Implementation with Interdependent Valuations*

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Abstract

It is well-known that the ability of the Vickrey-Clarke-Groves (VCG) mechanism to implement efficient outcomes for private value choice problems does not extend to interdependent value problems. When an agent's type affects other agents' utilities, it may not be incentive compatible for him to truthfully reveal his type when faced with CGV payments. We show that when agents are informationally small, there exist small modifications to CGV that restore incentive compatibility. We further show that truthful revelation is an approximate ex post equilibrium. Lastly, we show that in replicated settings aggregate payments suffcient to induce truthful revelation go to zero.

Keywords: Auctions, Incentive Compatibility, Mechanism Design, Interdependent Values, Ex Post Incentive Compatibility

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1. Introduction

There is a large literature aimed at characterizing the social choice functions that can be implemented in Bayes Nash equilibria. This literature typically takes agents' information as exogenous and fixed throughout the analysis. For some problems this may be appropriate, but the assumption is problematic for others. A typical analysis, relying on the revelation principle, maximizes some objective function subject to an incentive compatibility constraint requiring that truthful revelation be a Bayes-Nash equilibrium. It is often the case that truthful revelation is not ex post incentive compatible, that is, for a given agent, there are some profiles of the other agents' types for which the agent may be better off by misreporting his type than by truthfully revealing it. Truthful revelation, of course, may still be a Bayes equilibrium, because agents announce their types without knowing other agents' types: choices must be made on the basis of their beliefs about other agents' types. The assumption that agents' information is exogenous can lead to a difficulty: if truthful revelation is not ex post incentive compatible, then agents have incentives to learn other agents' types. To the extent that an agent can, at some cost, learn something about the types of other agents, then agents' beliefs at the stage at which agents actually participate in the mechanism must be treated as endogenous: if an agent can engage in preplay activities that provide him with some information about other agents' types, then that agent's beliefs when he actually plays the game are the outcome of the preplay activity.

A planner who designs a mechanism for which truthful revelation is ex post incentive compatible can legitimately ignore agents' incentives to engage in espionage to discover other agents' types, and consequently, ex post incentive compatibility is desirable. The Vickrey-Clarke-Groves- mechanism (hereafter VCG)[1] for private values environments is a classic example of a mechanism for which truthful revelation is ex post incentive compatible. For this mechanism, each agent submits his or her valuation. The mechanism selects the outcome that maximizes the sum of the agents' submitted valuations, and prescribes a transfer to each agent. These transfers can be constructed in such a way that it is a dominant strategy for each agent to reveal his valuation truthfully. Cremer and McLean (1985) (hereafter CM) consider a similar problem in which agents have private information, but interdependent valuations; that is, each agent's valuation can depend on other agents' information. They consider the mechanism design problem in which the aim is to maximize the revenue obtained from auctioning an

[1] See Clarke (1971), Groves (1973) and Vickrey (1961).

object. They analyze revelation games in which agents announce their types, and construct special transfers different from those in the VCG mechanism. Because each agent's valuation depends on other agents' announced types, truthful revelation will not generally be a dominant strategy in the CM mechanism. They show, however, that under certain conditions[2] truthful revelation will be ex post incentive compatible, i.e., the truth is an ex-post Nash equilibrium.

There has recently been renewed interest in mechanisms for which truthful revelation is ex post incentive compatible. Dasgupta and Maskin (2000), Perry and Reny (2002) and Ausubel (1999) (among others) have used the solution concept in designing auction mechanisms that assure an efficient outcome. Chung and Ely (2001) and Bergemann and Morris (2003) analyze the solution concept more generally. These papers (and Cremer and McLean), however, essentially restrict attention to the case in which an agent's private information is one dimensional[3], a serious restriction for many problems. Consider, for example, a problem in which an oil field is to be auctioned, and each agent may have private information about the quantity of the oil in the field, the chemical characteristics of the oil, the capacity of his refinery to handle the oil and the demand for the refined products in his market, all of which affect this agent's valuation (and potentially other agents' valuations as well). While the assumption that information is single dimensional is restrictive, it is necessary: Jehiel *et al.* (2006) show that for general mechanism design problems with interdependent values and multidimensional signals, for nearly all valuation functions, truthful revelation will be an ex post equilibrium only for trivial outcome functions.

Thus, it is only in the case of single dimensional information that we can hope for ex post equilibria for interdependent value problems. But even in the single dimensional case, there are difficulties. Most work on mechanism design in problems with asymmetric information begins with utilities of the form $u_i(c; t_i, t_{-i})$, where c is a possible outcome, t_i represents agent i's private information and t_{-i} is a vector representing other agents' private information. In the standard interpretation, u_i is a reduced form utility function that defines the utility of agent i for the outcome c under the particular circumstances likely to obtain given the agent's information. In the oil field problem above, for example, an agent's utility for the oil may depend on (among other things) the amount and chemical composition of the oil and the future demand for oil products, and the information of other

[2] The conditions are discussed in section 3.

[3] Formally, what is necessary is that agents' types are ordered in a particular way that typically fails in multidimensional information settings.

agents will affect i's (expected) value for the field insofar as i's beliefs about the quantity and composition of the oil and the demand for oil products are affected by their information. In this paper, we begin from more primitive data in which i has a utility function $v_i(c, \theta; t_i)$ where θ is a complete description of the state of nature and t_i represents his private information. For the oil example, θ would include those things that affect i's value for the oil – the amount and composition of the oil, the demand for oil, etc. The relationship between agents' private information and the state is given by a probability distribution P over $\Theta \times T$. This formulation emphasizes the fact that the information possessed by other agents will affect agent i precisely to the extent that the information of others provides information about the state of nature.

The reduced form utility function that is normally the starting point for mechanism design analysis can be calculated from this more primitive structure: $u(c, t) \equiv \Sigma_\theta v_i(c, \theta; t) P(\theta|t)$. Most work that employs ex post incentive compatibility makes additional assumptions regarding the reduced form utility functions u_i. It is typically assumed that each agent's types are ordered, and that agents' valuations are monotonic in any agent's type. Further, it is assumed that the utility function of each individual agent satisfies a classic single-crossing property and that, across agents, their utilities are linked by an "interagent crossing property." This latter property requires that a change in an agent's type from one type to a higher type causes his valuation to increase at least as much as any other agent's valuation. We show that the conditions on the primitive data of the problem that would ensure that the reduced form utility functions satisfy these crossing properties are stringent; the reduced form utility functions associated with very natural single dimensional information problems can fail to satisfy these properties.

In summary, while ex post incentive compatibility is desirable, nontrivial mechanisms for which truthful revelation is ex post incentive compatible fail to exist for a large set of important problems. We introduce in this paper a notion of *weak* $\varepsilon-$e*x post incentive compatibility:* a mechanism is weakly $\varepsilon-$e*x* post incentive compatible if truthful revelation is ex post incentive compatible with conditional probability at least $1 - \varepsilon$. If truthful revelation is weakly ε-ex post incentive compatible for a mechanism, then incentive that agents have to collect information about other agents is bounded by ε times the maximal gain from espionage. If espionage is costly, a mechanism designer can be relatively comfortable in taking agents' beliefs as exogenous when ε is sufficiently small. We show that the existence of mechanisms for which there are weakly ε-incentive compatible equilibria is related to the concept of informational size introduced in McLean and

Postlewaite (2002, 2004). When agents have private information, the posterior probability distribution on the set of states of the nature Θ will vary depending on a given agent's type. Roughly, an agent's informational size corresponds to the maximal expected change in the posterior on Θ as his type varies, fixing other agents' types. We show that for any ε, there exists a δ such that, if each agent's informational size is less than δ, then there exists an efficient mechanism for which truthful revelation is a weak ε-ex post incentive compatible equilibrium.

The weakly ε-ex post incentive compatible mechanism that is used in the proof of the result elicits agents' private information and employs payments to agents that depend on their own announcement and the announcements of others. The payments employed are nonnegative and are small when agents are informationally small. When there are many agents, each will typically be informationally small, and hence, the payment needed to elicit truthful revelation of any agent's private information will be small. But the accumulation of a large number of small payments may be large. We show, however, that for a replica problem in which the number of agents goes to infinity, agents' informational size goes to zero exponentially and the aggregate payments needed to elicit the private information necessary to ensure efficient outcomes goes to zero.

We describe the model in the next section and provide a brief history of ex post incentive compatibility in Section 3. In Section 4 we introduce a generalized VCG mechanism, along with an alternative efficient mechanism.

2. The Model

Let $\Theta = \{\theta_1, .., \theta_m\}$ represent the finite set of states of nature and let T_i denote the finite set of types of player i. Let C denote the set of social alternatives. Agent $i's$ payoff is represented by a nonnegative valued function $v_i : C \times \Theta \times T_i \to \Re_+$. We will assume that there exists $c_0 \in C$ such that $v_i(c_0, \theta, t_i) = 0$ for all $(\theta, t_i) \in \Theta \times T_i$ and that there exists $M > 0$ such that $v_i(\cdot, \cdot, \cdot) \leq M$ for each i. Since v_i takes on only nonnegative values, c_0 is the "uniformly worst" outcome for all agents. We will say that v_i satisfies the *pure common value property* if v_i depends only on $(c, \theta) \in C \times \Theta$ and the *pure private value property* if v_i depends only on $(c, t_i) \in C \times T_i$. Our notion of common value is more general than that typically found in the literature in that we do not require that all agents have the same value for a given decision. According to our definition of pure common value, an agent's "fundamental" valuation depends only on the state θ, and not on any private information he may have.

Let $(\widetilde{\theta}, \widetilde{t}_1, \widetilde{t}_2, ..., \widetilde{t}_n)$ be an (n+1)-dimensional random vector taking values in $\Theta \times T$ ($T \equiv T_1 \times \cdots \times T_n$) with associated distribution P where

$$P(\theta, t_1, .., t_n) = Prob\{\widetilde{\theta} = \theta, \widetilde{t}_1 = t_1, ..., \widetilde{t}_n = t_n\}.$$

We will make the following full support assumptions regarding the marginal distributions: $P(\theta) = \text{Prob}\{\widetilde{\theta} = \theta\} > 0$ for each $\theta \in \Theta$ and $P(t_i) = \text{Prob}\{\widetilde{t}_i = t_i\} > 0$ for each $t_i \in T_i$. If X is a finite set, let Δ_X denote the set of probability measures on X. The set of probability measures on $\Theta \times T$ satisfying the full support conditions will be denoted $\Delta^*_{\Theta \times T}$. If $P \in \Delta^*_{\Theta \times T}$, let $T^* := \{t \in T | P(t) > 0.\}$ (The set T^* depends on P but we will suppress this dependence to keep the notation lighter.)

In many problems with differential information, it is standard to assume that agents have utility functions $u_i : C \times T \to R_+$ that depend on other agents' types. It is worthwhile noting that, while our formulation takes on a different form, it is equivalent. Given a problem as formulated in this paper, we can define $u_i(c, t_{-i}, t_i) = \sum_{\theta \in \Theta} [v_i(c, \theta, t_i) P(\theta | t_{-i}, t_i)]$. Alternatively, given utility functions $u_i : C \times T \to R_+$, we can define $\Theta \equiv T$ and define $v_i(c, t, t'_i) = u_i(c, t_{-i}, t'_i)$. Our formulation will be useful in that it highlights the nature of the interdependence: agents care about other agents' types to the extent that they provide additional information about the state θ. Because of the separation of an agent's fundamental valuation function from other agents' information, this formulation allows an analysis of the effects of changing the information structure while keeping an agent's fundamental valuation function fixed.

A *social choice problem* is a collection $(v_1, .., v_n, P)$ where $P \in \Delta^*_{\Theta \times T}$. An *outcome function* is a mapping $q : T \to C$ that specifies an outcome in C for each profile of announced types. We will assume that $q(t) = c_0$ if $t \notin T^*$, where c_0 can be interpreted as a status quo point. A *mechanism* is a collection $(q, x_1, .., x_n)$ (written simply as $(q, (x_i))$ where $q : T \to C$ is an outcome function and the functions $x_i : T \to \Re$ are transfer functions. For any profile of types $t \in T^*$, let

$$\hat{v}_i(c; t) = \hat{v}_i(c; t_{-i}, t_i) = \sum_{\theta \in \Theta} v_i(c, \theta, t_i) P(\theta | t_{-i}, t_i).$$

Although $\hat{v}$ depends on P, we suppress this dependence for notational simplicity as well. Finally, we make the simple but useful observation that the pure private value model is mathematically indentical to a model in which $|\Theta| = 1$.

Definition: Let $(v_1, .., v_n, P)$ be a social choice problem. A mechanism $(q, (x_i))$ is:

ex post incentive compatible if truthful revelation is an *ex post Nash equilibrium*: for all $i \in N$, all $t_i, t_i' \in T_i$ and all $t_{-i} \in T_{-i}$ such that $(t_{-i}, t_i) \in T^*$,

$$\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + x_i(t_{-i}, t_i) \geq \hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + x_i(t_{-i}, t_i').$$

strongly ex post incentive compatible if truthful revelation is an *ex post dominant strategy* equilibrium: for all $i \in N$, all $t_i, t_i' \in T_i$, all $\sigma_{-i} \in T_{-i}$ and all $t_{-i} \in T_{-i}$ such that $(t_{-i}, t_i) \in T^*$,

$$\hat{v}_i(q(\sigma_{-i}, t_i); t_{-i}, t_i) + x_i(\sigma_{-i}, t_i) \geq \hat{v}_i(q(\sigma_{-i}, t_i'); t_{-i}, t_i) + x_i(\sigma_{-i}, t_i').$$

interim incentive compatible if truthful revelation is a *Bayes-Nash equilibrium*: for each $i \in N$ and all $t_i, t_i' \in T_i$

$$\sum_{\substack{t_{-i} \in T_{-i} \\ :(t_{-i}, t_i) \in T^*}} \left[\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + x_i(t_{-i}, t_i)\right] P(t_{-i}|t_i)$$
$$\geq \sum_{\substack{t_{-i} \in T_{-i} \\ :(t_{-i}, t_i) \in T^*}} \left[\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + x_i(t_{-i}, t_i')\right] P(t_{-i}|t_i)$$

ex post individually rational (XIR) if

$$\hat{v}_i(q(t); t) + x_i(t) \geq 0 \text{ for all } i \text{ and all } t \in T^*.$$

feasible if for each $t \in T^*$,

$$\sum_{j \in N} x_j(t) \leq 0.$$

balanced if for each $t \in T^*$,

$$\sum_{j \in N} x_j(t) = 0.$$

outcome efficient if for each $t \in T^*$,

$$q(t) \in \arg\max_{c \in C} \sum_{j \in N} \hat{v}_j(c; t).$$

Clearly, strong ex-post IC implies ex post IC and ex post IC implies interim IC. If, for all i, $\hat{v}_i(c; t)$ does not depend on t_{-i}, then the notions of ex post dominant strategy equilibrium and ex post Nash equilibrium coincide. In this private value setting, the two definitions actually reduce to the usual notion of dominant strategy equilibrium. There is, of course, a definition of dominant strategy equilibrium that is appropriate for the actual Bayesian game. This (interim) equilibrium concept is weaker than ex post dominant strategy equilibrium and stronger than Bayes-Nash equilibrium, but is not logically nested with respect to ex-post Nash equilibrium. For a discussion of the relationship between ex post dominant strategy equilibrium, dominant strategy equilibrium, ex post Nash equilibrium and Bayes-Nash equilibrium, see Cremer and McLean (1985).

We will need one more incentive compatibility concept.

Definition: Let $\varepsilon \geq 0$. A mechanism $(q,(x_i))$ is *weakly* $\varepsilon-$ ex post incentive compatible if for all i and all $t_i, t_i' \in T_i$,

$$\Pr ob\{(\tilde{t}_{-i},t_i) \in T^* \text{ and } \hat{v}_i(q(\tilde{t}_{-i},t_i');\tilde{t}_{-i},t_i)+x_i(\tilde{t}_{-i},t_i')) \leq \hat{v}_i(q(\tilde{t}_{-i},t_i);\tilde{t}_{-i},t_i)+x_i(\tilde{t}_{-i},t_i)+\varepsilon | \tilde{t}_i = t_i\} \geq 1$$

Note that $(q,(x_i))$ is a weakly $0-$ ex post incentive compatible mechanism if and only if $(q,(x_i))$ is an ex post incentive compatible mechanism.[4]

3. Historical Perspective

As mentioned in the introduction, the typical modeling approach to mechanism design with interdependent valuations begins with a collection of functions $u_i : C \times T \to \Re$ as the primitive objects of study. In this approach, the elements of each T_i are ordered and two "crossing" properties (see below) are imposed. To our knowledge, the earliest construction of an ex post IC mechanism in the interdependent framework appears in Cremer and McLean (1985). In their setup, $T_i = \{1,2,...,m_i\}$ and $C = [0,\bar{c}]$ is an interval. Let $u_i'(c,t_{-i},t_i)$ denote the derivative of $u_i(\cdot,t_{-i},t_i)$ evaluated at $c \in C$.

Definition: Let q be an outcome function. An *E(xtraction)- mechanism* is any mechanism $(q,(x_i))$ satisfying

$$x_i(t_{-i},t_i) = x_i(t_{-i},1) - \sum_{\sigma_i=2}^{t_i} [u_i(q(t_{-i},\sigma_i),t_{-i},\sigma_i) - u_i(q(t_{-i},\sigma_i - 1),t_{-i},\sigma_i)]$$

whenever $t_{-i} \in T_{-i}$ and $t_i \in T_i \backslash \{1\}$.

[4]For a discussion of *weak* $\varepsilon-$ ex post incentive compatibility and a related concept of $\varepsilon-$ ex post incentive compatibility, see Section 8.

There are many E- mechanisms, depending on the choice of $x_i(t_{-i}, 1)$ for each $t_{-i} \in T_{-i}$. In their 1985 paper, CM define such mechanisms and use them (in conjunction with a full rank condition) to derive their full extraction results. If q is monotonic and if each u_i satisfies the classic single crossing property, then an E-mechanism will implement q as an ex post Nash equilibrium. This is summarized in the next result (Lemma 2 in CM(1985)).

Theorem 1: Suppose that
(i)

$$u_i'(c, t_{-i}, t_i + 1) \geq u_i'(c, t_{-i}, t_i) \geq 0$$

for each $i \in N, t_{-i} \in T_{-i}, t_i \in T_i \backslash \{m_i\}$ and $c \in C$. [This is assumption 2 in CM(1985).]

(ii) The social choice rule q is monotonic in the sense that

$$q(t_{-i}, t_i + 1) \geq q(t_{-i}, t_i)$$

for each $i \in N, t_{-i} \in T_{-i}, t_i \in T_i \backslash \{m_i\}$.

Then any E-mechanism is ex post IC. If, in addition,

$$u_i(0, t) = 0 \text{ for all } t \in T,$$

then there exists an E-mechanism $(q, (x_i))$ satisfying feasibility, ex post IC and ex post IR.

Proof: If assumptions (i) and (ii) are satisfied, then any E-mechanism is ex post IC as a result of Lemma 2 in CM (1985). Suppose that, in addition, $u_i(0, t) = 0$ for all $t \in T$. For each t_{-i}, define

$$x_i(t_{-i}, 1) = -u_i(q(t_{-i}, 1), t_{-i}, 1).$$

Feasibility follows from the assumption that $u_i(q(t_{-i}, 1), t_{-i}, 1) \geq 0$ and the observation that $u_i(q(t_{-i}, \sigma_i), t_{-i}, \sigma_i) - u_i(q(t_{-i}, \sigma_i - 1), t_{-i}, \sigma_i) \geq 0$ for each σ_i. Since the resulting E-mechanism is ex post IC, it follows that

$$\begin{aligned}
u_i(q(t_{-i},t_i);t_{-i},t_i)+x_i(t_{-i},t_i) &\geq u_i(q(t_{-i},1);t_{-i},t_i)+x_i(t_{-i},1)\\
&= \int_0^{q(t_{-i},1)} u_i'(y;t_{-i},t_i)dy+x_i(t_{-i},1)\\
&\geq \int_0^{q(t_{-i},1)} u_i'(y;t_{-i},1)dy+x_i(t_{-i},1)\\
&= u_i(q(t_{-i},1);t_{-i},1)+x_i(t_{-i},1)\\
&= 0.
\end{aligned}$$

It is important to point out that the family of E-mechanisms includes ex post IC mechanisms that are ex post IR but do not extract the full surplus (such as the mechanism defined in the proof of Theorem 1 above) as well as ex post IC mechanisms that extract the full surplus but are not ex post IR (such as the surplus extracting mechanisms constructed in CM (1985) that satisfy interim IR but not ex post IR.)

If one is interested in implementing a specific outcome function (e.g., an ex post efficient outcome function), then one must make further assumptions that guarantee that q satisfies the monotonicity condition (ii). This is the point at which the interagent crossing property comes into play and we will illustrate this in the special case of a single object auction with interdependent valuations studied in CM (1985). In this case, a single object is to be allocated to one of n bidders. If i receives the object, his value is the nonnegative number $w_i(t)$. In this framework, $q(t)=(q_1(t),..,q_n(t))$ where each $q_i(t)\geq 0$ and $q_1(t)+\cdots+q_n(t)\leq 1$ and

$$u_i(q(t_{-i},t_i');t_{-i},t_i)+x_i(t_{-i},t_i')=q_i(t_{-i},t_i')w_i(t_{-i},t_i)+x_i(t_{-i},t_i').$$

Finally, outcome efficiency means that

$$\sum_{i\in N} q_i(t)w_i(t)=\max_{i\in N}\{w_i(t)\}.$$

Theorem 2: Suppose that
(i) for each $i\in N, t_{-i}\in T_{-i}, t_i\in T_i\backslash\{m_i\}$

$$w_i(t_{-i},t_i)\leq w_i(t_{-i},t_i+1)$$

(ii) For all $i,j\in N, t_{-i}\in T_{-i}, t_i\in T_i\backslash\{m_i\}$

$$w_i(t_{-i},t_i)\geq w_j(t_{-i},t_i)\Rightarrow w_i(t_{-i},t_i+1)\geq w_j(t_{-i},t_i+1)$$

$$w_i(t_{-i}, t_i) > w_j(t_{-i}, t_i) \Rightarrow w_i(t_{-i}, t_i + 1) > w_j(t_{-i}, t_i + 1)$$

Then there exists an outcome efficient, ex post IR, ex post IC auction mechanism.

Theorem 2 is extracted from Corollary 2 in CM(1985). Condition (i) is the single crossing property which, in the auction case, reduces to the simple assumption that an agent's valuation for the object is nondecreasing in his own type. Condition (ii) is the interagent crossing property that guarantees that i's probability of winning $q_i(t_{-i}, t_i)$ is nondecreasing in i's type t_i. Other authors have employed a marginal condition that implies (ii) when bidders' values are drawn from an interval. Dasgupta and Maskin (2000) and Perry and Reny (2002) (in their work on ex post efficient auctions) and Ausubel (1999) (in his work on auction mechanisms) assume that types are drawn from an interval and that the valuation functions are differentiable and satisfy

(i')

$$\frac{\partial w_i}{\partial t_i}(t) \geq 0$$

and (ii')

$$\frac{\partial w_i}{\partial t_i}(t) \geq \frac{\partial w_j}{\partial t_i}(t).$$

These are the exact continuum analogues of the discrete assumptions in Theorem 2 above. Indeed, (i') implies (i) and (ii') implies (ii). (To see the latter, simply integrate both sides of the inequality in (ii') over the interval $[t_i, t_i + 1]$.)

In this paper, we do not take the $u_i : C \times T \rightarrow \Re$ as the primitive objects of study. Instead, we derive the reduced form $\hat{v}_i : C \times T \rightarrow \Re$ from the function $v_i : C \times \Theta \times T_i \rightarrow R_+$ and the conditional distributions $P_\Theta(\cdot|t)$. In an auction framework (such as that studied in McLean and Postlewaite (2004)), this reduced form payoff for bidder i is defined by the reduced form valuation function

$$\hat{w}_i(t) = \sum_\theta w_i(\theta, t_i) P_\Theta(\theta|t).$$

In this special case, conditions like (i) and (ii) or (i') and (ii') can limit the applicability of results like Theorem 2. For example, suppose that $w_i(\theta, t_i) = \alpha_i\theta + \beta_i$ for each i where $\alpha_i > 0$. Then

$$\hat{w}_i(t) = \alpha_i \sum_\theta \theta P_\Theta(\theta|t) + \beta_i := \alpha_i\overline{\theta}(t) + \beta_i.$$

Assuming that $\overline{\theta}(\cdot)$ is differentiable, then conditions (i') and (ii') can only be satisfied if $\alpha_i = \alpha_j$. To see this, note that (ii') is satisfied only if

$$(\alpha_i - \alpha_j)\frac{\partial \overline{\theta}}{\partial t_i}(t) \geq 0$$

and

$$(\alpha_j - \alpha_i)\frac{\partial \overline{\theta}}{\partial t_j}(t) \geq 0$$

for each i and j. If it is also required that $\frac{\partial w_i}{\partial t_i}(t) = \alpha_i \frac{\partial \overline{\theta}}{\partial t_i}(t) \geq 0$ and $\frac{\partial w_j}{\partial t_j}(t) = \alpha_j \frac{\partial \overline{\theta}}{\partial t_j}(t) \geq 0$ with strict inequality for some t, then $\alpha_i = \alpha_j$.

In this paper, we do not investigate the assumptions that v_i and $P_\Theta(\cdot|t)$ would need to satisfy in order for Theorem 1 to be applicable to the reduced form $\hat{v}_i$. Instead, we take a complementary approach and make certain assumptions regarding the distribution $P \in \Delta^*_{\Theta \times T}$ but no assumptions regarding the primitive valuation function v_i.

4. A Generalized Vickrey-Clarke-Groves Mechanism

Let q be an outcome function and define transfers as follows:

$$\begin{aligned} \alpha_i^q(t) &= \sum_{j \in N \setminus i} \hat{v}_j(q(t); t) - \max_{c \in C} \left[\sum_{j \in N \setminus i} \hat{v}_j(c; t) \right] \text{ if } t \in T^* \\ &= 0 \text{ if } t \notin T^* \end{aligned}$$

The resulting mechanism $(q, (\alpha_i^q))$ is the *generalized VCG mechanism with interdependent valuations* (GVCG for short.) (Ausubel(1999) and Chung and Ely (2002) use the term generalized Vickrey mechanisms, but for different classes of mechanisms.) It is straightforward to show that the GVCG mechanism is ex post individually rational and feasible. If $\hat{v}_i$ depends only on t_i (as in the pure private value case case where $|\Theta| = 1$ or, more generally, when $\tilde{\theta}$ and $\tilde{t}$ are stochastically independent), then the GVCG mechanism reduces to the classical VCG mechanism for private value problems and it is well known that, in this case, the VCG mechanism satisfies strong ex post IC. In general, however, the GVCG mechanism will *not even* satisfy interim IC. However, we will show that the GVCG mechanism is ex post IC when P satisfies a property called nonexclusive information (Postlewaite and Schmeidler (1986).

Before proceeding to the main result for nonexclusive information, let us review the logic of the VCG mechanism in the case of pure private values. In that case, we obtain (abusing notation slightly),

$$\begin{aligned}\alpha_i^q(t) &= \sum_{j\in N\backslash i} \hat{v}_j(q(t);t_j) - \max_{c\in C}\left[\sum_{j\in N\backslash i} \hat{v}_j(c;t_j)\right] \text{ if } t \in T^* \\ &= 0 \text{ if } t \notin T^*\end{aligned}$$

In computing $\max_{c\in C}\left[\sum_{j\in N\backslash i} \hat{v}_j(c;t_j)\right]$, we maximize the total payoff of the players in $N\backslash i$ and, as a consequence of the pure private values assumption, only utilize the information of the agents in $N\backslash i$. Hence, the value of the optimum only depends on t_{-i}. In the interdependent case, this computation can be extended in two ways. First, we could maximize the total payoff of the players in $N\backslash i$ using the information of all agents. The associated transfer is equal to

$$\sum_{j\in N\backslash i} \hat{v}_j(q(t);t) - \max_{c\in C} \sum_{j\in N\backslash i}\left[\sum_{\theta\in\Theta} v_j(c,\theta,t_j)P(\theta|t_{-i},t_i)\right]$$

Alternatively, we could maximize the total payoff of the players in $N\backslash i$ using only the information of the agents in $N\backslash i$. The associated transfer is equal to

$$\sum_{j\in N\backslash i} \hat{v}_j(q(t);t) - \max_{c\in C} \sum_{j\in N\backslash i}\left[\sum_{\theta\in\Theta} v_j(c,\theta,t_j)P(\theta|t_{-i})\right].$$

In the first payment scheme, agent i pays the cost that he imposes on other agents assuming that they have access to his information even though he is not present. In the second scheme, agent i pays the cost that he imposes on other agents assuming that the other agents do not have access to his information. In the pure private values model, these two approaches yield the same transfer scheme.

These payment schemes induce different games in the case of interdependent values. We are interested in the first of the payment schemes that uses agent i's information when calculating the cost that he imposes on other agents. One can think of the designer's problem as encompassing two stages. In the first stage, the designer elicits the agents' information to determine the posterior probability distribution over the states and makes that probability distribution available to the agents. The second stage consists of a Vickrey auction, where the agents' values are computed with the probability distribution from the first stage. If the designer has elicited truthful revelation in the first stage, the auction in the second stage is a private values auction, and truthful revelation is a dominant strategy. The interdependence of agents matters only for the first stage; our method is to show how the designer can extract the information needed to compute the probability distribution over the states, following which the problem becomes a private value problem. In this private value problem, the first payment scheme mimics the standard VCG mechanism.

Definition: A measure $P \in \Delta^*_{\Theta \times T}$ satisfies *nonexclusive information* (NEI) if

$$t \in T^* \Rightarrow P_\Theta(\cdot|t) = P_\Theta(\cdot|t_{-i}) \text{ for all } i \in N.$$

Proposition A: Let $\{v_1, .., v_n\}$ be a collection of payoff functions. If $P \in \Delta^*_{\Theta \times T}$ exhibits nonexclusive information and if $q : T \to C$ is outcome efficient for the problem $(v_1, .., v_n, P)$, then the GVCG mechanism (q, α_i^q) is ex post IC and ex post IR.

Proof: Choose i, $t_i, t_i' \in T_i$ and $t_{-i} \in T_{-i}$ such that $(t_{-i}, t_i) \in T^*$. Defining $c_i^* \in C$ so that $\sum_{j \in N \setminus i} \hat{v}_j(c_i^*; t) = \max_{c \in C} \left[\sum_{j \in N \setminus i} \hat{v}_j(c; t) \right]$, then ex post IR follows from outcome efficiency, the assumption that $\hat{v}_i(c_i^*; t) \geq 0$ and the observation that

$$\hat{v}_i(q(t); t) + x_i(t) = \left[\sum_{j \in N} \hat{v}_j(q(t); t) - \sum_{j \in N} \hat{v}_j(c_i^*; t) \right] + \hat{v}_i(c_i^*; t).$$

If $(t_{-i}, t_i') \in T^*$, then $P_\Theta(\cdot|t_{-i}, t_i) = P_\Theta(\cdot|t_{-i}, t_i')$ so that

$$\max_{c \in C} \left[\sum_{j \in N \setminus i} \hat{v}_j(c; t_{-i}, t_i) \right] = \max_{c \in C} \left[\sum_{j \in N \setminus i} \hat{v}_j(c; t_{-i}, t_i') \right].$$

Furthermore, outcome efficiency implies that

$$\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + \sum_{j \in N \setminus i} \hat{v}_j(q(t_{-i}, t_i); t_{-i}, t_i) \geq \hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + \sum_{j \in N \setminus i} \hat{v}_j(q(t_{-i}, t_i'); t_{-i}, t_i).$$

Therefore,

$$\begin{aligned}
\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+x_i(t_{-i},t_i) &= \left[\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t_i);t_{-i},t_i)\right] \\
&\quad -\max_{c\in C}\left[\sum_{j\in N\setminus i}\hat{v}_j(c;t_{-i},t_i)\right] \\
&\geq \left[\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i)+\sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t_i');t_{-i},t_i)\right] \\
&\quad -\max_{c\in C}\left[\sum_{j\in N\setminus i}\hat{v}_j(c;t_{-i},t_i')\right] \\
&= \hat{v}_i(q(t_{-i},t_i');t_{-i},t_i)+x_i(t_{-i},t_i).
\end{aligned}$$

If $(t_{-i},t_i')\notin T^*$, then $q(t_{-i},t_i')=c_0$ and $x_i(t_{-i},t_i)=0$ and ex post IR implies that

$$\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+x_i(t_{-i},t_i)\geq 0=\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i)+x_i(t_{-i},t_i').$$

Nonexclusive information is a strong assumption. Note however, that the pure private values model is a special case: simply choose $|\Theta|=1$. Our goal in this paper is to identify conditions under which we can modify the GVCG payments so that the new mechanism is interim IC and approximately ex post IC. In the next section, we discuss the two main ingredients of our approximation results: informational size and the variability of agents' beliefs.

5. Informational Size and Variability of Beliefs

5.1. Informational Size

If $t\in T^*$, recall that $P_\Theta(\cdot|t)\in\Delta_\Theta$ denotes the induced conditional probability measure on Θ. A natural notion of an agent's informational size is one that measures the degree to which he can alter the best estimate of the state θ when other agents are announcing truthfully. In our setup, that estimate is the conditional probability distribution on Θ given a profile of types t. Any profile of agents' types $t = (t_{-i}, t_i) \in T^*$ induces a conditional distribution on Θ and, if agent i unilaterally changes his announced type from t_i to t'_i, this conditional distribution will (in general) change. We consider agent i to be informationally small if, for each t_i, there is a "small" probability that he can induce a "large" change in the induced conditional distribution on Θ by changing his announced type from t_i to some other t'_i. This is formalized in the following definition.

Definition: Let

$$I^i_\varepsilon(t'_i, t_i) = \{t_{-i} \in T_{-i} | (t_{-i}, t_i) \in T^*, (t_{-i}, t'_i) \in T^* \text{ and } ||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t'_i)|| > \varepsilon\}$$

The *informational size* of agent i is defined as

$$\nu^P_i = \max_{t_i \in T_i} \max_{t'_i \in T_i} \min\{\varepsilon \geq 0 | \text{ Prob}\{\tilde{t}_{-i} \in I^i_\varepsilon(t'_i, t_i) | \tilde{t}_i = t_i\} \leq \varepsilon\}.$$

Loosely speaking, we will say that agent i is *informationally small* with respect to P if his informational size ν^P_i is small. If agent i receives signal t_i but reports $t'_i \neq t_i$, the effect of this misreport is a change in the conditional distribution on Θ from $P_\Theta(\cdot|t_{-i}, t_i)$ to $P_\Theta(\cdot|t_{-i}, t'_i)$. If $t_{-i} \in I_\varepsilon(t'_i, t_i)$, then this change is "large" in the sense that $||P_\Theta(\cdot|\hat{t}_{-i}, t_i) - P_\Theta(\cdot|\hat{t}_{-i}, t'_i)|| > \varepsilon$. Therefore, $\text{Prob}\{\tilde{t}_{-i} \in I_\varepsilon(t'_i, t_i) | \tilde{t}_i = t_i\}$ is the probability that i can have a "large" influence on the conditional distribution on Θ by reporting t'_i instead of t_i when his observed signal is t_i. An agent is informationally small if for each of his possible types t_i, he assigns small probability to the event that he can have a "large" influence on the distribution $P_\Theta(\cdot|t_{-i}, t_i)$, given his observed type. Informational size is closely related to the notion of nonexclusive information: if all agents have zero informational size, then P must satisfy NEI. In fact, we have the following easily demonstrated result: $P \in \Delta^*_{\Theta \times T}$ satisfies NEI if and only if $\nu^P_i = 0$ for each $i \in N$. If $T^* = T$, then ν^P is the *Ky Fan distance* between the r.v.s $P_\Theta(\cdot|\tilde{t}_{-i}, t_i)$ and$P_\Theta(\cdot|\tilde{t}_{-i}, t_i)$ with respect to the probability measure $P_{T_{-i}}(\cdot|t_i)$ (see, e.g., Dudley (2002), Section 9.2) [5]

5.2. Variability of Agents' Beliefs

Whether an agent i can be given incentives to reveal his information will depend on the magnitude of the difference between $P_{T_{-i}}(\cdot|t_i)$ and $P_{T_{-i}}(\cdot|t'_i)$, the conditional

[5] If X and Y are random variables defined on a probability space (Ω, F, μ) taking values in a metric space (S, d), then the Ky Fan metric is defined as $\min[\varepsilon \geq 0 : \mu\{d(X, Y) > \varepsilon\} \leq \varepsilon]$. If $T^* = T$, then ν^P is the Ky Fan distance between the r.v.s $X = P_\Theta(\cdot|\tilde{t}_{-i}, t_i)$ and $Y = P_\Theta(\cdot|\tilde{t}_{-i}, t_i)$ with respect to the probability measure $\mu = P_{T_{-i}}(\cdot|t_i)$.

distributions on T_{-i} given different types t_i and t_i' for agent i. To define the measure of variability, we first define a metric d on Δ_Θ as follows: for each $\alpha, \beta \in \Delta_\Theta$, let

$$d(\alpha, \beta) = \left\| \frac{\alpha}{||\alpha||_2} - \frac{\beta}{||\beta||_2} \right\|_2$$

where $||\cdot||_2$ denotes the 2-norm. Hence, $d(\alpha, \beta)$ measures the Euclidean distance between the Euclidean normalizations of α and β. If $P \in \Delta_{\Theta \times T}$, let $P_\Theta(\cdot|t_i) \in \Delta_\Theta$ be the conditional distribution on Θ given that i receives signal t_i and define

$$\Lambda_i^P = \min_{t_i \in T_i} \min_{t_i' \in T_i \setminus t_i} d(P_\Theta(\cdot|t_i), P_\Theta(\cdot|t_i'))^2.$$

This is the measure of the "variability" of the conditional distribution $P_\Theta(\cdot|t_i)$ as a function of t_i.

As mentioned in the introduction, our work is related to that of Cremer and McLean (1985,1989). Those papers and subsequent work by McAfee and Reny (1992) demonstrated how one can use correlation to fully extract the surplus in certain mechanism design problems. The key ingredient there is the assumption that the collection of conditional distributions $\{P_{T_{-i}}(\cdot|t_i)\}_{t_i \in T_i}$ is a linearly independent set for each i. This of course, implies that $P_{T_{-i}}(\cdot|t_i) \neq P_{T_{-i}}(\cdot|t_i')$ if $t_i \neq t_i'$ and, therefore, that $\Lambda_i^P > 0$. While linear independence implies that $\Lambda_i^P > 0$, the actual (positive) size of Λ_i^P is not relevant in the Cremer-McLean constructions, and full extraction will be possible. In the present work, we do not require that the collection $\{P_{T_{-i}}(\cdot|t_i)\}_{t_i \in T_i}$ be linearly independent (or satisfy the weaker cone condition in Cremer and McLean (1988)). However, the "closeness" of the members of $\{P_{T_{-i}}(\cdot|t_i)\}_{t_i \in T_i}$ is an important issue. It can be shown that for each i, there exists a collection of numbers $\varsigma_i(t)$ satisfying $0 \leq \zeta_i(t) \leq 1$ and

$$\sum_{t_{-i} \in T_{-i}} [\varsigma_i(t_{-i}, t_i) - \varsigma_i(t_{-i}, t_i')] \, P_{T_{-i}}(t_{-i}|t_i) > 0$$

for each $t_i, t_i' \in T_i$ if and only if $\Lambda_i^P > 0$. The elements of the collection $\{\varsigma_i(t)\}_{i \in I, t \in T}$ can be thought of as "incentive payments" to the agents to reveal their information. The above inequality assures that, if the posteriors $\{P_{T_{-i}}(\cdot|t_i)\}_{t_i \in T_i}$ are all distinct, then the incentive compatibility inequalities above are strict. However, the expression on the left hand side decreases as $\Lambda^P \to 0$. Hence, the difference in the expected reward from a truthful report and from a false report will be very small if the conditional posteriors are very close to each other. Our results require that informational size and aggregate uncertainty be small relative to the variation in these posteriors.

6. Implementation and Informational Size

6.1. The Results

Let $(z_i)_{i\in N}$ be an n-tuple of functions $z_i : T \to \Re_+$ each of which assigns to each $t \in T$ a nonnegative number, interpreted as a "reward" to agent i. If $(q, x_1, ..x_n)$ is a mechanism, then the associated *augmented* mechanism is defined as $(q, x_1 + z_1, .., x_n + z_n)$ and will be written simply as $(q, (x_i + z_i))$.

Theorem A: Let $(v_1, .., v_n)$ be a collection of payoff functions.

(i) Suppose that $P \in \Delta^*_{\Theta\times T}$ satisfies $\Lambda_i^P > 0$ for each i and suppose that $q : T \to C$ is outcome efficient for the problem $(v_1, .., v_n, P)$. Then there exists an augmented GVCG mechanism $(q, \alpha_i^q + z_i)$ for the social choice problem problem $(v_1, .., v_n, P)$ satisfying ex post IR and interim IC.

(ii) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta^*_{\Theta\times T}$ satisfies

$$\max_i \nu_i^P \le \delta \min_i \Lambda_i^P,$$

and whenever $q : T \to C$ is outcome efficient for the problem $\{v_1, .., v_n, P\}$, there exists an augmented GVCG mechanism $(q, (\alpha_i^q + z_i))$ with $0 \le z_i(t) \le \varepsilon$ for every i and t satisfying ex post IR, interim IC and weak $\varepsilon-$ex post IC.

6.2. Discussion

The proof of Theorem A is deferred until the appendix but we will outline the logic now. Our results rely on the following key lemma, whose proof is also found in the appendix. .

Lemma A: Suppose that $q : T \to C$ is outcome efficient for the problem $(v_1, .., v_n, P)$. If $(t_{-i}, t_i), (t_{-i}, t_i') \in T^*$, then

$$\begin{aligned}&\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i') - \ \hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i)\\ &\le 2M(n-1)||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t_i')||\end{aligned}$$

where

$$M = \max_{i\in N} \max_{t_i\in T_i} \max_{c\in C} \max_{\theta\in\Theta} v_i(c; \theta, t_i).$$

In the case of the GVCG mechanism, Lemma A provides an upper bound on the "ex post gain" to agent i when i's true type is t_i but i announces t_i' and others announce truthfully. If agents have zero informational size – that is, if P exhibits nonexclusive information – then $||P_\Theta(\cdot|t_{-i},t_i) - P_\Theta(\cdot|t_{-i},t_i')|| = 0$ if $(t_{-i},t_i),(t_{-i},t_i') \in T^*$. Hence, truth is an ex post Nash equilibrium and Proposition A follows. If v_i does not depend on θ, then (letting $|\Theta| = 1$), we recover Vickrey's classic dominant strategy result for the VCG mechanism in the pure private values case.

If agent i is informationally small, then (informally) we can deduce from Lemma A that

$$\Pr ob\{||P_\Theta(\cdot|\tilde{t}_{-i},t_i) - P_\Theta(\cdot|\tilde{t}_{-i},t_i')|| \approx 0|\tilde{t}_i = t_i\} \approx 1$$

so truth is an *approximate* ex post equilibrium for the GVCG in the sense that

$$\Pr ob\{(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\alpha_i^q(t_{-i},t_i))-(\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i)+\alpha_i^q(t_{-i},t_i')) \underset{\approx}{\geq} 0|\tilde{t}_i = t_i\} \approx 1.$$

Consequently, we obtain the following continuity result: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that truth will be a weak $\varepsilon-$ex post Nash equilibrium whenever $\nu_i^P < \delta$ for each i.

Lemma A has a second important consequence: if agent i is informationally small, then truth is an *approximate* Bayes-Nash equilibrium in the GVCG mechanism so the mechanism is approximately interim incentive compatible. More precisely, we can deduce from Lemma A that the interim expected gain from misreporting one's type is essentially bounded from above by one's informational size. If we want the mechanism to be *exactly* interim incentive compatible, then we must alter the mechanism (specifically, construct an augmented GVCG mechanism) in order to provide the correct incentives for truthful behavior. It is in this step that variability of beliefs plays a crucial role. To see this, first note that interim incentive compatibility of the augmented GVCG mechanism requires that

$$\begin{aligned}
&\sum_{\substack{t_{-i}\in T_{-i} \\ :(t_{-i},t_i)\in T^*}} [(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + \alpha_i^q(t_{-i},t_i)) - (\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i) + \alpha_i^q(t_{-i},t_i'))]\, P(t_{-i}|t_i) \\
&+ \sum_{\substack{t_{-i}\in T_{-i} \\ :(t_{-i},t_i)\in T^*}} (z_i(t_{-i},t_i) - z_i(t_{-i},t_i'))\, P(t_{-i}|t_i) \\
&\geq 0
\end{aligned}$$

Lemma A implies that the first term is bounded from below by $-K\nu_i^P$ where K is a positive constant independent of P so that, as we stated above, the interim expected gain from misreporting one's type is essentially bounded from above by one's informational size. If $\Lambda_i^P > 0$, then there exists a collection of numbers $\{\varsigma_i(t)\}_{t\in T}$ satisfying $0 \leq \zeta_i(t) \leq 1$ and

$$\sum_{\substack{t_{-i}\in T_{-i} \\ :(t_{-i},t_i)\in T^*}} [\varsigma_i(t_{-i},t_i) - \varsigma_i(t_{-i},t_i')]\, P_{T_{-i}}(t_{-i}|t_i) > 0$$

for each $t_i, t_i' \in T_i$. By defining $z_i(t_{-i},t_i) = \eta\zeta_i(t_{-i},t_i)$ and choosing η sufficiently large, then we will obtain interim incentive compatibility of the augmented GVCG mechanism. This is part (i) of Theorem A. As the informational size of an agent decreases, the minimal reward required to induce the truth also decreases. If Λ_i^P large enough relative to an agent's informational size ν_i^P, then we can construct an augmented mechanism satisfying interim incentive compatibility. This is part (ii) of Theorem A.

Informally, Theorem A can be explained in the following way. If a problem is a pure private value problem, then the VCG mechanisms will implement efficient outcomes. In the presence of interdependent values, these mechanisms are no longer incentive compatible. With interdependent values, a given agent's utility depends on other agents' types, insofar as their types are correlated with the state θ. If there is correlation in the components of the agents' information that are related to θ, then those components can be truthfully elicited via payments to the agents that are of the magnitude of their informational sizes; this is the "augmented" part of the augmented GVCG mechanism. Once the part of an agent's information that affects the probability distribution over the states is obtained, the problem becomes a private value problem, and VCG-type payments can be used to extract the residual private information that agents may have, that is, their private values.

For pure common value problems, there is (by definition) no residual private information, so it might seem that the VCG-type payments can be dispensed with. However, simply dropping the GVCG payments introduces a problem. In the description of the intuition of the proof of Theorem A, we pointed out that the part of an agent's information that affects the utility of other agents can be extracted by augmenting the VCG payments. If agent i has true type t_i but announces t_i' when other agents announce t_{-i}, then the ex post payoff to agent i in the unaugmented GVCG mechanism is

$$U_i(t_{-i}, t_i'|t_i) := \hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i').$$

As a consequence of Lemma A, we know that the gain from a lie (i.e., $U_i(t_{-i}, t_i'|t_i) - U_i(t_{-i}, t_i|t_i)$) is small if $||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t_i')||$ is small. If we simply drop the GVCG transfers, then the gain to lying (i.e., $\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) - \hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i)$) will typically no longer be small when $||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t_i')||$ is small. Consequently, it will no longer be true that small informational size assures that an agent's information can be extracted with small payments. There are two important properties of the GVCG payments in our framework: they are used to elicit agents' private information and, in addition, they assure that an agent's ex post payoff behaves nicely with respect to the posterior distribution on Θ.

For pure private value problems, Green and Laffont (1979) show that the VCG payments are essentially unique. It may be the case that when there is a nontrivial private value component to agents' information, transfers that embody VCG payments are necessary, but for pure common value problems that is not the case. For pure common value problems with positive variability, there exist transfer schemes that have no relation to the GVCG mechanism. What is necessary is that the transfer payments accomplish what the GVCG payments accomplish: they must ensure that small changes in the posterior distribution on Θ do not translate into a large utility gain. This requires a "continuity" assumption on the mapping from posterior distributions on Θ into agents' utilities. We turn to this next.

6.3. Semi-Lipschitzean Mechanisms

In a typical implementation or mechanism design problem, one computes the mechanism for each instance of the data that defines the social choice problem. Therefore, in most cases of interest, the mechanism is parametrized by the valuation functions and probability structure that define the social choice problem. If we fix a profile $(v_1, .., v_n)$ of payoff functions, then we can analyze the parametric dependence of the mechanism on the probability distribution P and this dependence can be modelled as a mapping that associates a mechanism with each $P \in \Delta^*_{\Theta \times T}$. We will denote this mapping $P \mapsto (q^P, x_1^P, .., x_n^P)$. For example, the mapping naturally associated with the GVCG mechanism is defined by

$$q^P(t) \in \arg\max_{c\in C} \sum_{j\in N} \sum_{\theta\in\Theta} v_i(c,\theta,t_i)P(\theta|t_{-i},t_i) \text{ if } t \in T^*$$
$$q^P(t) = c_0 \text{ if } t \notin T^*$$

and

$$x_i^P(t) = \sum_{j\in N\setminus i} \sum_{\theta\in\Theta} v_i(q^P(t),\theta,t_i)P(\theta|t_{-i},t_i) - \max_{c\in C}\left[\sum_{j\in N\setminus i} \sum_{\theta\in\Theta} v_i(c,\theta,t_i)P(\theta|t_{-i},t_i)\right] \text{ if } t \in T^*$$
$$= 0 \text{ if } t \notin T^*.$$

Definition: Let $(v_1, .., v_n)$ be a profile of payoff functions. For each $P \in \Delta^*_{\Theta\times T}$, let $(q^P, x_1^P, .., x_n^P)$ be a mechanism for the social choice problem $(v_1, .., v_n, P)$. We will say that the mapping $P \mapsto (q^P, x_1^P, .., x_n^P)$ is *semi-Lipschitzean with respect to conditional probabilities*, or simply *semi-Lipschitzean*, if there exists a $K > 0$ such that for all $P \in \Delta^*_{\Theta\times T}$,

$$\hat{v}_i(q^P(t_{-i},t_i');t_{-i},t_i) + x_i^P(t_{-i},t_i') - \hat{v}_i(q^P(t_{-i},t_i);t_{-i},t_i) + x_i^P(t_{-i},t_i)$$
$$\le K||P_\Theta(\cdot|t_{-i},t_i) - P_\Theta(\cdot|t_{-i},t_i')||$$

whenever $(t_{-i},t_i), (t_{-i},t_i') \in T^*$.

Lemma A shows that the GVCG mechanism is semi-Lipschitzean with $(K = 2M(n-1))$ and this is the essential property of the GVCG mechanism that drives Theorem A. In fact, an important extension of Theorem A holds for any semi-Lipschitzean mechanism.

Theorem B: Let $(v_1, .., v_n)$ be a collection of payoff functions and suppose that $P \mapsto (q^P, x_1^P, .., x_n^P)$ is semi-Lipschitzean.

(i) If $\Lambda_i^P > 0$ for each i, then there exists an augmented mechanism $(q^P, (x_i^P + z_i^P))$ for the social choice problem problem $(v_1, .., v_n, P)$ satisfying ex post IR and interim IC.

(ii) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta^*_{\Theta\times T}$ satisfies

$$\max_i \nu_i^P \le \delta \min_i \Lambda_i^P,$$

there exists an augmented mechanism $(q^P, (x_i^P + z_i^P))$ with $0 \le z_i^P(t) \le \varepsilon$ for every i and t satisfying ex post IR, interim IC and weak $\varepsilon-$ex post IC.

6.4. Balanced Mecahanisms and Pure Common Value Problems

We now present an example of a balanced semi-Lipschitzean mechanism for pure common value models which is quite different from the GVCG mechanism. Let $(v_1, .., v_n)$ be a collection of payoff functions. For each $P \in \Delta^*_{\Theta\times T}$ suppose that $q^P : T \to C$ is a social choice function for the problem $(v_1, .., v_n, P)$ and define transfer payments associated with q^P as follows:

$$\beta_i^P(t) = \frac{1}{n}\sum_j \hat{v}_j(q^P(t), t) - \hat{v}_i(q^P(t), t).$$

In this simple scheme, agent i receives money if his individual payoff is less than the average payoff and he pays out money if his individual payoff is greater than the average payoff. Furthermore, note that

$$\sum_i \beta_i^P(t) = 0$$

so that the mechanism $(q^P, (\beta_i^P))$ is balanced for each $P \in \Delta^*_{\Theta\times T}$.

If q^P is outcome efficient for the problem $(v_1, .., v_n, P)$, then the associated mechanism with transfer payments $(\beta_i^P)_{i\in N}$ is semi-Lipschitzean in pure common value problems (though not for general problems). In fact, the mechanism $P \mapsto (q^P, \beta_1^P, .., \beta_n^P)$ is actually Lipschitzean.

Theorem C: Let $(v_1, .., v_n)$ be a collection of payoff functions satisfying the pure common value assumption. For each $P \in \Delta^*_{\Theta\times T}$ suppose that $q^P : T \to C$ is outcome efficient for the problem $(v_1, .., v_n, P)$ and let (β_i^P) be the transfer payments associated with q^P as defined as above.

(i) The mapping $P \mapsto (q^P, \beta_1^P, .., \beta_n^P)$ is semi-Lipschitzean.

(ii) If $\Lambda_i^P > 0$ for each i, then there exists an augmented mechanism $(q^P, \beta_i^P + z_i^P\}_{i\in N}$ for the social choice problem problem $(v_1, .., v_n, P)$ satisfying ex post IR and interim IC.

(iii) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta^*_{\Theta\times T}$ satisfies

$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^P,$$

there exists an augmented mechanism $(q^P, \beta_i^P + z_i^P)$ for the social choice problem problem $(v_1, .., v_n, P)$ with $0 \leq z_i^P(t) \leq \varepsilon$ for every i and t satisfying ex post IR, interim IC and weak $\varepsilon-$ex post IC.

The augmented mechanism of Theorem C is not balanced in general, but we do know that $0 \leq \sum_i(\beta_i^P + z_i^P) = \sum_i z_i^P \leq n\varepsilon$. If $n\varepsilon$ is small then this mechanism is "nearly" balanced. In the next section we show that, in a model with many agents, we can construct the $z_i's$ so that this mechanism is nearly balanced for pure common value problems.

7. Asymptotic Results

Informally, an agent is informationally small when the probability that he can affect the posterior distribution on Θ is small. One would expect, in general, that agents will be informationally small in the presence of many agents. For example, if agents receive conditionally independent signals regarding the state θ, then the announcement of one of many agents is unlikely to significantly alter the posterior distribution on Θ. Hence, it is reasonable to conjecture that (under suitable assumptions) an agent's informational size goes to zero in a sequence of economies with an increasing number of agents. Consequently, the required rewards z_i that induce truthful behavior will also go to zero as the number of agents grows. We will show below that this is in fact the case. Of greater interest, however, is the behavior of the *aggregate* reward necessary to induce truthful revelation. The argument sketched above only suggests that each individual's z_i becomes small as the number of agents goes to infinity, but does not address the asymptotic behavior of the sum of the z_i's. Roughly speaking, the size of the z_i that is necessary to induce agent i to reveal truthfully is of the order of magnitude of his informational size. Hence, the issue concerns the speed with which agents' informational size goes to zero as the number of agents increases. We will demonstrate below that, under reasonably general conditions, agents' informational size goes to zero at an exponential rate and that the total reward $\sum_{i\in N} z_i$ goes to zero as the number of agents increases.

7.1. Notation and Definitions:

We will assume that all agents have the same finite signal set $T_i = A$. Let $J_r = \{1, 2, ...r\}$. For each $i \in J_r$, let $v_i^r : C \times \Theta \times A \to \Re_+$ denote the payoff to agent i. For any positive integer r, let $T^r = A \times \cdots \times A$ denote the r-fold Cartesian product and let $t^r = (t_1^r, .., t_r^r)$ denote a generic element of T^r.

Definition: A sequence of probability measures $\{P^r\}_{r=1}^{\infty}$ with $P^r \in \Delta_{\Theta\times T^r}$ is a *conditionally independent sequence* if there exists $P \in \Delta_{\Theta\times A}$ such that

(a) $P(\theta, t) > 0$ for all $(\theta, t) \in \Theta \times A$ and for every $\theta, \hat{\theta}$ with $\theta \neq \hat{\theta}$, there exists a $t \in A$ such that $P(t|\theta) \neq P(t|\hat{\theta})$.

(b) For each r and each $(\theta, t_1, .., t_r) \in \Theta \times T^r$,

$$P^r(t_1^r, .., t_r^r|\theta) = \text{Prob}\{\widetilde{t}_1^r = t_1, \widetilde{t}_2^r = t_2, ..., \widetilde{t}_r^r = t_r|\tilde{\theta} = \theta\} = \prod_{i=1}^{r} P(t_i|\theta).$$

Because of the symmetry in the objects defining a conditionally independent sequence, it follows that, for fixed r, the informational size of each $i \in J_r$ is the same. In the remainder of this section we will drop the subscript i and will write ν^{P^r} for the value of the informational size of agents in J_r.

Lemma D: Suppose that $\{P^r\}_{r=1}^{\infty}$ is a conditionally independent sequence. For every $\varepsilon > 0$ and every positive integer k, there exists an $\hat{r}$ such that

$$r^k \nu^{P^r} \leq \varepsilon$$

whenever $r > \hat{r}$.

The proof is provided in the appendix and is an application of a large deviations result due to Hoeffding (1960). With this lemma, we can prove the following asymptotic result, the proof of which is also in the appendix.

Theorem D: Suppose that $\{P^r\}_{r=1}^{\infty}$ is a conditionally independent sequence. Let M and ε be positive numbers. Let $\{(v_1^r, .., v_r^r)\}_{r \geq 1}$ be a sequence of payoff function profiles and for each r, let $\{q^{P^r}(r), x_1^{P^r}(r), .., x_r^{P^r}(r)\}$ be an ex post IR mechanism for the SCP $(v_1^r, .., v_r^r, P^r)$. Suppose that

(1) $|v_i^r(\cdot, \cdot, \cdot)| \leq M$ for all r and $i \in J_r$

(2) For each r, $(q^{P^r}(r), x_1^{P^r}(r), .., x_r^{P^r}(r))$ is a semi-Lipschitz mechanism with constant K_r and for some positive integer L, $r^{-L}K_r \to 0$ as $r \to \infty$.

(3) The marginal measure of P^2 on T^2 exhibits positive variability.

Then there exists an $\hat{r}$ such that for all $r > \hat{r}$, there exists an augmented mechanism $(q^{P^r}(r), x_1^{P^r}(r) + z_1^r, .., x_r^{P^r}(r) + z_r^r)$ for the social choice problem $(v_1^r, .., v_r^r, P^r)$ satisfying ex post IR, interim IC weak $\varepsilon-$ex post IC. Furthermore, for each $t^r \in T^r$, $z_i^r(t^r) \geq 0$ and $\sum_{i \in J_r}^r z_i^r(t^r) \leq \varepsilon$.

Corollary: Suppose that $\{P^r\}_{r=1}^{\infty}$ is a conditionally independent sequence. Let M and ε be positive numbers. Let $\{(v_1^r, .., v_r^r)\}_{r\geq 1}$ be a sequence of payoff function profiles and for each r, let $\{q^{P^r}(r), \alpha_1^{P^r}(r), .., \alpha_r^{P^r}(r)\}$ denote the GVCG mechanism for the SCP $(v_1^r, .., v_r^r, P^r)$. Suppose that $|v_i^r(\cdot,\cdot,\cdot)| \leq M$ for all r and $i \in J_r$ and that the marginal measure of P^2 on T^2 exhibits positive variability.

Then there exists an $\hat{r}$ such that for all $r > \hat{r}$, there exists an augmented GVCG mechanism $(q^{P^r}(r), \alpha_1^{P^r}(r) + z_1^r, .., \alpha_r^{P^r}(r) + z_r^r)$ for the social choice problem $(v_1^r, .., v_r^r, P^r)$ satisfying ex post IR, interim IC and weak $\varepsilon-$ex post IC. Furthermore, for each $t^r \in T^r$, $z_i^r(t^r) \geq 0$ and $\sum_{i\in J_r}^r z_i^r(t^r) \leq \varepsilon$.

7.2. An Auction Application

The significance of Theorem D can be illustrated in the case of a Vickrey auction with interdependent valuations as studied in McLean and Postlewaite (2004). For simplicity, suppose that $T^* = T$. If i receives the object, his value is the nonnegative number $w_i(\theta, t_i)$ and his "reduced form" value is

$$\hat{w}_i(t) = \sum_{\theta} w_i(\theta, t_i) P_\Theta(\theta|t).$$

In this framework, $q(t) = (q_1(t), .., q_n(t))$ where each $q_i(t) \geq 0$ and $q_1(t) + \cdots + q_n(t) \leq 1$ and

$$\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + x_i(t_{-i}, t_i') = q_i(t_{-i}, t_i')\hat{w}_i(t_{-i}, t_i) + x_i(t_{-i}, t_i').$$

Finally, outcome efficiency means that

$$\sum_{i\in N} q_i(t) w_i(t) = \max_{i\in N}\{w_i(t)\}.$$

Let $w^*(t) := \max_i \hat{w}_i(t)$ and let $I(t) := \{i \in N | \hat{w}_i(t) = w^*(t)\}$. If

$$\begin{aligned} q_i^*(t) &= \frac{1}{|I(t)|} \text{ if } i \in I(t) \\ &= 0 \text{ if } i \notin I(t) \end{aligned}$$

then q^* is outcome efficient. Defining $w_{-i}^*(t) := \max_{j:j\neq i}\{w_j(t)\}$, it is easy to verify that the GVCG transfers associated with q^* are given by

$$\begin{aligned} \alpha_i^*(t) &= -\frac{w_{-i}^*(t)}{|I(t)|} \text{ if } i \in I(t) \\ &= 0 \text{ if } i \notin I(t). \end{aligned}$$

If the GVCG mechanism $(q^*, (\alpha_i^*))$ were ex post IC (as in the pure private value case or, more generally, the case of nonexclusive information), then the auctioneer's ex post revenue would be exactly

$$-\sum_{i=1}^{n} \alpha_i^*(t) = \sum_{i \in I(t)} \frac{w_{-i}^*(t)}{|I(t)|}.$$

In an augmented mechanism $(q^*, (\alpha_i^* + z_i))$, the auctioneer's ex post revenue is

$$-\sum_{i=1}^{n} \alpha_i^*(t) = \sum_{i \in I(t)} \frac{w_{-i}^*(t)}{|I(t)|} - \sum_{i=1}^{n} z_i(t)$$

so the auctioneeer's ex post revenue is reduced by the total of the reward payments necessary to elicit truthful revelation of types. In a large conditionally independent model of an auction, we know that the rewards z_i can be constructed so that the augmented mechanism $(q^*, (\alpha_i^* + z_i))$ is ex post IR, interim IC and approximately ex post IC. Furthermore, the sum $\sum_{i=1}^{n} z_i(t)$ is converging to zero as the number of bidders grows. Consequently, the auctioneer's ex post revenue will be close the auctioneer's ex post revenue from the unaugmented GVCG auction in the presence of many bidders.

8. Discussion

1. Our results are related to the work on surplus extraction (see, e.g., Cremer and McLean 1985, 1988) and McAfee and Reny (1992). For auction problems, our results say that a seller of an object can extract the information about θ by making payments to each agent of the order of magnitude of that agent's informational size. We discussed above the interpretation of the mechanism as extracting agents' information necessary to determine the probability distribution on the states concatenated with a private value auction with agents' values determined by that probability distribution. Under the mechanism in Theorem A, the seller will extract all surplus except for the payments necessary to extract the information necessary to determine the probability distribution, and the information rents associated with the private value auction.[6]

[6] Of course, if the part of agents' information that determines their private values is correlated, one could extract some (or all) of the agents' information rents associated with the private value auction.

2. It is worth pointing out one further aspect of an agent's informational size in expanding economies. Roughly speaking, when an agent has informational size ε, then that agent's (conditional) probability that he can change the posterior distribution on Θ by more than ε is at most ε. One might consider an alternative definition of informational size whereby an agent's informational size is ε if *with probability one* he cannot change the posterior distribution on Θ by more than ε.

Definition: The *strict informational size* of agent i is defined as

$$\sigma_i^P = \max_{t_i \in T_i} \max_{t_i' \in T_i} \max_{t_{-i} \in T_{-i}} \{||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t_i')|| : (t_{-i}, t_i), (t_{-i}, t_i') \in T^*\}.$$

We will refer to an agent as strictly informationally small if his strict informational size is small. From the definitions, it is clear that $\nu_i^P \leq \sigma_i^P$. For economic problems with a small number of agents, it is often the case that every agent is informationally small but no agent is strictly informationally small. For example, consider a problem with two equiprobable states, θ_1 and θ_2, and three agents, each of whom receives a noisy signal about the state θ. With very accurate signals, each agent's signal is the true state θ with high probability. In this case, it is easy to verify that any agent who unilaterally misreports his signal will, with high probability, have only a small effect on the posterior distribution and, consequently, agents are informationally small. However, it is also easy to see that agents will not be *strictly* informationally small. When the agents' signals are very accurate, then all agents' signals will correspond to the true state θ with high probability. However, the probability that two agents, say agent 1 and agent 2, receive different signals is positive. In this case, agent 3's announcement will have a large effect on the posterior distribution: whether he announces θ_1 or θ_2, one of the other two agents' announcements will match his announcement and one will not. Consequently, agent 3 cannot be strictly informationally small in this case.

The discussion above illustrates the advantage of results that employ the weaker notion of informational size rather than strict informational size: a large and interesting class of problems is covered by the former notion that will not be covered by the latter. There is, of course, a cost: theorems employing the weaker hypothesis will have weaker consequences. If a mechanism satisfies our notion of *weak* $\varepsilon-$ex post IC, then with (conditional) probability at most ε, a change in an agent's reported type (given other agents' types) will increase his utility by more than ε. This, of course allows the possibility that a change could lead to a large increase in his utility for some (low probability) profiles of other agents' types. The small probability of large utility gains is connected to the fact that, with small probability, an agent's report will a large effect on the posterior distribution. In interdependent type mechanisms, an agent's transfer depends on other agents' valuations, and those valuations depend on the posterior distribution on Θ; large changes in the posterior distribution can translate into large changes in utility.

The above discussion suggests a stronger notion of approximate ex post incentive compatibility:

Definition: Let $\varepsilon \geq 0$. A mechanism is ε- ex post incentive compatible if truthful revelation is an $\varepsilon-$ex post Nash equilibrium*:* if for all i, all $t_i, t_i' \in T_i$ and all $t_{-i} \in T_{-i}$ such that $(t_{-i}, t_i) \in T^*$,

$$(\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + x_i(t_{-i}, t_i')) - (\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + x_i(t_{-i}, t_i)) \leq \varepsilon.$$

That is, a mechanism is ε- ex post incentive compatible if, with probability one, no agent can increase his utility by more than ε regardless of other agents' types. Ex post incentive compatibility is stronger than ε- ex post incentive compatibility, and ε- ex post incentive compatibility is stronger than weak ε- ex post incentive compatibility.

As a consequence of Lemma A, we have the following continuity result for the GVCG mechanism: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that truth will be an weak $\varepsilon-$ex post Nash equilibrium whenever $\nu_i^P < \delta$ for each i. Strict informational size is related to ε- ex post incentive compatibility in the same way: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that truth will be an $\varepsilon-$ex post Nash equilibrium whenever $\sigma_i^P < \delta$ for each i.

9. Proofs:

We begin with a simple result regarding Lipschitz continuity of the optimal value function.

Lemma 1: For each $S \subseteq N$ and for each $p \in \Delta_\Theta$, let

$$F_S(p) = \max_{\hat{c} \in C} \sum_{\theta \in \Theta} \sum_{i \in S} v_i(\hat{c}, \theta, t_i) p(\theta).$$

Then for each $p, p' \in \Delta_\Theta$,

$$|F_S(p) - F_S(p')| \leq |S| M ||p - p'||.$$

Proof: Choose $S \subseteq N$ and $p, p' \in \Delta_\Theta$. Choose c and c' so that

$$\begin{aligned}
\sum_{\theta\in\Theta}\sum_{i\in S} v_i(c,\theta,t_i)p(\theta) &= \max_{\hat{c}\in C}\sum_{\theta\in\Theta}\sum_{i\in S} v_i(\hat{c},\theta,t_i)p(\theta) \\
\sum_{\theta\in\Theta}\sum_{i\in S} v_i(c',\theta,t_i)p(\theta) &= \max_{\hat{c}\in C}\sum_{\theta\in\Theta}\sum_{i\in S} v_i(\hat{c},\theta,t_i)p'(\theta)
\end{aligned}$$

Then,

$$\begin{aligned}
F_S(p) - F_S(p') &= \sum_{\theta\in\Theta}\sum_{i\in S} v_i(c,\theta,t_i)\left[p(\theta) - p'(\theta)\right] + \sum_{\theta\in\Theta}\sum_{i\in S}\left[v_i(c,\theta,t_i) - v_i(c',\theta,t_i)\right]p'(\theta) \\
&\leq \sum_{\theta\in\Theta}\sum_{i\in S} v_i(c,\theta,t_i)\left[p(\theta) - p'(\theta)\right] \\
&\leq |S|M||p - p'||.
\end{aligned}$$

Reversing the roles of p and p' yields the result.

9.1. Proof of Lemma A:

Choose $(t_{-i}, t_i), (t_{-i}, t_i') \in T^*$. Then

$$\begin{aligned}
\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + \alpha_i(t_{-i},t_i) = \hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) &+ \sum_{j\in N\backslash i}\hat{v}_j(q(t_{-i},t_i);t_{-i},t_i) \\
&- \max_{c\in C}\left[\sum_{j\in N\backslash i}\hat{v}_j(c;t_{-i},t_i)\right]
\end{aligned}$$

and

$$\begin{aligned}
\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i) + \alpha_i(t_{-i},t_i') = \hat{v}_i(q(t_{-i},t_i');t_{-i},t_i) &+ \sum_{j\in N\backslash i}\hat{v}_j(q(t_{-i},t_i');t_{-i},t_i) \\
&- \sum_{j\in N\backslash i}\hat{v}_j(q(t_{-i},t_i');t_{-i},t_i) \\
&+ \sum_{j\in N\backslash i}\hat{v}_j(q(t_{-i},t_i');t_{-i},t_i') - \max_{c\in C}\left[\sum_{j\in N\backslash i}\hat{v}_j(c;t_{-i},t_i')\right]
\end{aligned}$$

Since

$$\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t_i);t_{-i},t_i) \geq \hat{v}_i(q(t_{-i},t_i');t_{-i},t_i)+\sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t_i');t_{-i},t_i)$$

we conclude that

$$\begin{aligned}
&(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\alpha_i(t_{-i},t_i))-(\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i)+\alpha_i(t_{-i},t_i'))\\
&\geq \max_{c\in C}\left[\sum_{j\in N\setminus i}\hat{v}_j(c;t_{-i},t_i')\right]-\max_{c\in C}\left[\sum_{j\in N\setminus i}\hat{v}_j(c;t_{-i},t_i)\right]\\
&\quad -\sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t_i');t_{-i},t_i')+\sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t_i');t_{-i},t_i)
\end{aligned}$$

Lemma 1 implies that

$$\max_{c\in C}\left[\sum_{j\in N\setminus i}\hat{v}_j(c;t_{-i},t_i')\right]-\max_{c\in C}\left[\sum_{j\in N\setminus i}\hat{v}_j(c;t_{-i},t_i)\right] \geq -(n-1)M||P_\Theta(\cdot|t_{-i},t_i)-P_\Theta(\cdot|t_{-i},t_i')||.$$

so the result follows from the observation that

$$\left|\sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t_i');t_{-i},t_i)-\sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t_i');t_{-i},t_i')\right| \leq (n-1)M||P_\Theta(\cdot|t_{-i},t_i)-P_\Theta(\cdot|t_{-i},t_i')||.$$

9.2. Proof of Theorem A:

We prove part (ii) first. Choose $\varepsilon > 0$. Let

$$M = \max_\theta \max_i \max_{t_i} \max_{q\in C} v_i(q,\theta,t_i)$$

and let $|T|$ denote the cardinality of T. Choose δ so that

$$0 < \delta < \min\left\{\frac{\varepsilon}{4M(n+1)\sqrt{|T|}},\frac{\varepsilon}{4}\right\}$$

Suppose that $P \in \Delta^*_{\Theta\times T}$ satisfies

$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^P.$$

Define $\hat{\nu}^P = \max_i \nu_i^P$and $\Lambda^P = \min_i \Lambda_i^P$. Therefore $\hat{\nu}^P \leq \delta\Lambda^P$.

Now we define an augmented GVCG mechanism. For each $t \in T$, define

$$z_i(t_{-i}, t_i) = \varepsilon \frac{P_{T_{-i}}(t_{-i}|t_i)}{||P_{T_{-i}}(\cdot|t_i)||_2}.$$

Since $0 \leq \frac{P_{T_{-i}}(t_{-i}|t_i)}{||P_{T_{-i}}(\cdot|t_i)||_2} \leq 1$, it follows that

$$0 \leq z_i(t_{-i}, t_i) \leq \varepsilon$$

for all i, t_{-i} and t_i.

The augmented VCG mechanism $\{q, \alpha_i^q + z_i\}_{i \in N}$ is clearly ex post efficient. Individual rationality follows from the observations that

$$\hat{v}_i(q(t); t) + \alpha_i^q(t) \geq 0$$

and

$$z_i(t) \geq 0.$$

Claim 1: For i and for each $t_i, t_i' \in T_i$,

$$\sum_{t_{-i}:(t_{-i},t_i)\in T^*} (z_i(t_{-i}|t_i) - z_i(t_{-i}|t_i'))\, P(t_{-i}|t_i) = \sum_{t_{-i}} (z_i(t_{-i}|t_i) - z_i(t_{-i}|t_i'))\, P(t_{-i}|t_i) \geq \frac{\varepsilon}{2\sqrt{|T|}} \Lambda_i^P$$

Proof of Claim 1:

$$\begin{aligned}
\sum_{t_{-i}} (z_i(t_{-i}|t_i) - z_i(t_{-i}|t_i'))\, P(t_{-i}|t_i) = & \sum_{t_{-i}} \varepsilon \left[\frac{P_{T_{-i}}(t_{-i}|t_i)}{||P_{T_{-i}}(\cdot|t_i)||_2} - \frac{P_{T_{-i}}(t_{-i}|t_i')}{||P_{T_{-i}}(\cdot|t_i')||_2} \right] P(t_{-i}|t_i) \\
&= \frac{\varepsilon ||P_{T_{-i}}(\cdot|t_i)||_2}{2} \left\| \frac{P_{T_{-i}}(\cdot|t_i)}{||P_{T_{-i}}(\cdot|t_i)||_2} - \frac{P_{T_{-i}}(\cdot|t_i')}{||P_{T_{-i}}(\cdot|t_i')||_2} \right\|^2 \\
&\geq \frac{\varepsilon}{2\sqrt{|T|}} \Lambda_i^P
\end{aligned}$$

This completes the proof of Claim 1.

Claim 2: For each i and for each $t_i, t_i' \in T_i$,

$$\begin{aligned}
& \sum_{t_{-i}:(t_{-i},t_i)\in T^*} [(\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + \alpha_i(t_{-i}, t_i)) - (\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + \alpha_i(t_{-i}, t_i'))]\, P(t_{-i}|t_i) \\
\geq \; & -(n+1)2M\hat{\nu}^P
\end{aligned}$$

Proof of Claim 2: Define

$$A_i(t_i', t_i) = \{t_{-i} \in T_{-i} |\ (t_{-i}, t_i) \in T^*, (t_{-i}, t_i') \in T^*, ||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i} t_i')|| > \hat{\nu}^P\}$$

and

$$B_i(t_i', t_i) = \{t_{-i} \in T_{-i} |\ (t_{-i}, t_i) \in T^*, (t_{-i}, t_i') \in T^*, ||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i} t_i')|| \leq \hat{\nu}^P\}$$

and

$$C_i(t_i', t_i) = \{t_{-i} \in T_{-i} |\ (t_{-i}, t_i) \in T^*, (t_{-i} t_i') \notin T^*\}$$

Since $\nu_i^P \leq \hat{\nu}^P$, we conclude that

$$\text{Prob}\{\tilde{t}_{-i} \in A_i(t_i', t_i) | \tilde{t}_i = t_i\} \leq \nu_i^P \leq \hat{\nu}^P.$$

In addition,

$$0 \leq \hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i) \leq \hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) \leq M$$

for all i, t_i and t_{-i}. Therefore,

$$\begin{aligned}
|\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i')| &= |\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) - \hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i') \\
&\quad + \hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i') + \alpha_i^q(t_{-i}, t_i')| \\
&\leq |\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) - \hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i')| \\
&\quad + |\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i') + \alpha_i^q(t_{-i}, t_i')| \\
&\leq 3M
\end{aligned}$$

for all i, t_i, t_i' and t_{-i}. Applying the definitions and Lemma A, it follows that

$$\begin{aligned}
&\sum_{t_{-i} \in A_i(t_i', t_i)} [(\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i)) - (\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i'))] P(t_{-i}|t_i) \\
&\geq -4M \sum_{t_{-i} \in A_i(t_i', t_i)} P(t_{-i}|t_i) \\
&\geq -4M \hat{\nu}^P.
\end{aligned}$$

In addition,

$$\begin{aligned}
&\sum_{t_{-i} \in B_i(t_i', t_i)} [(\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i)) - (\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i'))] P(t_{-i}|t_i) \\
&\geq -2M(n-1) \sum_{t_{-i} \in B_i(t_i', t_i)} ||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i} t_i')|| P(t_{-i}|t_i) \\
&\geq -2M(n-1) \hat{\nu}^P
\end{aligned}$$

and, finally,

$$
\begin{aligned}
&\sum_{t_{-i}\in C_i(t_i',t_i)} \left[(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\alpha_i^q(t_{-i},t_i)) - (\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i)+\alpha_i^q(t_{-i},t_i'))\right] P(t_{-i}|t_i) \\
&= \sum_{t_{-i}\in C_i(t_i',t_i)} \left[(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\alpha_i^q(t_{-i},t_i)) - (\hat{v}_i(c_0;t_{-i},t_i)+0)\right] P(t_{-i}|t_i) \\
&= \sum_{t_{-i}\in C_i(t_i',t_i)} (\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\alpha_i^q(t_{-i},t_i))\, P(t_{-i}|t_i) \\
&\geq 0.
\end{aligned}
$$

Combining these observations completes the proof of the claim 2.

Applying Claims 1 and 2, it follows that

$$
\begin{aligned}
&= \sum_{t_{-i}:(t_{-i},t_i)\in T^*} \left[(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\alpha_i^q(t_{-i},t_i)) - (\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i)+\alpha_i^q(t_{-i},t_i'))\right] P(t_{-i}|t_i) \\
&\quad + \sum_{t_{-i}:(t_{-i},t_i)\in T^*} (z_i(t_{-i},t_i)-z_i(t_{-i},t_i'))\, P(t_{-i}|t_i) \\
&\geq \frac{\varepsilon}{2\sqrt{|T|}}\Lambda_i^P - (n+1)2M\hat{\nu}^P \\
&\geq 0.
\end{aligned}
$$

and the mechanism is interim incentive compatible. If $(t_{-i},t_i)\in T^*$ but $t_{-i}\notin A_i(t_i',t_i)$, then $\Lambda^P\leq 2$ implies that

$$
\begin{aligned}
(\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i)+\alpha_i^q(t_{-i},t_i')) - (\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\alpha_i^q(t_{-i},t_i)) &\leq 2M(n-1)\nu^P \\
&\leq 2M(n-1)\frac{\varepsilon}{4M(n+1)\sqrt{|T|}} \\
&\leq \varepsilon.
\end{aligned}
$$

In addition, $\Lambda^P\leq 2$ implies that $\nu^P\leq\frac{\varepsilon}{2}\Lambda^P\leq\varepsilon$. Therefore,

$$
Prob\{\tilde{t}_{-i}\notin A_i(t_i',t_i)|\tilde{t}_i=t_i\} = 1 - Prob\{\tilde{t}_{-i}\in A_i(t_i',t_i)|\tilde{t}_i=t_i\} \geq 1-\nu^P \geq 1-\varepsilon
$$

and it follows that the mechanism is weakly ε-ex post incentive compatible. This completes the proof of part (ii).

Part (i) follows from the computations in part (ii). We have shown that, for any positive number α, there exists an augmented GVCG mechanism $\{q, \alpha_i^q + z_i\}_{i \in N}$ satisfying

$$\sum_{t_{-i}:(t_{-i},t_i)\in T^*} \left[(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + \alpha_i^q(t_{-i},t_i)) - (\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i) + \alpha_i^q(t_{-i},t_i'))\right] P(t_{-i}|t_i)$$
$$\geq \frac{\alpha}{2\sqrt{|T|}}\Lambda_i^P - (n+1)2M\hat{\nu}^P$$

for each i and each t_i, t_i'. If $\Lambda_i^P > 0$ for each i, then α can be chosen large enough so that incentive compatibility is satisfied. This completes the proof of part (i).

9.3. Proofs of Theorems B and C

The proof of Theorem B is identical to that of Theorem A except that the constant K in the definition of semi-Lipschitzean mechanism replaces the constant $M(n-1)$. Theorem C is a corollary of of Theorem B once we estabish that the mechanism is semi-Lipschitzean. But this is simply an application of Lemma 1 of the appendix.

9.4. Proof of Lemma D

For each $\theta \in \Theta$, let $P(\cdot|\theta)$ denote the conditional measure on A. For each $\alpha \in A$, let $f_\alpha(t^r) = \#\{i \in J_r | t_i^r = \alpha\}$ and define $f(t^r) = (f_\alpha(t^r))_{\alpha \in A}$.

For each θ, let

$$\rho(\theta) := \max_{\hat{\theta}\neq\theta} \prod_{\alpha\in A} \left[\frac{P(\alpha|\hat{\theta})}{P(\alpha|\theta)}\right]^{P(\alpha|\theta)}$$

Assumption (a) in the definition of conditionally independent sequence and the strict concavity of the function $ln(\cdot)$ imply that $\rho(\theta) < 1$. It is easy to show (again by computing the logarithm) that there exists a $\delta > 0$ such that

$$\prod_{\alpha\in A} \left[\frac{P_\Theta(\alpha|\hat{\theta})}{P_\Theta(\alpha|\theta)}\right]^{\frac{f_\alpha(t^r)}{r} - P(\alpha|\theta)} \leq \frac{1}{\sqrt{\rho(\theta)}}$$

whenever $\hat{\theta} \neq \theta$ and $||\frac{f(t^r)}{r} - P(\cdot|\theta)|| < \delta$. Letting $R = \max_\theta \rho(\theta)$, we conclude that $||\frac{f(t^r)}{r} - P(\cdot|\theta)|| < \delta$ implies that

$$\frac{P_\Theta(\hat{\theta}|t^r)P(\theta)}{P_\Theta(\theta|t^r)P(\hat{\theta})} = \left[\prod_{\alpha\in A}\left[\frac{P_\Theta(\alpha|\hat{\theta})}{P_\Theta(\alpha|\theta)}\right]^{P(\alpha|\theta)}\prod_{\alpha\in A}\left[\frac{P_\Theta(\alpha|\hat{\theta})}{P_\Theta(\alpha|\theta)}\right]^{\frac{f_\alpha(t^r)}{r}-P(\alpha|\theta)}\right]^r \leq \left[\rho(\theta)\frac{1}{\sqrt{\rho(\theta)}}\right]^r \leq R^{r/2}$$

whenever $\hat{\theta} \neq \theta$. Therefore, $||\frac{f(t^r)}{r} - P(\cdot|\theta)|| < \delta$ implies that

$$||\chi_\theta - P_\Theta(\cdot|t^r)|| = 2\sum_{\hat{\theta}\neq\theta} P_\Theta(\hat{\theta}|t^r) \leq 2\sum_{\hat{\theta}\neq\theta}\frac{P(\hat{\theta})}{P(\theta)}P_\Theta(\theta|t^r)R^{r/2} \leq \frac{2R^{r/2}}{\beta}$$

where χ_θ is the Dirac measure with $\chi_\theta(\theta) = 1$ and $\beta := \min_{\alpha\in A} P(\alpha)$. To complete the argument, choose $t_i, t_i' \in A$ and note that for all r sufficiently large,

$$\begin{aligned}
&\Pr ob\{||P_\Theta(\cdot|\tilde{t}^r_{-i}, t_i) - P_\Theta(\cdot|\tilde{t}^r_{-i}, t_i')|| > \frac{4R^{r/2}}{\beta}|\tilde{\theta} = \theta\} \\
&\leq \Pr ob\{\exists\alpha \in A : ||\chi_\theta - P_\Theta(\cdot|\tilde{t}^r_{-i}, \alpha)|| > \frac{2R^{r/2}}{\beta}|\tilde{\theta} = \theta\} \\
&\leq \Pr ob\{\exists\alpha \in A : ||\frac{f(\tilde{t}^r_{-i}, \alpha)}{r} - P_\Theta(\cdot|\theta)|| \geq \delta|\tilde{\theta} = \theta\} \\
&\leq \Pr ob\{||\frac{f(\tilde{t}^r)}{r} - P_\Theta(\cdot|\theta)|| \geq \delta/2|\tilde{\theta} = \theta\} \\
&\leq 2\exp(\frac{-r\delta^2}{2})
\end{aligned}$$

where the last inequality is due to Hoeffding (1963). Hence, for all r sufficiently large,

$$\nu_i^{P^r} \leq \max\{\frac{4R^{r/2}}{\beta}, \frac{2\exp(\frac{-r\delta^2}{2})}{\beta}\}$$

where

$$\beta := \min_{\alpha\in A} P(\alpha).$$

9.5. Proof of Theorem D

The proof is essentially identical to that of Theorem A. First, note that $(T^r)^* = T^r$. For notational ease, we will write T, t, t_{-i} and t_i instead of T^r, t^r, t^r_{-i} and t^r_i.

Choose $\varepsilon > 0$. Let M be the bound defined in the statement of the Theorem. For each $t \in T$, define

$$\begin{aligned} z_i(t_{-i}, t_i) &= \frac{\varepsilon}{r} \frac{P^2(t_{i+1}|t_i)}{||P^2(\cdot|t_i)||_2} \text{ if } i = 1, .., r-1 \\ &= \frac{\varepsilon}{r} \frac{P^2(t_1|t_r)}{||P^2(\cdot|t_r)||_2} \text{ if } i = r \end{aligned}$$

Therefore,

$$0 \leq z_i(t_{-i}, t_i) \leq \frac{\varepsilon}{r}$$

for all i, t_{-i} and t_i. Individual rationality of the augmented mechanism follows from the observations that

$$\hat{v}_i(q(t); t) + x_i(t) \geq 0$$

and

$$z_i(t) \geq 0.$$

Claim 1: Let $|A|$ denote the cardinality of A. Then

$$\sum_{t_{-i}} (z_i(t_{-i}|t_i) - z_i(t_{-i}|t_i'))\, P^r(t_{-i}|t_i) \geq \frac{\varepsilon}{2r\sqrt{|A|}} \Lambda_i^{P^2}$$

Proof of Claim 1:

$$\begin{aligned} \sum_{t_{-i}} (z_i(t_{-i}|t_i) - z_i(t_{-i}|t_i'))\, P^r(t_{-i}|t_i) &= \sum_{t_{-i}} \sum_{(t_{-i}, t_i) \in T^r} (z_i(t_{-i}|t_i) - z_i(t_{-i}|t_i'))\, P^r(t_{-i}|t_i) \\ &= \sum_{t_{-i}} \frac{\varepsilon}{r} \left[\frac{P^2(t_{i+1}|t_i)}{||P^2(\cdot|t_i)||_2} - \frac{P^2(t_{i+1}|t_i')}{||P^2(\cdot|t_i)||_2} \right] P^r(t_{-i}|t_i) \\ &= \sum_{\alpha \in A} \frac{\varepsilon}{r} \left[\frac{P^2(\alpha|t_i)}{||P^2(\cdot|t_i)||_2} - \frac{P^2({}^{t_{i+1}}a|t_i')}{||P^2(\cdot|t_i)||_2} \right] P^2(\alpha|t_i) \\ &\geq \frac{\varepsilon}{2r\sqrt{|A|}} \Lambda_i^{P^2}. \end{aligned}$$

This completes the proof of Claim 1.

Claim 2:

$$\sum_{t_i} \left[(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + x_i(t_{-i},t_i)) - (\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i) + x_i(t_{-i},t_i')) \right] P^r(t_{-i}|t_i) \geq -4M\nu^{P^r} - K_r\nu^{P}$$

Proof of Claim 2: Define

$$A_i(t_i',t_i) = \{t_{-i} \in T_{-i} |\ ||P_\Theta^r(\cdot|t_{-i},t_i) - P_\Theta^r(\cdot|t_{-i}t_i')|| > \hat{\nu}^{P^r}\}$$

and

$$B_i(t_i',t_i) = \{t_{-i} \in T_{-i} |\ ||P_\Theta^r(\cdot|t_{-i},t_i) - P_\Theta^r(\cdot|t_{-i}t_i')|| \leq \hat{\nu}^{P^r}\}.$$

We conclude that

$$\text{Prob}\{\tilde{t}_{-i} \in A_i(t_i',t_i)|\tilde{t}_i = t_i\} \leq \nu^{P^r}.$$

In addition,

$$0 \leq \hat{v}_i^r(q(t_{-i},t_i);t_{-i},t_i) + x_i(t_{-i},t_i) \leq \hat{v}_i^r(q(t_{-i},t_i);t_{-i},t_i) \leq M$$

for all i, t_i and t_{-i}. Therefore,

$$\begin{aligned}
|\hat{v}_i^r(q(t_{-i},t_i');t_{-i},t_i) + x_i(t_{-i},t_i')| &= |\hat{v}_i^r(q(t_{-i},t_i');t_{-i},t_i) - \hat{v}_i^r(q(t_{-i},t_i');t_{-i},t_i') \\
&+ \hat{v}_i^r(q(t_{-i},t_i');t_{-i},t_i') + x_i(t_{-i},t_i')| \\
&\leq |\hat{v}_i^r(q(t_{-i},t_i');t_{-i},t_i) - \hat{v}_i^r(q(t_{-i},t_i');t_{-i},t_i')| \\
&+ |\hat{v}_i^r(q(t_{-i},t_i');t_{-i},t_i') + x_i(t_{-i},t_i')| \\
&\leq 3M
\end{aligned}$$

for all i, t_i, t_i' and t_{-i}. Applying the definitions, it follows that

$$\begin{aligned}
&\sum_{t_{-i} \in A_i(t_i',t_i)} \left[(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + x_i(t_{-i},t_i)) - (\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i) + x_i(t_{-i},t_i')) \right] P^r(t_{-i}|t_i) \\
&\geq -4M \sum_{t_{-i} \in A_i(t_i',t_i)} P^r(t_{-i}|t_i) \\
&\geq -4M\hat{\nu}^P.
\end{aligned}$$

and that

$$\begin{aligned}
&\sum_{t_{-i} \in B_i(t_i',t_i)} \left[(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + x_i(t_{-i},t_i)) - (\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i) + x_i(t_{-i},t_i')) \right] P^r(t_{-i}|t_i) \\
&\geq -K_r \sum_{t_{-i} \in B_i(t_i',t_i)} ||P_\Theta^r(\cdot|t_{-i},t_i) - P_\Theta^r(\cdot|t_{-i}t_i')|| P^r(t_{-i}|t_i) \\
&\geq -K_r\nu^{P^r}.
\end{aligned}$$

Combining these observations completes the proof of the claim 2.

Applying Claims 1 and 2, it follows that for sufficiently large r,

$$\begin{aligned}
&\sum_{t_{-i}} \left(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + x_i(t_{-i},t_i) + z_i(t_{-i},t_i)\right) P^r(t_{-i}|t_i) \\
&- \sum_{t_{-i}} \left(\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i) + x_i(t_{-i},t_i') + z_i(t_{-i},t_i')\right) P^r(t_{-i}|t_i) \\
&= \sum_{t_{-i}} \left[\left(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + x_i(t_{-i},t_i)\right) - \left(\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i) + x_i(t_{-i},t_i')\right)\right] P^r(t_{-i}|t_i) \\
&\qquad + \sum_{t_{-i}} \left(z_i(t_{-i},t_i) - z_i(t_{-i},t_i')\right) P^r(t_{-i}|t_i) \\
&\geq \frac{\varepsilon}{2r\sqrt{|A|}} \Lambda_i^{P^2} - 4M\nu^{P^r} - K^r\nu^{P^r} \\
&= \frac{1}{r}\left[\frac{\varepsilon}{2\sqrt{|A|}} \Lambda_i^{P^2} - 4Mr\nu^{P^r} - \left(\frac{K_r}{r^L}\right)\left(r^{L+1}\nu^{P^r}\right)\right] \\
&\geq 0.
\end{aligned}$$

and the proof of interim IC is complete.

10. Bibliography

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