the simulated moments of the model. For the case of multiple invariant distributions, however, Theorem 4.5 shows that the moments from numerical approximations with eventually be located (up to small error) within the range of variation of the simulated moments of the original model.

5 Numerical Examples

Since most interesting dynamic economic models cannot be solved analytically, numerical approximations are employed to derive the quantitative implications of the theory. Moreover, the vast majority of existing quantitative work ignores the technical issues discussed this far. Instead, it is assumed that the correspondence driving the dynamical system (3.1) is a continuous function on a minimal state space constituted by the vector of predetermined endogenous variables and shocks. This section shows models where this common practice results in substantial biases. The theoretical developments of this paper yield foundations for the correct simulation of the very same examples where the standard approach fails.

A Growth Model with Taxes

The economy is made up of a representative household and a single firm. For a given sequence of interest rates $\{r_t\}$ and taxes $\{\tau_t\}$ the representative household solves the following optimization problem

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

$$s.t.$$

$$c_t + k_{t+1} \leq \pi_t + (1-\tau_t)\, r_t k_t + T_t$$

$$k_0 \text{ given, } 0 < \beta < 1,$$

$$c_t \geq 0,\ k_{t+1} \geq 0 \text{ for all } t \geq 0.$$

Here, c_t denotes consumption, k_t is the individual capital holdings. Taxes on capital income $\{\tau_t\}$ are functions of the aggregate capital stock K_t. All tax revenues are rebated back to the consumer as lump-sum transfers T_t.

The representative firm is subject to a sequence of productivity shocks $\{z_t\}$ and seeks to maximize one-period profits by employing the optimal amount of capital

$$\pi_t = \max_{K_t} z_t\ f(K_t) - r_t K_t.$$

Let

$$f(K) = K^{1/3},\ \beta = 0.95,\ \ z_t = 1 \text{ for all } t.$$

Also, consider the following piecewise linear tax schedule on capital rents:

$$\tau(K) = \begin{cases} 0.10 & \text{if } \ K \leq 0.160002 \\ 0.05 - 10(K - 0.165002) & \text{if } \ 0.160002 \leq K \leq 0.170002 \\ 0 & \text{if } \ K \geq 0.170002. \end{cases}$$

This specification for the model is useful for our purposes because Santos (2002, Prop. 3.4) shows that a continuous Markov equilibrium fails to exist. This deterministic version of the model can also be solved up-to machine accuracy by a simple "shooting" algorithm. There are three steady states: The middle steady state is unstable and has two complex eigenvalues while the other two steady states are saddle-path stable.

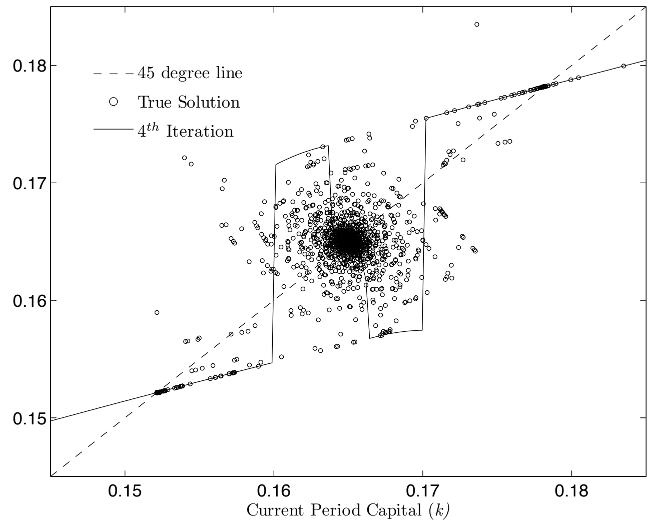
Standard algorithms approximating the Euler equation would solve for a continuous policy function of the form

$$k_{t+1} = \varphi(k_t, \xi),$$

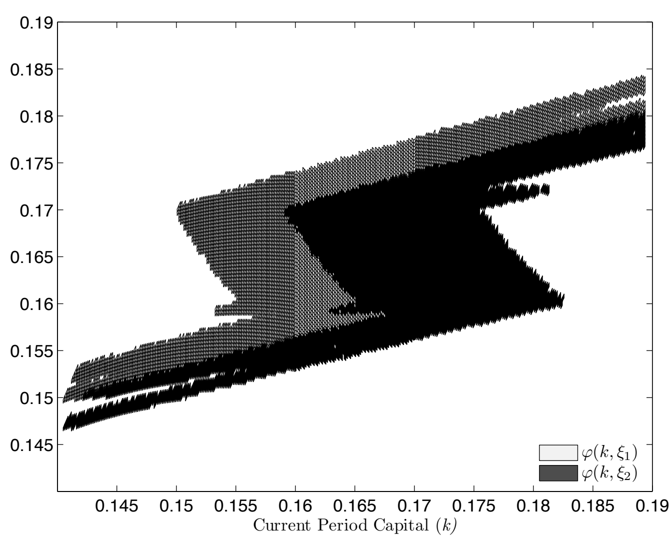
where φ belongs to a finite dimensional space of continuous functions as defined by a vector of parameters ξ. We obtain an estimate for ξ by forming a discrete system of Euler equations over as many grid points k^i as the dimensionality of the parameter space:

$$u'(k^i, \varphi(k^i, \xi)) = \beta u'(\varphi(k^i, \xi), \varphi(\varphi(k^i, \xi), \xi)) \cdot \left[f'(\varphi(k^i, \xi))(1 - \tau(\varphi(k^i, \xi))) \right].$$

We assume that $\varphi(k^i, \xi)$ belongs to the class of piecewise linear functions, and employ a uniform grid of 5000 points over the domain $k \in [0.14..0.19]$. The resulting approximation, together with the highly accurate solution using our algorithm are illustrated in Figure 3. This approximation of the Euler equation over piecewise continuous functions converged up to computer precision in only 3 iterations. This fast convergence is actually deceptive because as pointed out above no continuous policy function does exist. Indeed, the dynamic behavior implied by the continuous function approximation is quite different from the true one as it displays four more steady states, and changes substantially the basins of attraction of the original steady states (see Figure 3). A further test of the fixed-point solution of this algorithm based on the Euler equation residuals produced mixed results. First, the average Euler equation residual (a standard accuracy measure) over the domain of feasible capitals is fairly small, i.e. it is equal to 0.0073. Second, the maximum Euler equation residual is slightly more pronounced in a small area near the unstable steady state. But even in that area, the error is not extremely large: In three tiny intervals the Euler equation residuals are just around 0.06. Therefore, from these computational tests a researcher may be led to conclude that the purported continuous policy function should mimic well the true equilibrium dynamics.

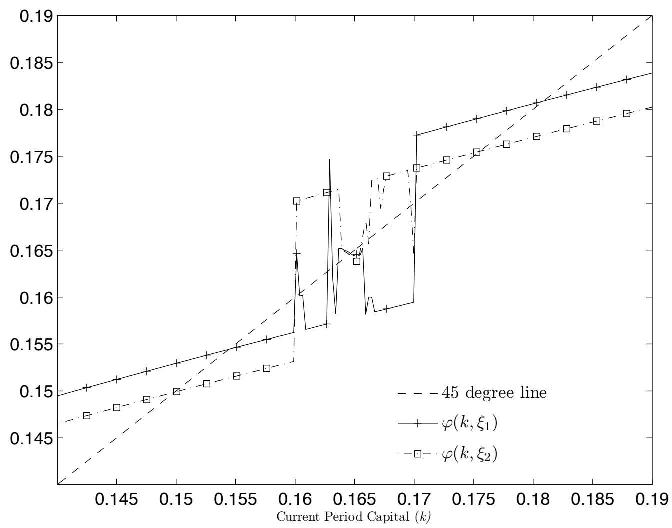


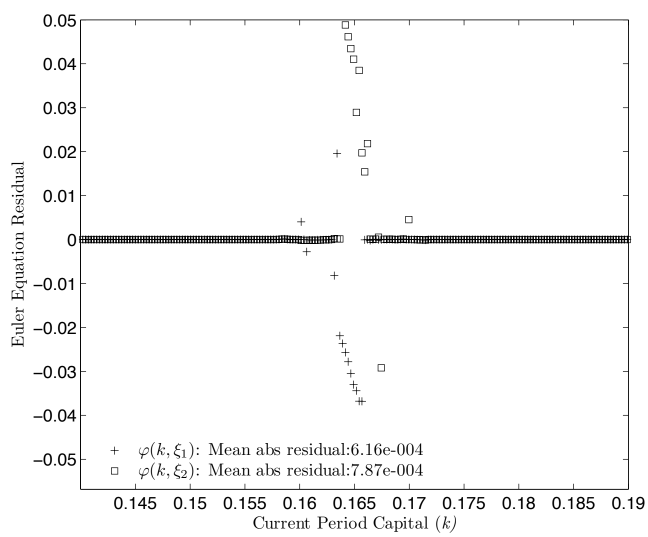
Shooting can no loger be used to derive a solution for a stochastic version of the model. Specifically, consider the case where $z_t \in \{0.99, 1.01\}$ for all t, and it also iid. The only reliable method for the solution of such an economy we are aware of is that developed by Feng et. al. (2009). We applied that method to solve for approximated equilibrium correspondences. Figure 4 summarizes the resulting correspondences.



From these correspondences, we were able to generate arbitrarily long time series with Euler equation residuals close to zero at every point of the simulation. We generated many of such time series, departing from different initial conditions. In the long run, these approximate equilibrium time series always converge (quite rapidly) to oscillate around either the highest or the lowest steady state of the deterministic version of the model. This gives numerical evidence of the existence of two invariant distributions.

From a standard algorithm that iterates over functions of the form $\varphi(k_t, z_t, \xi)$, we can construct the functions in Figures 5, which have relativley low average Euler equation residuals, as can be seen in Figure 6, and three invariant distributions. One of them close to the middle, unstable, steady state of the deterministic version of the model.





An overlapping generations economy

We study a version of the economy analyzed by Kubler and Polemarchakis (2004). At each date event 2 individuals commence their lives. They live for 2 periods. At each date-event there are 2 perishable commodities available for consumption and two assets. The first asset is a one-period risk-free bond, which pays one unit of the numeraire at all successors of z^t. This asset is in zero net supply. A Lucas tree might also be available at time zero. There is one share of ownership to this tree and it entitles the holder to a random stream of dividends equal to $d(z_t)$ units of consumption good 1. The market value (or price per share) of this asset is $q^s(z^t)$. We denote by $\theta^{b,i,z^\tau}, \theta^{s,i,z^\tau}$ the holdings of bonds and shares of agent (i, z^τ). Shares cannot be sold short.

The intertemporal objective of agent of type one is

$$-\frac{1024}{\left(c_1^1\left(z^t\right)\right)^4} + E_{z_{t+1}|z^t}\left[-\frac{1024}{\left(c_1^1\left(z^{t+1}\right)\right)^4} - \frac{1}{\left(c_2^1\left(z^{t+1}\right)\right)}\right]$$

while that of agent of type 2 is given by

$$-\frac{1}{\left(c_1^2\left(z^t\right)\right)^4} + E_{z_{t+1}|z^t}\left[-\frac{1}{\left(c_1^2\left(z^{t+1}\right)\right)^4} - \frac{1024}{\left(c_2^2\left(z^{t+1}\right)\right)}\right]$$

In the first period of life of each agent, endowments are stochastic (and depend only on the current state z_t), while in the second period they are deterministic. In particular, $e^{1,z^t}\left(z^{t+1}\right) = (12, 1)$; and $e^{2,z^t}\left(z^{t+1}\right) = (1, 12)$. Each individual faces the following budget constraint

$$\begin{aligned} p(z^t) \cdot c^{i,z^\tau}(z^t) + q^s(z^t)[\theta^{s,i,z^\tau}(z^{t+1}) - \theta^{s,i,z^\tau}(z^t)] + q^b(z^t)[\theta^{b,i,z^\tau}(z^{t+1})] - \\ (p(z^t)e^{i,z^\tau}(z_t) + \theta^{b,i,z^\tau}(z^t) + \theta^{s,i,z^\tau}(z^t)d(z_t)) = 0, \quad (1) \\ 0 \leq \theta^{s,i,z^\tau}(z^{t+1}) \quad (2) \end{aligned}$$

An **equilibrium** for this economy is sequences of prices, $(q^b(z^t), q^s(z^t), p(z^t))$, consumption allocations, and asset holdings for all agents, such that each agent maximizes her expected utility subject to individual budget constraints, bond holdings add up to zero, share holdings add up to one, and consumption allocations add to the aggregate endowment at all possible date-events.

For this model, it is easy to show that competitive equilibrium exists. Practitioners are, however, interested competitive equilibrium that have a recursive structure on the space of shocks and distribution of wealth. Specifically, standard computational methods employed in quantitative analyses search for continuous, time invariant, functions f such that:

$$\begin{aligned}
\hat{c}_1^{i,\sigma_t}\left(z^t\right) &= f_c\left(z_t, \theta^{1,z^{t-1}}\right) \\
\hat{c}_l^{i,\sigma_{t-1}}\left(z^t\right) &= f_{cl}\left(z_{t,}\theta^{1,z^{t-1}}\right) \\
\hat{\theta}^{1,z^t} &= f_\theta\left(z_t, \theta^{1,z^{t-1}}\right) \\
\hat{p}\left(z^t\right) &= f_\theta\left(z_t, \theta^{1,z^{t-1}}\right) \\
\hat{q}\left(z^t\right) &= f_q\left(z_t, \theta^{1,z^{t-1}}\right).
\end{aligned}$$

Kubler and Polemarchakis show that, if no shares are available, such a representation does not exist for this economy. Specifically, equilibria can be characterized as follows.

1. There is a unique equilibrium.

2. $\theta^1 = 0$ at all z^t.

3. Given node z^{t-1} with $z_{t-1} = z_1$, we have that for all successors of z^{t-1}, namely $z^t = z^{t-1}s_1$ or $z^t = z^{t-1}s_2$: $\left(c_1^{1,z^{t-1}}\left(z^t\right), c_2^{1,z^{t-1}}\left(z^t\right)\right) = (10.4, 2.6)$, $\left(c_1^{2,z^{t-1}}\left(z^t\right), c_2^{2,z^{t-1}}\left(z^t\right)\right) = (2.6, 10.4)$, and $p = 1$.

4. Given node z^{t-1} with $z_{t-1} = z_2$, we have that for all successors of z^{t-1}, namely $z^t = z^{t-1}s_1$ or $z^t = z^{t-1}s_2$: $\left(c_1^{1,z^{t-1}}\left(z^t\right), c_2^{1,z^{t-1}}\left(z^t\right)\right) = (8.4, 1.4)$, $\left(c_1^{1,z^{t-1}}\left(z^t\right), c_2^{1,z^{t-1}}\left(z^t\right)\right) = (4.6, 11.6)$, and $p = 7.9$.

Hence, knowledge of the current shock and wealth distribution are not enough to characterize consumption of the old. Information about the shock that took place when the old were born is required.

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