Implementation with Interdependent Valuations PRELIMINARY AND INCOMPLETE

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1. Introduction

There is a large literature aimed at characterizing the social choice functions that can be implemented in Bayes Nash equilibria. This literature typically takes agents' information as exogenously given and fixed throughout the analysis. While for some problems this may be appropriate, the assumption is problematic for others. A typical analysis, relying on the revelation principle, maximizes some objective function subject to truthful revelation being a Bayes equilibrium. It is often the case that truthful revelation is not ex post incentive compatible, that is, for a given agent, there are some vectors of the other agents' types for which the agent may be better off by misreporting his type than truthfully revealing it. Truthful revelation, of course, may still be a Bayes equilibrium, because agents announce their types without knowing other agents types: choices must be made on the basis of their beliefs about other agents' types. The difficulty with assuming that agents' information is exogenous is that when truthful revelation is not ex post incentive compatible, agents have incentives to learn other agents' types. To the extent that an agent can, at some cost, learn something about other agents' types, agents' beliefs when a mechanism is applied must be treated as endogenous.

A planner who designs a mechanism for which truthful revelation is ex post incentive compatible can legitimately ignore agents' incentives to engage in espionage to discover other agents' types, and consequently, ex post incentive compatibility is desirable. The Clarke-Groves-Vickrey mechanism (hereafter CGV)[1] for

[1] See Clarke (1971), Groves (1973) and Vickrey (1961).

private values environments is a classic example of a mechanism for which truthful revelation is ex post incentive compatible. For this mechanism, each agent submits his or her valuation for each possible choice. The mechanism selects the outcome that maximizes the sum of the agents' submitted valuations, and prescribes a transfer to each agent an amount equal to the sum of the values of the other agents for the outcome. With these transfers, each agent has a dominant strategy to reveal his valuation truthfully. Cremer and McLean (1985) (hereafter CM) consider a similar problem in which agents have private information, but interdependent valuations; that is, each agent's valuation can depend on other agents' information. They consider the mechanism design problem in which the aim is to maximize the revenue obtained from auctioning an object. They analyze revelation games in which agents announce their types, and construct transfers similar to those in the CGV mechanism. The transfers are such that for each outcome, (roughly) each agent receives a transfer equal to the sum of the valuations of the other agents. Because each agent's valuation depends on other agents' announced types, truthful revelation will not generally be a dominant strategy. Cremer and McLean show, however, that under certain conditions[2] truthful revelation will, as in the CGV mechanism, be ex post incentive compatible.

There has recently been renewed interest in mechanisms for which truthful revelation is ex post incentive compatible. Dasgupta and Maskin (2000), Perry and Reny (2002) and Ausubel (1999) (among others) have used the solution concept in designing auction mechanisms that assure an efficient outcome. Chung and Ely (2001) and Bergemann and Morris (2003) analyze the solution concept more generally. These papers (and Cremer and McLean), however, restrict attention to the case that agents' private information is one dimensional, a serious restriction for many problems. Consider, for example, a problem in which an oil field is to be auctioned, and each agent may have private information about the quantity of the oil in the field, the chemical characteristics of the oil, the capacity of his refinery to handle the oil and the demand for the refined products in his market, all of which affect this agent's valuation (and potentially other agents' valuations as well). While the assumption that information is single dimensional is restrictive, it is necessary: Jehiel *et. al.* (2002) show that for general mechanism design problems with interdependent values and multidimensional signals, for nearly all valuation functions, truthful revelation will be an ex post equilibrium only for trivial outcome functions.

Thus, it is only in the case of single dimensional information that we can

[2] The conditions are discussed in section 3.

hope for ex post equilibria for interdependent value problems. But even in the single dimensional case, there are difficulties. Most work on mechanism design in problems with asymmetric information begin with utilities of the form $u_i(c; t_i, t_{-i})$, where c is a possible outcome, t_i represents agent i's private information and t_{-i} is a vector representing other agents' private information. In the standard interpretation, u_i is a reduced form utility function that gives agent i's utility of the outcome c under the particular circumstances likely to obtain given the agents' information. In the oil field problem above, for example, an agent's utility for the oil may depend on (among other things) the amount and chemical composition of the oil and the future demand oil products, and other agents' information affects i's (expected) value for the field insofar as i's beliefs about the quantity and compostion of the oil and the demand for oil products are affected by their information. In this paper, we begin from this more primitive data in which i has a utility function $v_i(\theta; t_i)$, where θ is a complete description of the state of the world and t_i is his private information. For the oil example, θ would include those things that affect i's value for the oil – the amount and compostion of the oil, the demand for oil, etc. The relationship between agents' private information and the state of the world is given by a probability distribution P over $\Theta \times T$. This formulation emphasizes the fact that other agents' information affects agent i precisely to the extent that it provides information about the state of the world.

The reduced form utility function that is normally the starting point for mechanism design analysis can be calculated from this more primitive structure: $u(c, t) \equiv \Sigma_\theta v_i(\theta; t) P(\theta|t)$. Most work that employs ex post incentive compatibility makes additional assumptions on the reduced form utility functions u_i. It is typically assumed that each agent's types are ordered, and that agents' valuations are monotonic in any agent's type. Further, it is assumed that the utility functions satisfy a single-crossing property: a movement of a given agent from one type to a higher type causes his valuation to increase at least as much as any other agent's valuation. We show that the conditions on the primitive data of the problem that would ensure that the reduced form utility functions satisfy the single crossing property are very stringent; the reduced form utility functions associated with very natural single dimensional information problems can fail to satisfy the single crossing property.

In summary, while ex post incentive compatibility is desirable, nontrivial mechanisms for which truthful revelation is ex post incentive compatible fail to exist for a large set of important problems. We introduce in this paper a notion of $\varepsilon-$ex *post incentive compatibility:* a mechanism is $\varepsilon-$ex post incentive compatibile if truthful revelation is ex post incentive compatible with probability at least $1-\varepsilon$. If truthful revelation is ε-ex post incentive compatible for a mechanism, agents' incentive to collect information about other agents' is bounded by ε times the maximal gain from espionage. If espionage is costly, a mechanism designer can be relatively comfortable in taking agents' beliefs as exogenous when ε is sufficiently small. We show that the existence of mechanisms for which there are ε-incentive compatible equilibria is related to the concept of informational size introduced in McLean and Postlewaite (2001, 2002). When agents have private information, the posterior probability distribution on the set of states of the world Θ will vary depending on a given agent's type. Roughly, an agent's informational size corresponds to the maximal expected change in the posterior on Θ as his type varies, fixing other agents' types. We show that for any ε, there exists δ such that if each agent's informational size is less than δ, there exists an efficient mechanism for which truthful revelation is an ε-incentive compatible equilibrium.

The ε-ex post incentive compatible mechanism that is used in the proof of the result elicits agents' private information and employs payments to agents that depend on their own announcement and the announcements of others. The payments employed are nonnegative and are small when agents are informationally small. When there are many agents, each will typically be informationally small, and hence, the payment needed to elicit truthful revelation of any agent's private information will be small. But the accumulation of a large number of small payments may be large. We show, however, that for a replica problem in which the number of agents goes to infinity, agents' informational size goes to zero exponentially and the aggregate payments needed to elicit the private information necessary to ensure efficient outcomes goes to zero.

We describe the model in the next section and provide a brief history of ex post incentive compatibility in Section 3. In Section 4 we introduce a generalized CGV mechanism, along with an alternative efficient mechanism.

2. The Model

Let $\Theta = \{\theta_1, .., \theta_m\}$ represent the finite set of states of nature and let T_i be the finite set of types of player i. Let C denote the set of social alternatives. Agent $i's$ payoff is represented by a nonnegative function $v_i : C \times \Theta \times T_i \rightarrow \Re_+$. We will assume that there exists $c_0 \in C$ such that $v_i(c_0, \theta, t_i) = 0$ for all $(\theta, t_i) \in \Theta \times T_i$ and that there exists $M > 0$ such that $v_i(\cdot, \cdot, \cdot) \leq M$ for each i. We will say that v_i satisfies the *pure common value property* if v_i depends only on $(c, \theta) \in C \times \Theta$

and the *pure private value property* if v_i depends only on $(c, t_i) \in C \times T_i$. Note that a problem satisfying the pure common value property does not imply that all agents have the same value for a given decision, but only that each agent's value depends only on the state θ, and not on any private information he may have.

Let $(\widetilde{\theta}, \widetilde{t}_1, \widetilde{t}_2, ..., \widetilde{t}_n)$ be an (n+1)-dimensional random vector taking values in $\Theta \times T$ ($T \equiv T_1 \times \cdots \times T_n$) with associated distribution P where

$$P(\theta, t_1, .., t_n) = Prob\{\widetilde{\theta} = \theta, \widetilde{t}_1 = t_1, ..., \widetilde{t}_n = t_n\}.$$

We will make the following full support assumptions regarding the marginal distributions: $P(\theta)$ =Prob$\{\widetilde{\theta} = \theta\} > 0$ for each $\theta \in \Theta$ and $P(t_i)$ =Prob$\{\widetilde{t}_i = t_i\} > 0$ for each $t_i \in T_i$. If X is a finite set, let Δ_X denote the set of probability measures on X. The set of probability measures on $\Theta \times T$ satisfying the full support conditions will be denoted $\Delta^*_{\Theta \times T}$. If $P \in \Delta^*_{\Theta \times T}$, let $T^* := \{t \in T | P(t) > 0.\}$ (The set T^* depends on P but we will suppress this dependence to keep the notation lighter.)

In many problems with differential information, it is standard to assume that agents have utility functions $u_i : C \times T \to R_+$ that depend on other agents' types. It is worthwhile noting that, while our formulation takes on a different form, it is equivalent. Given a problem as formulated in this paper, we can define $u_i(c, t_{-i}, t_i) = \sum_{\theta \in \Theta} [v_i(c, \theta, t_i) P(\theta | t_{-i}, t_i)]$. Alternatively, given utility functions $u_i : C \times T \to R_+$, we can define $\Theta \equiv T$ and define $v_i(c, t, t'_i) = u_i(c, t_{-i}, t'_i)$. Our formulation will be useful in that it highlights the nature of the interdependence: agents care about other agents' types to the extent that they provide additional information about the state θ.

A *social choice problem* is a collection $(v_1, .., v_n, P)$ where $P \in \Delta^*_{\Theta \times T}$. An *outcome function* is a mapping $q : T \to C$ that specifies an outcome in C for each profile of announced types. We will assume that $q(t) = c_0$ if $t \notin T^*$, where c_0 can be interpreted as a status quo point. A *mechanism* is a collection $(q, x_1, ..x_n)$ (written simply as $(q, (x_i))$ where $q : T \to C$ is an outcome function and the functions $x_i : T \to \Re$ are transfer functions. For any profile of types $t \in T^*$, let

$$\hat{v}_i(c; t) = \hat{v}_i(c; t_{-i}, t_i) = \sum_{\theta \in \Theta} v_i(c, \theta, t_i) P(\theta | t_{-i}, t_i).$$

Although $\hat{v}$ depends on P, we suppress this dependence for notational simplicity as well.

Definition: Let $(v_1, .., v_n, P)$ be a social choice problem. A mechanism $(q, (x_i))$ is:

ex post incentive compatible if truthful revelation is an ex post Nash equilibrium: for all i, all $t_i, t_i' \in T_i$ and all $t_{-i} \in T_{-i}$ such that $(t_{-i}, t_i) \in T^*$,

$$\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + x_i(t_{-i}, t_i) \geq \hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + x_i(t_{-i}, t_i').$$

interim incentive compatible (IC) if for each $i \in N$ and all $t_i, t_i' \in T_i$

$$\sum_{(t_{-i}, t_i) \in T^*} \left[\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + x_i(t_{-i}, t_i)\right] P(t_{-i}|t_i)$$
$$\geq \sum_{(t_{-i}, t_i) \in T^*} \left[\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + x_i(t_{-i}, t_i')\right] P(t_{-i}|t_i)$$

ex post individually rational (XIR) if

$$\hat{v}_i(q(t); t) + x_i(t) \geq 0 \text{ for all } i \text{ and all } t \in T^*.$$

feasible if for each $t \in T^*$,

$$\sum_{j \in N} x_j(t) \leq 0.$$

balanced if for each $t \in T^*$,

$$\sum_{j \in N} x_j(t) = 0.$$

outcome efficient if for each $t \in T^*$,

$$q(t) \in \arg\max_{c \in C} \sum_{j \in N} \hat{v}_j(c; t).$$

Clearly, strong ex post IC implies ex post IC which in turn implies interim IC. If $\hat{v}_i(c; t)$ does not depend on t_{-i}, then the notions of ex post dominant strategy and ex post Nash equilibrium coincide.[3] We will need one more incentive compatibility concept.

[3] For a discussion of the relationship between ex post dominant strategy equilibrium, dominant strategy equilibrium, ex post Nash equilibrium and Bayes-Nash equilibrium, see Cremer and McLean (1985).

Definition: Let $\varepsilon \geq 0$. A mechanism $(q, (x_i))$ is $\varepsilon-$ ex post incentive compatible if for all i and all $t_i, t_i' \in T_i$,

$$\Pr ob\{\hat{v}_i(q(\tilde{t}_{-i}, t_i'); \tilde{t}_{-i}, t_i) + x_i(\tilde{t}_{-i}, t_i')) \leq \hat{v}_i(q(\tilde{t}_{-i}, t_i); \tilde{t}_{-i}, t_i) + x_i(\tilde{t}_{-i}, t_i) + \varepsilon | \tilde{t}_i = t_i\} \geq 1 - \varepsilon.$$

Note that $(q, (x_i)_{i \in N})$ is a $0-$ ex post incentive compatible mechanism if and only if $(q, (x_i))$ is an ex post incentive compatible mechanism.

3. Historical Perspective

As mentioned in the introduction, the typical modeling approach to mechanism design with interdependent valuations begins with a collection of functions $u_i : C \times T \to \Re$ as the primitive objects of study. In this approach, the elements of each T_i are ordered and a single crossing property (see below) is imposed. To our knowledge, the earliest construction of an ex post IC mechanism in this framework appears in Cremer and McLean (1985). In their setup, $T_i = \{1, 2, ..., m_i\}$ and $C = [0, \bar{c}]$ is an interval. Let $u_i'(c, t_{-i}, t_i)$ denote the derivative of $u_i(\cdot, t_{-i}, t_i)$ evaluated at $c \in C$.

Definition: Let q be an outcome function. An E(xtraction)- mechanism is any mechanism $(q, (x_i)_{i \in N})$ satisfying

$$x_i(t_{-i}, t_i) = x_i(t_{-i}, 1) - \sum_{\sigma_i = 2}^{t_i} [u_i(q(t_{-i}, \sigma_i), t_{-i}, \sigma_i) - u_i(q(t_{-i}, \sigma_i - 1), t_{-i}, \sigma_i)]$$

whenever $t_{-i} \in T_{-i}$ and $t_i \in T_i \backslash \{1\}$.

There are many E- mechanisms, depending on the choice of $x_i(t_{-i}, 1)$ for each $t_{-i} \in T_{-i}$. In their 1985 paper, CM define such mechanisms and use them (in conjunction with a full rank condition) to derive their full extraction results. If q and u_i satisfy certain assumptions, then there exists an E-mechanism that implements q as an ex post Nash equilibrium and is also ex post individually rational. This is summarized in the next result.

Theorem 1: Suppose that
(i)

$$u_i'(c, t_{-i}, t_i + 1) \geq u_i'(c, t_{-i}, t_i) \geq 0$$

for each $i \in N, t_{-i} \in T_{-i}, t_i \in T_i \backslash \{m_i\}$ and $c \in C$. [This is assumption 2 in CM.]

(ii) The social choice rule q is monotonic in the sense that

$$q(t_{-i}, t_i + 1) \geq q(t_{-i}, t_i)$$

for each $i \in N, t_{-i} \in T_{-i}, t_i \in T_i \backslash \{m_i\}$.

Then any E-mechanism is ex post IC. If, in addition,

$$u_i(0, t) = 0 \text{ for all } t \in T,$$

then there exists an E-mechanism $\{q, (x_i)_{i \in N}\}$ satisfying feasibility, ex post IC and ex post IR.

Proof: If assumptions (i) and (ii) are satisfied, then any E-mechanism is ex post IC as a result of Lemma 2 in CM (1985). Suppose that, in addition, $u_i(0, t) = 0$ for all $t \in T$. For each t_{-i}, define

$$x_i(t_{-i}, 1) = -u_i(q(t_{-i}, 1), t_{-i}, 1).$$

Feasibility follows from the assumption that $u_i(q(t_{-i}, 1), t_{-i}, 1) \geq 0$ and the observation that $u_i(q(t_{-i}, \sigma_i), t_{-i}, \sigma_i) - u_i(q(t_{-i}, \sigma_i - 1), t_{-i}, \sigma_i) \geq 0$ for each σ_i. Since the resulting E-mechanism is ex post IC, it follows that

$$\begin{aligned} u_i(q(t_{-i}, t_i); t_{-i}, t_i) + x_i(t_{-i}, t_i) &\geq u_i(q(t_{-i}, 1); t_{-i}, t_i) + x_i(t_{-i}, 1) \\ &= \int_0^{q(t_{-i}, 1)} u_i'(y; t_{-i}, t_i) dy + x_i(t_{-i}, 1) \\ &\geq \int_0^{q(t_{-i}, 1)} u_i'(y; t_{-i}, 1) dy + x_i(t_{-i}, 1) \\ &= u_i(q(t_{-i}, 1); t_{-i}, 1) + x_i(t_{-i}, 1) \\ &= 0. \end{aligned}$$

It is important to point out that the family of E-mechanisms includes ex post IC mechanisms that are ex post IR but do not extract the full surplus (such as the mechanism defined in the proof of Theorem 1 above) as well as ex post IC mechanisms that extract the full surplus but are not ex post IR (such as the surplus extracting mechanisms constructed in CM (1985) that satisfy interim IR but not ex post IR.)

If one is interested in implementing a specific outcome function (e.g., an ex post efficient outcome function), then one must make further assumptions that guarantee that q satisfies the monotonicity condition (ii). This can be illustrated in the special case of a single object auction with interdependent valuations studied in CM (1985). In this case, a single object is to be allocated to one of n bidders. If i receives the object, his value is the nonnegative number $w_i(t)$. In this framework, $q(t) = (q_1(t), .., q_n(t))$ where each $q_i(t) \geq 0$ and $q_1(t) + \cdots + q_n(t) \leq 1$ and

$$u_i(q(t_{-i}, t_i'); t_{-i}, t_i) + x_i(t_{-i}, t_i') = q_i(t_{-i}, t_i')w_i(t_{-i}, t_i) + x_i(t_{-i}, t_i').$$

Finally, outcome efficiency means that

$$\sum_{i \in N} q_i(t)w_i(t) = \max_{i \in N}\{w_i(t)\}.$$

Theorem 2: Suppose that
(i) for each $i \in N, t_{-i} \in T_{-i}, t_i \in T_i \backslash \{m_i\}$

$$w_i(t_{-i}, t_i) \leq w_i(t_{-i}, t_i + 1)$$

(ii) For all $i, j \in N, t_{-i} \in T_{-i}, t_i \in T_i \backslash \{m_i\}$

$$w_i(t_{-i}, t_i) \geq w_j(t_{-i}, t_i) \Rightarrow w_i(t_{-i}, t_i + 1) \geq w_j(t_{-i}, t_i + 1)$$

$$w_i(t_{-i}, t_i) > w_j(t_{-i}, t_i) \Rightarrow w_i(t_{-i}, t_i + 1) > w_j(t_{-i}, t_i + 1)$$

Then there exists an outcome efficient, ex post IR, ex post IC auction mechanism.

Condition (ii) in Theorem 2 guarantees that i's probability of winning $q_i(t_{-i}, t_i)$ is nondecreasing in i's type t_i. Other authors have employed a marginal condition that implies (ii) when bidders' values are drawn from an interval. Dasgupta and Maskin (2000) and Perry and Reny (2002) (in their work on ex post efficient auctions) and Ausubel (1999) (in his work on auction mechanisms) assume that types are drawn from an interval and that the valuation functions are differentiable and satisfy
(i′)

$$\frac{\partial w_i}{\partial t_i}(t) \geq 0$$

and (ii′)

$$\frac{\partial w_i}{\partial t_i}(t) \geq \frac{\partial w_j}{\partial t_i}(t).$$

These are the continuum analogues of the discrete assumptions in Theorem 2 above.

In this paper, we do not take the $u_i : C \times T \to \Re$ as the primitive objects of study. Instead, we derive the reduced form $\hat{v}_i : C \times T \to \Re$ from the function $v_i : C \times \Theta \times T_i \to R_+$ and the conditional distributions $P_\Theta(\cdot|t)$. In an auction framework (such as that studied in McLean and Postlewaite (2002)), this reduced form is defined by the reduced form valuation function

$$w_i(t) = \sum_\theta w_i(\theta, t_i) P_\Theta(\theta|t).$$

In this special case, the second condition is quite restrictive. For example, suppose that $v_i(\theta, t_i) = \alpha_i \theta + \beta_i$ for each i where $\alpha_i > 0$. Then

$$w_i(t) = \alpha_i \sum_\theta \theta P_\Theta(\theta|t) + \beta_i := \alpha_i \overline{\theta}(t) + \beta_i.$$

Assuming that $\overline{\theta}(\cdot)$ is differentiable, then the second condition (ii′) is satisfied only if

$$(\alpha_i - \alpha_j) \frac{\partial \overline{\theta}}{\partial t_i}(t) \geq 0$$

and

$$(\alpha_j - \alpha_i) \frac{\partial \overline{\theta}}{\partial t_j}(t) \geq 0$$

for each i and j. If it is also required that $\frac{\partial w_i}{\partial t_i}(t) = \alpha_i \frac{\partial \overline{\theta}}{\partial t_i}(t) \geq 0$ and $\frac{\partial w_j}{\partial t_j}(t) = \alpha_j \frac{\partial \overline{\theta}}{\partial t_j}(t) \geq 0$ with strict inequality for some t, then $\alpha_i = \alpha_j$.

In this paper, we do not investigate the assumptions that v_i and $P_\Theta(\cdot|t)$ would need to satisfy in order for Theorem 1 to be applicable to the reduced form $\hat{v}_i$. Indeed, we believe that such assumptions are prohibitively restrictive. Instead, we make certain assumptions regarding the distribution $P \in \Delta^*_{\Theta \times T}$ but no assumptions regarding the primitive valuation function v_i.

4. A Generalized Clarke-Groves-Vickrey Mechanism

Let q be an outcome function and define transfers as follows:

$$\alpha_i^q(t) = \sum_{j \in N \backslash i} \hat{v}_j(q(t); t) - \max_{c \in C} \left[\sum_{j \in N \backslash i} \hat{v}_j(c; t) \right] \text{ if } t \in T^*$$
$$= 0 \text{ if } t \notin T^*$$

The resulting mechanism $(q, (\alpha_i^q))$ is the *generalized CGV mechanism with interdependent valuations* (GCGV for short.) (Ausubel(1999) and Chung and Ely (2002) use the term generalized Vickrey mechanisms, but for different classes of mechanisms.) If $\hat{v}_i$ depends only on t_i (as in the case when $\tilde{\theta}$ and $\tilde{t}$ are stochastically independent), then the GCGV mechanism reduces to the classical CGV mechanism and it is well known that, in this case, the CGV mechanism satisfies strong ex post IC. It is straightforward to show that the GCGV mechanism is ex post individually rational and feasible. However, it will generally *not even* satisfy interim IC. First, we show that the GCGV mechanism is ex post IC when P satisfies a property called nonexclusive information (Postlewaite and Schmeidler (1986).

Before proceeding to the main result for nonexclusive information, let us recap the logic of the CGV mechanism in the case of pure private values. In that case, we obtain (abusing notation slightly),

$$\alpha_i^q(t) = \sum_{j \in N \backslash i} \hat{v}_j(q(t); t_j) - \max_{c \in C} \left[\sum_{j \in N \backslash i} \hat{v}_j(c; t_j) \right] \text{ if } t \in T^*$$
$$= 0 \text{ if } t \notin T^*$$

In computing $\max_{c \in C} \left[\sum_{j \in N \backslash i} \hat{v}_j(c; t_j) \right]$, we maximize the total payoff of the players in $N \backslash i$ and, as a consequence of the pure private values assumption, only utilize the information of the agents in $N \backslash i$. Hence, the value of the optimum only depends on t_{-i}. In the interdependent case, this computation can be extended in two ways. First, we could maximize the total payoff of the players in $N \backslash i$ using the information of all agents. The associated transfer is equal to

$$\sum_{j \in N \backslash i} \hat{v}_j(q(t); t) - \max_{a \in A} \sum_{j \in N \backslash i} \left[\sum_{\theta \in \Theta} v_j(a, \theta, t_j) P(\theta | t_{-i}, t_i) \right]$$

Alternatively, we could maximize the total payoff of the players in $N\backslash i$ using only the information of the agents in $N\backslash i$. The associated transfer is equal to

$$\sum_{j\in N\backslash i}\hat{v}_j(q(t);t)-\max_{a\in A}\sum_{j\in N\backslash i}\left[\sum_{\theta\in\Theta}v_j(a,\theta,t_j)P(\theta|t_{-i})\right].$$

The first method yields a transfer function that depends on t_{-i} and t_i while the second method yields a function that depends only on t_{-i}. In the first payment scheme, agent i pays the cost that he imposes on other agents assuming that they have access to his information even though he is not present. In the second scheme, agent i pays the cost that he imposes on other agents assuming that the other agents do not have access to his information. In the pure private values model, these two approaches yield the same transfer scheme. Not surprisingly,

$$\max_{a\in A}\sum_{j\in N\backslash i}\left[\sum_{\theta\in\Theta}v_j(a,\theta,t_j)P(\theta|t_{-i})\right]\leq\max_{a\in A}\sum_{j\in N\backslash i}\left[\sum_{\theta\in\Theta}v_j(a,\theta,t_j)P(\theta|t_{-i},t_i)\right]$$

whenever $t\in T^*$.

These payment schemes induce different games in the case of interdependent values. We are interested in the first of the payment schemes that uses agent i's information when calculating the cost that he imposes on other agents. One can think of the designer's problem as extracting two different components of an agent's information, his "external" information about the state θ and the purely private part of the information determining his utility given the state. Our method is to show how the designer can extract the information about θ from the agents, following which the problem becomes a private value problem. In this private value problem, the first payment scheme mimics the standard CGV mechanism.

Under the next assumption, however, the two payment schemes defined above are equivalent.

Definition: A measure $P\in\Delta^*_{\Theta\times T}$ satisfies *nonexclusive information* (NEI) if

$$t\in T^*\Rightarrow P_\Theta(\cdot|t)=P_\Theta(\cdot|t_{-i})\text{ for all }i\in N.$$

Proposition A: Let $\{v_1,..,v_n\}$ be a collection of payoff functions. If $P\in\Delta^*_{\Theta\times T}$ exhibits nonexclusive information and if $q:T\to C$ is outcome efficient for the problem $\{v_1,..,v_n,P\}$, then the GCGV mechanism (q,α_i^q) is ex post IC and ex post IR.

Proof: See appendix.

Nonexclusive information is a strong assumption. Note however, that the pure private values model is a special case: simply choose $|\Theta| = 1$. Our goal in this paper is to identify conditions under which we can modify the GCGV payments so that the new mechanism is interim IC and approximately ex post IC. In the next section, we discuss the two main ingredients of our approximation results: informational size and the variability of agents' beliefs.

5. Informational Size and Variability of Beliefs

5.1. Informational Size

If $t \in T^*$, recall that $P_\Theta(\cdot|t) \in \Delta_\Theta$ denotes the induced conditional probability measure on Θ. A natural notion of an agent's informational size is the degree to which he can alter the best estimate of the state θ when other agents are announcing truthfully. In our setup, that estimate is the conditional probability distribution on Θ given a profile of types t. Any profile of agents' types $t = (t_{-i}, t_i) \in T^*$ induces a conditional distribution on Θ and, if agent i unilaterally changes his announced type from t_i to t'_i, this conditional distribution will (in general) change. We consider agent i to be informationally small if, for each t_i, there is a "small" probability that he can induce a "large" change in the induced conditional distribution on Θ by changing his announced type from t_i to some other t'_i. This is formalized in the following definition.

Definition: Let

$$I^i_\varepsilon(t'_i, t_i) = \{t_{-i} \in T_{-i} | (t_{-i}, t_i) \in T^*, (t_{-i}, t'_i) \in T^* \text{ and } ||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t'_i)|| > \varepsilon\}$$

The *informational size* of agent i is defined as

$$\nu^P_i = \max_{t_i \in T_i} \max_{t'_i \in T_i} \min\{\varepsilon \geq 0 | \text{ Prob}\{\tilde{t}_{-i} \in I^i_\varepsilon(t'_i, t_i) | \tilde{t}_i = t_i\} \leq \varepsilon\}.$$

Loosely speaking, we will say that agent i is *informationally small* with respect to P if his informational size ν^P_i is small. If agent i receives signal t_i but reports $t'_i \neq t_i$, the effect of this misreport is a change in the conditional distribution on Θ from $P_\Theta(\cdot|t_{-i}, t_i)$ to $P_\Theta(\cdot|t_{-i}, t'_i)$. If $t_{-i} \in I_\varepsilon(t'_i, t_i)$, then this change is "large" in the sense that $||P_\Theta(\cdot|\hat{t}_{-i}, t_i) - P_\Theta(\cdot|\hat{t}_{-i}, t'_i)|| > \varepsilon$. Therefore, Prob$\{\tilde{t}_{-i} \in I_\varepsilon(t'_i, t_i) | \tilde{t}_i = t_i\}$

is the probability that i can have a "large" influence on the conditional distribution on Θ by reporting t'_i instead of t_i when his observed signal is t_i. An agent is informationally small if for each of his possible types t_i, he assigns small probability to the event that he can have a "large" influence on the distribution $P_\Theta(\cdot|t_{-i}, t_i)$, given his observed type. Informational size is closely related to the notion of nonexclusive information: if all agents have zero informational size, then P must satisfy NEI. In fact, we have the following easily demonstrated result: $P \in \Delta^*_{\Theta\times T}$ satisfies NEI if and only if $\nu_i^P = 0$ for each $i \in N$.

5.2. Variability of Agents' Beliefs

Whether an agent i can be given incentives to reveal his information will depend on the magnitude of the difference between $P_{T_{-i}}(\cdot|t_i)$ and $P_{T_{-i}}(\cdot|t'_i)$, the conditional distributions on T_{-i} given different types t_i and t'_i for agent i. To define the measure of variability, we first define a metric d on Δ_Θ as follows: for each $\alpha, \beta \in \Delta_\Theta$, let

$$d(\alpha, \beta) = \left\| \frac{\alpha}{||\alpha||_2} - \frac{\beta}{||\beta||_2} \right\|_2$$

where $||\cdot||_2$ denotes the 2-norm. Hence, $d(\alpha, \beta)$ measures the Euclidean distance between the Euclidean normalizations of α and β. If $P \in \Delta_{\Theta\times T}$, let $P_\Theta(\cdot|t_i) \in \Delta_\Theta$ be the conditional distribution on Θ given that i receives signal t_i and define

$$\Lambda_i^P = \min_{t_i \in T_i} \min_{t'_i \in T_i \setminus t_i} d(P_\Theta(\cdot|t_i), P_\Theta(\cdot|t'_i))^2.$$

This is the measure of the "variability" of the conditional distribution $P_\Theta(\cdot|t_i)$ as a function of t_i.

As mentioned in the introduction, our work is related to that of Cremer and McLean (1985,1989). Those papers and subsequent work by McAfee and Reny (1992) demonstrated how one can use correlation to fully extract the surplus in certain mechanism design problems. The key ingredient there is the assumption that the collection of conditional distributions $\{P_{T_{-i}}(\cdot|t_i)\}_{t_i\in T_i}$ is a linearly independent set for each i. This of course, implies that $P_{T_{-i}}(\cdot|t_i) \neq P_{T_{-i}}(\cdot|t'_i)$ if $t_i \neq t'_i$ and, therefore, that $\Lambda_i^P > 0$. While linear independence implies that $\Lambda_i^P > 0$, the actual (positive) size of Λ_i^P is not relevant in the Cremer-McLean constructions, and full extraction will be possible. In the present work, we do not require that the collection $\{P_{T_{-i}}(\cdot|t_i)\}_{t_i\in T_i}$ be linearly independent (or satisfy the weaker cone condition in Cremer and McLean (1988)). However, the "closeness" of the members of $\{P_{T_{-i}}(\cdot|t_i)\}_{t_i \in T_i}$ is an important issue. It can be shown that for each i, there exists a collection of numbers $\varsigma_i(t)$ satisfying $0 \leq \zeta_i(t) \leq 1$ and

$$\sum_{t_{-i}} \left[\varsigma_i(t_{-i}, t_i) - \varsigma_i(t_{-i}, t_i')\right] P_{T_{-i}}(t_{-i}|t_i) > 0$$

for each $t_i, t_i' \in T_i$ if and only if $\Lambda_i^P > 0$. The elements of the collection $\{\varsigma_i(t)\}_{i \in I, t \in T}$ can be thought of as "incentive payments" to the agents to reveal their information. The above inequality assures that, if the posteriors $\{P_{T_{-i}}(\cdot|t_i)\}_{t_i \in T_i}$ are all distinct, then the incentive compatibility inequalities above are strict. However, the expression on the left hand side decreases as $\Lambda^P \to 0$. Hence, the difference in the expected reward from a truthful report and from a false report will be very small if the conditional posteriors are very close to each other. Our results require that informational size and aggregate uncertainty be small relative to the variation in these posteriors.

6. Implementation and Informational Size

6.1. The Results

Let $\{z_i\}_{i \in N}$ be an n-tuple of functions $z_i : T \to \Re_+$ each of which assigns to each $t \in T$ a nonnegative number, interpreted as a "reward" to agent i. If $(q, x_1, ..x_n)$ is a mechanism, then the associated *augmented* mechanism is defined as $(q, x_1 + z_1, ..x_n + z_n)$ and will be written simply as $(q, x_i + z_i)$.

Theorem A: Let $(v_1, .., v_n)$ be a collection of payoff functions.

(i) Suppose that $P \in \Delta^*_{\Theta \times T}$ satisfies $\Lambda_i^P > 0$ for each i and suppose that $q : T \to C$ is outcome efficient for the problem $\{v_1, .., v_n, P\}$. Then there exists an augmented GCGV mechanism $(q, \alpha_i^q + z_i)$ for the social choice problem problem $(v_1, .., v_n, P)$ satisfying ex post IR and interim IC.

(ii) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta^*_{\Theta \times T}$ satisfies

$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^P,$$

and whenever $q : T \to C$ is outcome efficient for the problem $\{v_1, .., v_n, P\}$, there exists an augmented GCGV mechanism $(q, \alpha_i^q + z_i)$ with $0 \leq z_i(t) \leq \varepsilon$ for every i

and t satisfying ex post IR, interim IC and $\varepsilon-$ex post IC. Furthermore, for each $t \in T^*$,

$$0 \leq \sum_{i \in N} (\alpha_i(t) + z_i(t)) = \sum_{i \in N} \alpha_i(t) + n\varepsilon.$$

6.2. Discussion

Our results rely on the following key lemma.

Lemma A: Suppose that $q : T \to C$ is an efficient outcome function for the problem $\{v_1, .., v_n, P\}$. If $(t_{-i}, t_i), (t_{-i}, t_i') \in T^*$, then

$$\begin{aligned}&(\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i')) - \ (\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i)) \\ &\leq 2M(n-1)||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t_i')||\end{aligned}$$

In the case of the GCGV mechanism, Lemma A provides an upper bound on the "ex post gain" to agent i when i's true type is t_i but i announces t_i' and others announce truthfully. If agents have zero informational size – that is, if P exhibits nonexclusive information – then $||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t_i')|| = 0$ if $(t_{-i}, t_i), (t_{-i}, t_i') \in T^*$. Hence, truth is an ex post Nash equilibrium and Proposition A follows. If v_i does not depend on θ, then (letting $|\Theta| = 1$), we recover Vickrey's classic dominant strategy result for the CGV mechanisms in the pure private values case.

If agent i is informationally small, then (informally) we can deduce from Lemma A that

$$\Pr ob\{||P_\Theta(\cdot|\tilde{t}_{-i}, t_i) - P_\Theta(\cdot|\tilde{t}_{-i}, t_i')|| \approx 0 | \tilde{t}_i = t_i\} \approx 1$$

so truth is an "approximate" ex post equilibrium for the CGCV in the sense that

$$\begin{aligned}&\Pr ob\{(\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i)) \\ &-(\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i')) \underset{\approx}{>} 0 | \tilde{t}_i = t_i\} \approx 1.\end{aligned}$$

Lemma A has a second important consequence: if agent i is informationally small, then truth is an approximate Bayes-Nash equilibrium in the GCGV mechanism so the mechanism is *approximately* interim incentive compatible. More precisely, we can deduce from Lemma A that the interim expected gain from misreporting one's type is essentially bounded from above by one's informational size. If we want the mechanism to be *exactly* interim incentive compatible, then we must alter the mechanism (specifically, construct an augmented GCGV mechanism) in order to provide the correct incentives for truthful behavior. It is in this step that variability of beliefs plays a crucial role. To see this, first note that incentive compatibility of the augmented GCGV mechanism requires that

$$\begin{aligned}
&\sum_{(t_{-i},t_i)\in T^*} \left[(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + \alpha_i^q(t_{-i},t_i)) - (\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i) + \alpha_i^q(t_{-i},t_i')) \right] P(t_{-i}|t_i) \\
&+ \sum_{:(t_{-i},t_i)\in T^*} (z_i(t_{-i},t_i) - z_i(t_{-i},t_i'))\, P(t_{-i}|t_i) \\
&\geq 0
\end{aligned}$$

Lemma A implies that the first term is bounded from below by $-K\nu_i^P$ where K is a positive constant independent of P. If $\Lambda_i^P > 0$, then there exists a collection of numbers $\varsigma_i(t)$ satisfying $0 \leq \zeta_i(t) \leq 1$ and

$$\sum_{t_{-i}} \left[\varsigma_i(t_{-i},t_i) - \varsigma_i(t_{-i},t_i') \right] P_{T_{-i}}(t_{-i}|t_i) > 0$$

for each $t_i, t_i' \in T_i$. By defining $z_i(t_{-i},t_i) = \eta\zeta_i(t_{-i},t_i)$ and choosing η sufficiently large, then we will obtain interim incentive compatibility of the augmented GCGV mechanism. This is part (i) of Theorem A. As the informational size of an agent decreases, the minimal reward required to induce the truth also decreases. If Λ_i^P large enough relative to an agent's informational size ν_i^P, then we can construct an augmented mechanism satisfying interim incentive compatibility. This is part (ii) of Theorem A.

Heuristically, Theorem A can be described in the following way. If a problem is pure private value problem , then CGV mechanisms will implement efficient outcomes. When there are interdependent values, these mechanisms no longer are incentive compatible. When there are interdependent values, a given agent's utility depends on other agents' types insofar as their types are correlated to the state θ. If there is correlation in the parts of agents' information that affects θ, that part can be elicited via payments to the agents that are of the magnitude of their informational size; this is the "augmented" part of the augmented GCGV mechanism. Once the part of agents' information that affects other agents' values is obtained, the problem essentially becomes a private value problem, and CGV-type payments can be used to extract the residual private information agents may have.

For pure private value problems, Green and Laffont (1979) show that the CGV payments are essentially unique. While we cannot be sure that the particular characteristics of the payment scheme for pure private good problems carries over to our framework, we conjecture that the only efficient social choice functions that can be implemented in our framework embody CGV payments. For pure common value problems, once agents' information about θ is elicited, there is no residual uncertainty since by definition agents' utilities depend only on θ. In this case, the only transfers that are needed are our augmented transfers that elicit information about θ.

The above discussion suggests an analogue of Theorem A for purely common value problems with positive variability that has no relation to the CGV mechanism. There is a difficulty in establishing such a result. It is true that when an agent is informationally small, his expected effect on the posterior distribution on Θ will be small. In order to guarantee that this does not translate into a large utility gain, we require a continuity assumption on the mapping from posterior distributions on Θ into agents' utilities. We turn to this next.

6.3. Gain bounded Mechanisms

In a typical implementation or mechanism design problem, one computes the mechanism for each instance of the data that defines the social choice problem. Therefore, in many if not most cases of interest, the mechanism is parametrized by the data defining the social choice problem. If we fix a profile $(v_1, .., v_n)$ of payoff functions, then we can analyze the parametric dependence of the mechanism on the probability distribution P and this dependence can be modelled as a mapping that associates a mechanism with each $P \in \Delta^*_{\Theta \times T}$. We will denote this mapping $P \mapsto (q^P, x_1^P, .., x_n^P)$. For example, the mapping naturally associated with the GCGV mechanism is defined by

$$q^P(t) \in \arg\max_{c \in C} \sum_{j \in N} \sum_{\theta \in \Theta} v_i(c, \theta, t_i) P(\theta | t_{-i}, t_i) \text{ if } t \in T^*$$
$$q^P(t) = c_0 \text{ if } t \notin T^*$$

and

$$x_i^P(t) = \sum_{j \in N \setminus i} \sum_{\theta \in \Theta} v_i(q^P(t), \theta, t_i) P(\theta | t_{-i}, t_i) - \max_{c \in C} \left[\sum_{j \in N \setminus i} \sum_{\theta \in \Theta} v_i(c, \theta, t_i) P(\theta | t_{-i}, t_i) \right] \text{ if } t \in T^*$$
$$= 0 \text{ if } t \notin T^*$$

Definition: Let $(v_1, .., v_n)$ be a profile of payoff functions. For each $P \in \Delta^*_{\Theta\times T}$, let $(q^P, x_1^P, .., x_n^P)$ be a mechanism for the social choice problem $(v_1, .., v_n, P)$. We will say that the mapping $P \mapsto (q^P, x_1^P, .., x_n^P)$ is *gain bounded* if there exists a $K > 0$ such that for all $P \in \Delta^*_{\Theta\times T}$,

$$\begin{aligned}&\big(\hat{v}_i(q^P(t_{-i}, t_i'); t_{-i}, t_i) + x_i^P(t_{-i}, t_i')\big) - \big(\hat{v}_i(q^P(t_{-i}, t_i); t_{-i}, t_i) + x_i^P(t_{-i}, t_i)\big)\\ &\leq K||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t_i')||\end{aligned}$$

whenever $(t_{-i}, t_i), (t_{-i}, t_i') \in T^*$.

If q is outcome efficient, the GCGV mechanism is gain bounded. In addition, a large class of gain bounded mechanisms are associated with efficient outcome functions generated by payoff functions satisfying the pure common value assumption.

Theorem B: Let $(v_1, .., v_n)$ be a collection of payoff functions satisfying the pure common value assumption.

(i) Suppose that $P \in \Delta^*_{\Theta\times T}$ satisfies $\Lambda_i^P > 0$ for each i and suppose that $q^P : T \to C$ is outcome efficient for the problem $\{v_1, .., v_n, P\}$ and $P \mapsto (q^P, x_1^P, .., x_n^P)$ is gain bounded. Then there exists an augmented mechanism $\{q^P, x_i^P + z_i^P\}_{i\in N}$ for the social choice problem problem $(v_1, .., v_n, P)$ satisfying ex post IR and interim IC.

(ii) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta^*_{\Theta\times T}$ satisfies

$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^P,$$

and whenever $q^P : T \to C$ is outcome efficient for the problem $\{v_1, .., v_n, P\}$, there exists an augmented mechanism $(q^P, x_i^P + z_i^P)$ for the social choice problem problem $(v_1, .., v_n, P)$ with $0 \leq z_i^P(t) \leq \varepsilon$ for every i and t satisfying ex post IR, interim IC and $\varepsilon-$ex post IC. Furthermore,

$$0 \leq \sum_{i\in N} z_i^P \leq n\varepsilon.$$

7. Asymptotic Results

Informally, an agent is informationally small when the probability that he can affect the posterior distribution on Θ is small. One would expect, in general, that agents will be informationally small in the presence of many agents. For example, if agents receive conditionally independent signals regarding the state θ, the announcement of one of many agents is unlikely to significantly alter the posterior distribution on Θ. Hence, it is reasonable to conjecture that (under suitable assumptions) an agent's informational size goes to zero in a sequence of economies with an increasing number of agents. Consequently, the required rewards z_i that induce truthful behavior will also go to zero as the number of agents grows. We will show below that this is in fact the case. Of greater interest, however, is the beavior of the *aggregate* reward necessary to induce truthful revelation. The argument sketched above only suggests that each individual's z_i goes to 0 as the number of agents goes to infinity but does not address the asymptotic behavior of the sum of the z_i's. Roughly speaking, the size of the z_i that is necessary to induce agent i to reveal truthfully is of the order of magnitude of his informational size. Hence, the issue is the speed with which agents' informational size goes to 0 as the number of agents increases. We will demonstrate below that, under reasonably general conditions, agents' informational size goes to 0 at an exponential rate and that the total reward $\sum_{i\in N} z_i$ goes to zero as the number of agents increases.

7.1. Notation and Definitions:

We will assume that all agents have the same finite signal set $T_i = A$. Let $J_r = \{1, 2, ...r\}$. For each $i \in J_r$, let $v_i^r : C \times \Theta \times A \to \Re_+$ denote the payoff to agent i. For any positive integer r, let $T^r = A \times \cdots \times A$ denote the r-fold Cartesian product and let $t^r = (t_1^r, .., t_r^r)$ denote a generic element of T^r.

Definition: A sequence of prob measures $\{P^r\}_{r=1}^{\infty}$ with $P^r \in \Delta_{\Theta\times T^r}$ is a *conditionally independent sequence* if there exists $P \in \Delta_{\Theta\times A}$ such that

(a) For each r and each $(\theta, t_1, .., t_r) \in \Theta \times T^r$,

$$P^r(t_1^r, .., t_r^r|\theta) = \text{Prob}\{\widetilde{t_1^r} = t_1, \widetilde{t_2^r} = t_2, ..., \widetilde{t_r^r} = t_r|\tilde{\theta} = \theta\} = \prod_{i=1}^{r} P(t_i|\theta).$$

(b) For every $\theta, \hat{\theta}$ with $\theta \neq \hat{\theta}$, there exists a $t \in A$ such that $P(t|\theta) \neq P(t|\hat{\theta})$.

Because of the symmetry in the objects defining a conditionally independent sequence, it follows that, for fixed r, the informational size of each $i \in J_r$ is the same. In the remainder of this section we will drop the subscript i and will write ν^{P^r} for the value of the informational size of agents in J_r.

Lemma C: Suppose that $\{P^r\}_{r=1}^{\infty}$ is a conditionally independent sequence. For every $\varepsilon > 0$ and every positive integer k, there exists an $\hat{r}$ such that

$$r^k \nu^{P^r} \leq \varepsilon$$

whenever $r > \hat{r}$.

The proof is provided in the appendix and is an application of a large deviations result due to Hoeffding (1960). With this lemma, we can prove the following asymptotic result, the proof of which is also in the appendix.

Theorem C: Suppose that $\{P^r\}_{r=1}^{\infty}$ is a conditionally independent sequence. Let M and ε be positive numbers. Let $\{(v_1^r, .., v_r^r)\}_{r \geq 1}$ be a sequence of payoff function profiles and for each r, let $\{q^{P^r}(r), x_1^{P^r}(r), .., x_r^{P^r}(r)\}$ be an ex post IR mechanism for the SCP $(v_1^r, .., v_r^r, P^r)$. Suppose that

(1) $|v_i^r(\cdot, \cdot, \cdot)| \leq M$ for all r and $i \in J_r$

(2) For each r, $(q^{P^r}(r), x_1^{P^r}(r), .., x_r^{P^r}(r))$ is lower bounded mechanism with constant $K(P^r)$ and for some positive integer L, $r^{-L}K(P^r) \to 0$ as $r \to \infty$.

(3) The marginal measure of P^2 on T^2 exhibits positive variability.

Then there exists an $\hat{r}$ such that for all $r > \hat{r}$, there exists an augmented mechanism $(q^{P^r}(r), x_1^{P^r}(r) + z_1^r, .., x_r^{P^r}(r) + z_r^r)$ for the social choice problem $(v_1^r, .., v_r^r, P^r)$ satisfying ex post IR and interim IC. Furthermore, for each $t^r \in T^r$, $z_i^r(t^r) \geq 0$ and $\sum_{i \in J_r}^r z_i^r(t^r) \leq \varepsilon$.

Corollary: Suppose that $\{P^r\}_{r=1}^{\infty}$ is a conditionally independent sequence. Let M and ε be positive numbers. Let $\{(v_1^r, .., v_r^r)\}_{r \geq 1}$ be a sequence of payoff function profiles and for each r, let $\{q^{P^r}(r), \alpha_1^{P^r}(r), .., \alpha_r^{P^r}(r)\}$ denote the GCGV mechanism for the SCP $(v_1^r, .., v_r^r, P^r)$. Suppose that $|v_i^r(\cdot, \cdot, \cdot)| \leq M$ for all r and $i \in J_r$.

Then there exists an $\hat{r}$ such that for all $r > \hat{r}$, there exists an augmented GCGV mechanism $(q^{P^r}(r), \alpha_1^{P^r}(r) + z_1^r, .., \alpha_r^{P^r}(r) + z_r^r)$ for the social choice problem $(v_1^r, .., v_r^r, P^r)$ satisfying ex post IR and interim IC. Furthermore, for each $t^r \in T^r$, $z_i^r(t^r) \geq 0$ and $\sum_{i \in J_r}^r z_i^r(t^r) \leq \varepsilon$.

8. Discussion

1. Our results are related to the work on surplus extraction (see, e.g., Cremer and McLean 1985, 1988) and McAfee and Reny (1992). For auction problems, our results say that a seller of an object can extract the information about θ by making payments to each agent of the order of magnitude of that agent's informational size. Under the mechanism in Theorem A, the seller will extract all surplus except for the payments necessary to elicit the private information about θ and the surplus associated with the purely private component of their information. As in the case of a purely private value problem with indpendent types, the seller will not be able to extract all surplus. Of course, if the purely private value components of the agents' information are correlated, one could extract some of the surplus associated with the private values.

For the asymptotic problem, the asymptotic revenue is full extraction from the highest value agent. This is because we extract all the surplus except the payments in the augmentation, and the augmentation payments go to zero. This plus the fact that the surplus the high value guy gets goes to zero since he's getting the object at the second highest value and the difference between the highest value and the second highest value goes to zero.

2. It is worth pointing out one further aspect of agents' informational size in expanding economies. Roughly speaking, when an agent has informational size ε, the probability that he can change the posterior distribution on Θ by more than ε is less than ε. One might consider an alternative definition of informational size whereby an agent's informational size is ε if with probability one he cannot change the posterior distribution on Θ by more than ε. Formally,

Definition: The *strong informational size* of agent i is defined as

$$\sigma_i^P = \max_{t_i \in T_i} \max_{t_i' \in T_i} \max_{t_{-i} \in T_{-i}} ||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t_i')||.$$

We will refer to an agent as strongly informationally small if his strong informational size is small. From the definitions, it is clear that $\nu_i^P \leq \sigma_i^P$. For economic problems with a small number of agents, it is often the case that an agent may be informationally small but not be strongly informationally small. For example, consider a problem in with two states, θ_1 and θ_2, and three agents, each of whom receives a noisy signal about the state θ. With very accurate signals, each agent's signal is the true state θ with high probability. In this case, it is easy to verify that any agent who unilaterally misreports his signal will, with high probability, have only a small effect on the posterior distribution and, consequently, agents are informationally small. However, it is easy to see that agents will not be *strongly* informationally small. Although with very accurate signals all agents' signals will be the correct state θ, there is positive probability that two agents, say agent 1 and agent 2, receive different signals. In this case, agent 3's announcement will have a large effect on the posterior distribution; whether he announces θ_1 or θ_2, one of the other two agents' announcements will match his announcement and one will not. When the signals are very accurate, the posterior distribution on Θ will put very high probability on agent 3's announced state, and hence, his announcement will have a large effect on the posterior distribution in this case.

The discussion above illustrates the advantage of results that employ the weaker notion of informational size rather than strong informational size: a large and interesting class of problems is covered by the former notion that will not be covered by the latter. There is, of course, a cost: theorems employing the weaker hypothesis will have weaker consequences. If a mechanism satisfies our notion of $\varepsilon-$ex post IC, the probability that a change in an agent's reported type (given other agents' types) would increase his utility by more than ε is less than ε. This, of course allows the possibility that a change could lead to a large increase in his utility for some (low probability) vectors of other agents' types. The small probability of large utility gains is connected to the small probability of an agent's report having a large effect on the posterior distribution. In interdependent type mechanisms, an agent's transfer depends on other agents' valuations, and those valuations depend on the posterior distribution on Θ; large changes in the posterior distribution can translate into large changes in utility.

The above discussion suggests a stronger notion of approximate ex post incentive compatibility:

Definition: Let $\varepsilon \geq 0$. A mechanism $\{q, x_i\}_{i \in N}$ is *strongly* ε*- ex post incentive compatible* if for all i, all $t_i, t_i' \in T_i$ and all $t_{-i} \in T_{-i}$ such that $(t_{-i}, t_i) \in T^*$

$$(\hat{v}_i(q(t_{-i}, t_i^{'}); t_{-i}, t_i) + x_i(t_{-i}, t_i^{'})) - (\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + x_i(t_{-i}, t_i)) \leq \varepsilon.$$

That is, a mechanism is strongly ε- ex post incentive compatible if, with probability one, no agent can increase his utility by more than ε regardless of other agents' types.

On might hope that for problems in which agents are strongly informationally small, mechanisms that yield efficient outcomes might be strongly ex post incentive compatible. We argued above that for many problems with small numbers of agents, requiring that agents be strongly informationally small might be too demanding. When there are many agents, however, there is some reason to think that agents might be strongly informationally small. The three person example discussed above illustrated how an agent's reported type might have a large effect on the posterior distribution: when two agents "tie" in their signals about the two states, the third agent's announcement makes a large difference. If a given agent is one of several thousand agents receiving a noisy signal of the two alternative states **that has low accuracy**, however, there will be no distribution of other agents' signals that will make the given agent's signal important.

9. Proofs:

9.1. Proof of Lemma A:

First, consider the GCGV mechanism. Choose $(t_{-i}, t_i), (t_{-i}, t_i') \in T^*$. Then

$$\begin{aligned}\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + \alpha_i(t_{-i}, t_i) &= \hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + \sum_{j \in N \setminus i} \hat{v}_j(q(t_{-i}, t_i); t_{-i}, t_i) \\ &\quad - \max_{c \in C} \left[\sum_{j \in N \setminus i} \hat{v}_j(c; t_{-i}, t_i) \right]\end{aligned}$$

and

$$\begin{aligned}\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + \alpha_i(t_{-i}, t_i') &= \hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + \sum_{j \in N \setminus i} \hat{v}_j(q(t_{-i}, t_i'); t_{-i}, t_i) \\ &\quad - \sum_{j \in N \setminus i} \hat{v}_j(q(t_{-i}, t_i'); t_{-i}, t_i) \\ &\quad + \sum_{j \in N \setminus i} \hat{v}_j(q(t_{-i}, t_i'); t_{-i}, t_i') - \max_{c \in C} \left[\sum_{j \in N \setminus i} \hat{v}_j(c; t_{-i}, t_i') \right]\end{aligned}$$

Since

$$\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + \sum_{j \in N \setminus i} \hat{v}_j(q(t_{-i}, t_i); t_{-i}, t_i) \geq \hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + \sum_{j \in N \setminus i} \hat{v}_j(q(t_{-i}, t_i'); t_{-i}, t_i)$$

it follows that

$$
\begin{aligned}
&(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\alpha_i(t_{-i},t_i))-(\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i)+\alpha_i(t_{-i},t_i'))\\
&\geq \max_{c\in C}\left[\sum_{j\in N\setminus i}\hat{v}_j(c;t_{-i},t_i')\right]-\max_{c\in C}\left[\sum_{j\in N\setminus i}\hat{v}_j(c;t_{-i},t_i)\right]\\
&\quad-\sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t_i');t_{-i},t_i')+\sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t_i');t_{-i},t_i)
\end{aligned}
$$

Let

$$
q^*(t_{-i},t_i)\in\arg\max_{c\in C}\left[\sum_{j\in N\setminus i}\hat{v}_j(c;t_{-i},t_i)\right]
$$

and let

$$
q^*(t_{-i},t_i')\in\arg\max_{c\in C}\left[\sum_{j\in N\setminus i}\hat{v}_j(c;t_{-i},t_i')\right].
$$

Then

$$
\begin{aligned}
&\max_{q\in C}\left[\sum_{j\in N\setminus i}\hat{v}_j(q;t_{-i},t_i')\right]-\max_{q\in C}\left[\sum_{j\in N\setminus i}\hat{v}_j(q;t_{-i},t_i)\right]\\
&=\left[\sum_{j\in N\setminus i}\hat{v}_j(q^*(t_{-i},t_i');t_{-i},t_i')\right]-\left[\sum_{j\in N\setminus i}\hat{v}_j(q^*(t_{-i},t_i);t_{-i},t_i)\right]\\
&=\left[\sum_{j\in N\setminus i}\hat{v}_j(q^*(t_{-i},t_i');t_{-i},t_i')-\sum_{j\in N\setminus i}\hat{v}_j(q^*(t_{-i},t_i);t_{-i},t_i')\right]\\
&\quad+\left[\sum_{j\in N\setminus i}\hat{v}_j(q^*(t_{-i},t_i);t_{-i},t_i')-\sum_{j\in N\setminus i}\hat{v}_j(q^*(t_{-i},t_i);t_{-i},t_i)\right]\\
&\geq\sum_{j\in N\setminus i}\hat{v}_j(q^*(t_{-i},t_i);t_{-i},t_i')-\sum_{j\in N\setminus i}\hat{v}_j(q^*(t_{-i},t_i);t_{-i},t_i)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+x_i(t_{-i},t_i))-(\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i)+x_i(t_{-i},t_i'))\\
&\geq \sum_{j\in N\setminus i}\hat{v}_j(q^*(t_{-i},t_i);t_{-i},t_i')-\sum_{j\in N\setminus i}\hat{v}_j(q^*(t_{-i},t_i);t_{-i},t_i)\\
&-\sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t_i');t_{-i},t_i')+\sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t_i');t_{-i},t_i)\\
&\geq -2M(n-1)||P_\Theta(\cdot|t_{-i},t_i)-P_\Theta(\cdot|t_{-i},t_i')||
\end{aligned}
$$

9.2. Proof of Theorem A:

We prove part (ii) first. Choose $\varepsilon > 0$. Let

$$M=\max_\theta\max_i\max_{t_i}\max_{q\in C}v_i(q,\theta,t_i)$$

and let K be the cardinality of T. Choose δ so that

$$0<\delta<\frac{\varepsilon}{4M(n+1)\sqrt{K}}.$$

Suppose that $P\in\Delta^*_{\Theta\times T}$ satisfies

$$\max_i \nu_i^P\leq\delta\min_i\Lambda_i^P.$$

Define $\hat{\nu}^P=\max_i\nu_i^P$and $\Lambda^P=\min_i\Lambda_i^P$. Therefore $\hat{\nu}^P\leq\delta\Lambda^P$. Since

Now we define an augmented GCGV mechanism. For each $t\in T$, define

$$z_i(t_{-i},t_i)=\varepsilon\frac{P_{T_{-i}}(t_{-i}|t_i)}{||P_{T_{-i}}(\cdot|t_i)||_2}.$$

Since $0\leq\frac{P_{T_{-i}}(t_{-i}|t_i)}{||P_{T_{-i}}(\cdot|t_i)||_2}\leq 1$, it follows that

$$0\leq z_i(t_{-i},t_i)\leq\varepsilon$$

for all i, t_{-i} and t_i.

The augmented CGV mechanism $\{q,\alpha_i^q+z_i\}_{i\in N}$ is clearly ex post efficient. Individual rationality follows from the observations that

$$\hat{v}_i(q(t);t)+\alpha_i^q(t)\geq 0$$

and

$$z_i(t) \geq 0.$$

Claim 1: Let $K = |T|$. Then

$$\sum_{(t_{-i},t_i)\in T^*} \left(z_i(t_{-i}|t_i) - z_i(t_{-i}|t'_i)\right) P(t_{-i}|t_i) \geq \frac{\varepsilon}{2\sqrt{K}}\Lambda_i^P$$

Proof of Claim 1:

$$\begin{aligned}
\sum_{(t_{-i},t_i)\in T^*} \left(z_i(t_{-i}|t_i) - z_i(t_{-i}|t'_i)\right) P(t_{-i}|t_i) &= \sum_{(t_{-i},t_i)\in T^*} \left(z_i(t_{-i}|t_i) - z_i(t_{-i}|t'_i)\right) P(t_{-i}|t_i) \\
&= \sum_{(t_{-i},t_i)\in T^*} \varepsilon \left[\frac{P_{T_{-i}}(t_{-i}|t_i)}{||P_{T_{-i}}(\cdot|t_i)||_2} - \frac{P_{T_{-i}}(t_{-i}|t'_i)}{||P_{T_{-i}}(\cdot|t'_i)||_2}\right] P(t_{-i}|t_i) \\
&= \frac{\varepsilon||P_{T_{-i}}(\cdot|t_i)||_2}{2} \left\|\frac{P_{T_{-i}}(\cdot|t_i)}{||P_{T_{-i}}(\cdot|t_i)||_2} - \frac{P_{T_{-i}}(\cdot|t'_i)}{||P_{T_{-i}}(\cdot|t'_i)||_2}\right\|^2 \\
&\geq \frac{\varepsilon}{2\sqrt{K}}\Lambda_i^P
\end{aligned}$$

This completes the proof of Claim 1.

Claim 2:

$$\sum_{(t_{-i},t_i)\in T^*} \left[(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + x_i(t_{-i},t_i)) - (\hat{v}_i(q(t_{-i},t'_i);t_{-i},t_i) + x_i(t_{-i},t'_i))\right] P(t_{-i}|t_i) \geq -5M\hat{\nu}^P$$

Proof of Claim 2: Define

$$A_i(t'_i,t_i) = \{t_{-i} \in T_{-i}|\ (t_{-i},t_i) \in T^*, (t_{-i},t'_i) \in T^*, ||P_\Theta(\cdot|t_{-i},t_i) - P_\Theta(\cdot|t_{-i}t'_i)|| > \hat{\nu}^P\}$$

and

$$B_i(t'_i,t_i) = \{t_{-i} \in T_{-i}|\ (t_{-i},t_i) \in T^*, (t_{-i},t'_i) \in T^*, ||P_\Theta(\cdot|t_{-i},t_i) - P_\Theta(\cdot|t_{-i}t'_i)|| \leq \hat{\nu}^P\}$$

and

$$C_i(t'_i,t_i) = \{t_{-i} \in T_{-i}|\ (t_{-i},t_i) \in T^*, (t_{-i}t'_i) \notin T^*\}$$

Since $\nu_i^P \leq \hat{\nu}^P$, we conclude that

$$\text{Prob}\{\tilde{t}_{-i} \in A_i(t'_i,t_i)|\tilde{t}_i = t_i\} \leq \nu_i^P \leq \hat{\nu}^P.$$

In addition,

$$0 \leq \hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + \alpha_i^q(t_{-i},t_i) \leq \hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) \leq M$$

for all i, t_i and t_{-i}. Therefore,

$$\begin{aligned}
|\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i) + \alpha_i^q(t_{-i},t_i')| &= |\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i) - \hat{v}_i(q(t_{-i},t_i');t_{-i},t_i') \\
&\quad + \hat{v}_i(q(t_{-i},t_i');t_{-i},t_i') + \alpha_i^q(t_{-i},t_i')| \\
&\leq |\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i) - \hat{v}_i(q(t_{-i},t_i');t_{-i},t_i')| \\
&\quad + |\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i') + \alpha_i^q(t_{-i},t_i')| \\
&\leq 3M
\end{aligned}$$

for all i, t_i, t_i' and t_{-i}. Applying the definitions, it follows that

$$\begin{aligned}
&\sum_{t_{-i}\in A_i(t_i',t_i)} \left[(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + \alpha_i^q(t_{-i},t_i)) - (\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i) + \alpha_i^q(t_{-i},t_i'))\right] P(t_{-i}|t_i) \\
&\geq -3M \sum_{t_{-i}\in A_i(t_i',t_i)} P(t_{-i}|t_i) \\
&\geq -3M\hat{\nu}^P.
\end{aligned}$$

$$\begin{aligned}
&\sum_{t_{-i}\in B_i(t_i',t_i)} \left[(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + \alpha_i^q(t_{-i},t_i)) - (\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i) + \alpha_i^q(t_{-i},t_i'))\right] P(t_{-i}|t_i) \\
&\geq -2M(n-1) \sum_{t_{-i}\in B_i(t_i',t_i)} ||P_\Theta(\cdot|t_{-i},t_i) - P_\Theta(\cdot|t_{-i}t_i')||P(t_{-i}|t_i) \\
&\geq -2M(n-1)\hat{\nu}^P.
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{t_{-i}\in C_i(t_i',t_i)} \left[(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + \alpha_i^q(t_{-i},t_i)) - (\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i) + \alpha_i^q(t_{-i},t_i'))\right] P(t_{-i}|t_i) \\
&= \sum_{t_{-i}\in C_i(t_i',t_i)} \left[(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + \alpha_i^q(t_{-i},t_i)) - (\hat{v}_i(c_0;t_{-i},t_i) + 0)\right] P(t_{-i}|t_i) \\
&= \sum_{t_{-i}\in C_i(t_i',t_i)} (\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + \alpha_i^q(t_{-i},t_i))\, P(t_{-i}|t_i) \\
&\geq 0.
\end{aligned}$$

Combining these observations completes the proof of the claim 2.

Applying Claims 1 and 2, it follows that

$$\begin{aligned}
&\sum_{(t_{-i},t_i)\in T^*} \left(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\alpha_i(t_{-i},t_i)+z_i(t_{-i},t_i)\right) P(t_{-i}|t_i) \\
&-\sum_{(t_{-i},t_i)\in T^*} \left(\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i)+\alpha_i(t_{-i},t_i')+z_i(t_{-i},t_i')\right) P(t_{-i}|t_i) \\
&=\sum_{(t_{-i},t_i)\in T^*} \left[\left(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\alpha_i^q(t_{-i},t_i)\right)-\left(\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i)+\alpha_i^q(t_{-i},t_i')\right)\right] P(t_{-i}|t_i) \\
&\quad +\sum_{:(t_{-i},t_i)\in T^*} \left(z_i(t_{-i},t_i)-z_i(t_{-i},t_i')\right) P(t_{-i}|t_i) \\
&\geq \frac{\varepsilon}{2\sqrt{K}}\Lambda_i^P - 2(n+1)M\hat{\nu}^P \\
&\geq 0.
\end{aligned}$$

and the proof of part (ii) is complete.

Part (i) follows from the computations in part (ii). We have shown that, for any positive number α, there exists an augmented GCGV mechanism $\{q,\alpha_i^q+z_i\}_{i\in N}$ satisfying

$$\begin{aligned}
&\sum_{(t_{-i},t_i)\in T^*} \left[\left(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\alpha_i^q(t_{-i},t_i)\right)-\left(\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i)+\alpha_i^q(t_{-i},t_i')\right)\right] P(t_{-i}|t_i) \\
&\geq \frac{\alpha}{2\sqrt{K}}\Lambda_i^P - 5M\hat{\nu}^P
\end{aligned}$$

for each i and each t_i, t_i'. If $\Lambda_i^P > 0$ for each i, then α can be chosen large enough so that incentive compatibility is satisfied. This completes the proof of part (i).

9.3. Proof of Lemma C

Let $P(\cdot|\theta)$ denote the conditional measure on A and we assume that $P(\cdot|\theta) \neq P(\cdot|\hat{\theta})$. Let $t^r = (t_1^r, .., t_r^r)$ so that $\Pr ob\{\tilde{t}^r = t^r|\tilde{\theta} = \theta)\} = P(t_1^r|\theta)\cdots P(t_r^r|\theta)$. For each $\alpha \in A$, let $f(t^r,\alpha) = \#\{i \leq r | t_i^r = \alpha\}$ and define $f(t^r) = (f(t^r,\alpha))_{\alpha\in A}$.

For each θ, let

$$\rho(\theta) := \max_{\hat{\theta}\neq\theta} \prod_{\alpha\in A}\left[\frac{P(\alpha|\hat{\theta})}{P(\alpha|\theta)}\right]^{P(\alpha|\theta)}$$

Using the same argument found in Gul and Postlewaite (see their equation 9) we deduce that $\rho(\theta) < 1$. It is easy to show (simply compute the logarithm) that there exists a $\delta > 0$ such that

$$\prod_{\alpha \in A} \left[\frac{P_\Theta(\alpha|\hat{\theta})}{P_\Theta(\alpha|\theta)} \right]^{\frac{f(t^r|\alpha)}{r} - P(\alpha|\theta)} \leq \frac{1}{\sqrt{\rho(\theta)}}$$

whenever $\hat{\theta} \neq \theta$ and $||\frac{f(t^r)}{r} - P(\cdot|\theta)|| < \delta$. Letting $R = \max_\theta \rho(\theta)$, we conclude that $||\frac{f(t^r)}{r} - P(\cdot|\theta)|| < \delta$ implies that

$$\frac{P_\Theta(\hat{\theta}|t^r)}{P_\Theta(\theta|t^r)} = \left[\prod_{\alpha \in A} \left[\frac{P_\Theta(\alpha|\hat{\theta})}{P_\Theta(\alpha|\theta)} \right]^{P(\alpha|\theta)} \prod_{\alpha \in A} \left[\frac{P_\Theta(\alpha|\hat{\theta})}{P_\Theta(\alpha|\theta)} \right]^{\frac{f(t^r|\alpha)}{r} - P(\alpha|\theta)} \right]^r \leq \left[\rho(\theta) \frac{1}{\sqrt{\rho(\theta)}} \right]^r \leq R^{r/2}$$

whenever $\hat{\theta} \neq \theta$. This in turn implies that

$$||\chi_\theta - P_\Theta(\cdot|t^r)|| \leq 2(m-1)R^{r/2}$$

where χ_θ is the Dirac measure with $\chi_\theta(\theta) = 1$ and $|\Theta| = m$. To complete the argument, choose $t_i, t_i' \in A$ and note that for all r sufficiently large,

$$\begin{aligned}
&\Pr ob\{||P_\Theta(\cdot|\tilde{t}^r_{-i}, t_i) - P_\Theta(\cdot|\tilde{t}^r_{-i}, t_i')|| > 4(m-1)R^{r/2}|\tilde{\theta} = \theta\} \\
&\qquad \leq \Pr ob\{\exists \alpha \in A : ||\chi_\theta - P_\Theta(\cdot|\tilde{t}^r_{-i}, \alpha)|| > 2(m-1)R^{r/2}|\tilde{\theta} = \theta\} \\
&\qquad \leq \Pr ob\{\exists \alpha \in A : ||\frac{f(\tilde{t}^r_{-i}, \alpha)}{r} - P_\Theta(\cdot|\theta)|| \geq \delta|\tilde{\theta} = \theta\} \\
&\qquad \leq \Pr ob\{||\frac{f(\tilde{t}^r)}{r} - P_\Theta(\cdot|\theta)|| \geq \delta/2|\tilde{\theta} = \theta\} \\
&\qquad \leq 2\exp(\frac{-r\delta^2}{2})
\end{aligned}$$

where the last inequality is due to Hoeffding (JASA, 1963). Hence, for all r sufficiently large,

$$\nu_i^P \leq \max\{4(m-1)R^{r/2}, \frac{2\exp(\frac{-r\delta^2}{2})}{\beta}\}$$

where

$$\beta := \min_{\alpha \in A} P(\alpha).$$

9.4. Proof of Theorem C

For each $t \in T$, define

$$
\begin{aligned}
z_i(t_{-i}, t_i) &= \frac{\varepsilon}{r} \frac{P_{T_{i+1}}(t_{i+1}|t_i)}{||P_{T_{i+1}}(\cdot|t_i)||_2} \text{ if } i = 1, .., r-1 \\
&= \frac{\varepsilon}{r} \frac{P_{T_1}(t_1|t_r)}{||P_{T_1}(\cdot|t_r)||_2} \text{ if } i = r
\end{aligned}
$$

Since

$$0 \leq z_i(t_{-i}, t_i) \leq \frac{\varepsilon}{r}$$

for all i, t_{-i} and t_i so individual rationality of the augmented mechanism follows from the observations that

$$\hat{v}_i(q(t); t) + x_i(t) \geq 0$$

and

$$z_i(t) \geq 0.$$

Claim 1: Let $K = |T^2|$. Then

$$\sum_{(t_{-i}, t_i) \in T^*} \left(z_i(t_{-i}|t_i) - z_i(t_{-i}|t_i')\right) P(t_{-i}|t_i) \geq \frac{\varepsilon}{2\sqrt{K}} \Lambda_i^P$$

Proof of Claim 1:

$$
\begin{aligned}
\sum_{(t_{-i}, t_i) \in T^r} \left(z_i(t_{-i}|t_i) - z_i(t_{-i}|t_i')\right) P^r(t_{-i}|t_i) &= \sum_{(t_{-i}, t_i) \in T^r} \left(z_i(t_{-i}|t_i) - z_i(t_{-i}|t_i')\right) P^r(t_{-i}|t_i) \\
&= \sum_{(t_{-i}, t_i) \in T^r} \frac{\varepsilon}{r} \left[\frac{P_{T_{i+1}}(t_{i+1}|t_i)}{||P_{T_{i+1}}(\cdot|t_i)||_2} - \frac{P_{T_{i+1}}(t_{i+1}|t_i')}{||P_{T_{i+1}}(\cdot|t_i)||_2} \right] P(t_{-i}|t_i) \\
&= \sum_{(t_{i+1}, t_i) \in T^r} \frac{\varepsilon}{r} \left[\frac{P_{T_{i+1}}(t_{i+1}|t_i)}{||P_{T_{i+1}}(\cdot|t_i)||_2} - \frac{P_{T_{i+1}}(t_{i+1}|t_i')}{||P_{T_{i+1}}(\cdot|t_i)||_2} \right] P(t_{i+1}|t_i) \\
&\geq \frac{\varepsilon}{2r\sqrt{K}} \Lambda_i^{P^2}
\end{aligned}
$$

This completes the proof of Claim 1.

Claim 2:

$$\sum_{(t_{-i},t_i)\in T^r} [(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + x_i(t_{-i},t_i)) - (\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i) + x_i(t_{-i},t_i'))]\, P^r(t_{-i}|t_i) \geq -5M\nu^{P^r}$$

Proof of Claim 2: Define

$$A_i(t_i',t_i) = \{t_{-i} \in T_{-i}^r |\ ||P_\Theta^r(\cdot|t_{-i},t_i) - P_\Theta^r(\cdot|t_{-i}t_i')|| > \hat{\nu}^{P^r}\}$$

and

$$B_i(t_i',t_i) = \{t_{-i} \in T_{-i}^r |\ ||P_\Theta^r(\cdot|t_{-i},t_i) - P_\Theta^r(\cdot|t_{-i}t_i')|| \leq \hat{\nu}^{P^r}\}.$$

We conclude that

$$\text{Prob}\{\tilde{t}_{-i} \in A_i(t_i',t_i)|\tilde{t}_i = t_i\} \leq \nu^{P^r}.$$

In addition,

$$0 \leq \hat{v}_i^r(q(t_{-i},t_i);t_{-i},t_i) + x_i(t_{-i},t_i) \leq \hat{v}_i^r(q(t_{-i},t_i);t_{-i},t_i) \leq M$$

for all i, t_i and t_{-i}. Therefore,

$$\begin{aligned}
|\hat{v}_i^r(q(t_{-i},t_i');t_{-i},t_i) + x_i(t_{-i},t_i')| &= |\hat{v}_i^r(q(t_{-i},t_i');t_{-i},t_i) - \hat{v}_i^r(q(t_{-i},t_i');t_{-i},t_i') \\
&\quad + \hat{v}_i^r(q(t_{-i},t_i');t_{-i},t_i') + x_i(t_{-i},t_i')| \\
&\leq |\hat{v}_i^r(q(t_{-i},t_i');t_{-i},t_i) - \hat{v}_i^r(q(t_{-i},t_i');t_{-i},t_i')| \\
&\quad + |\hat{v}_i^r(q(t_{-i},t_i');t_{-i},t_i') + x_i(t_{-i},t_i')| \\
&\leq 3M
\end{aligned}$$

for all i, t_i, t_i' and t_{-i}. Applying the definitions, it follows that

$$\begin{aligned}
&\sum_{t_{-i}\in A_i(t_i',t_i)} [(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + x_i(t_{-i},t_i)) - (\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i) + x_i(t_{-i},t_i'))]\, P^r(t_{-i}|t_i) \\
&\geq -3M \sum_{t_{-i}\in A_i(t_i',t_i)} P^r(t_{-i}|t_i) \\
&\geq -3M\hat{\nu}^P.
\end{aligned}$$

$$\begin{aligned}
&\sum_{t_{-i}\in B_i(t_i',t_i)} [(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + x_i(t_{-i},t_i)) - (\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i) + x_i(t_{-i},t_i'))]\, P^r(t_{-i}|t_i) \\
&\geq -2MK^r \sum_{t_{-i}\in B_i(t_i',t_i)} ||P_\Theta^r(\cdot|t_{-i},t_i) - P_\Theta^r(\cdot|t_{-i}t_i')|| P^r(t_{-i}|t_i) \\
&\geq -2MK^r\nu^{P^r}.
\end{aligned}$$

Combining these observations completes the proof of the claim 2.

Applying Claims 1 and 2, it follows that for sufficiently large r,

$$\begin{aligned}
&\sum_{(t_{-i},t_i)\in T^r} \left(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\alpha_i(t_{-i},t_i)+z_i(t_{-i},t_i)\right) P^r(t_{-i}|t_i) \\
&- \sum_{(t_{-i},t_i)\in T^r} \left(\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i)+\alpha_i(t_{-i},t_i')+z_i(t_{-i},t_i')\right) P^r(t_{-i}|t_i) \\
&= \sum_{(t_{-i},t_i)\in T^r} \left[\left(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\alpha_i^q(t_{-i},t_i)\right)-\left(\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i)+\alpha_i^q(t_{-i},t_i')\right)\right] P^r(t_{-i}|t_i) \\
&\qquad + \sum_{:(t_{-i},t_i)\in T^r} \left(z_i(t_{-i},t_i)-z_i(t_{-i},t_i')\right) P^r(t_{-i}|t_i) \\
&\geq \frac{\varepsilon}{2r\sqrt{K}}\Lambda_i^{P^2} - 3M\nu^{P^r} - 2MK^r\nu^{P^r} \\
&= \frac{1}{r}\left[\frac{\varepsilon}{2\sqrt{K}}\Lambda_i^{P^2} - 3Mr\nu^{P^r} - 2M\left(\frac{K^r}{r^L}\right)\left(r^{L+1}\nu^{P^r}\right)\right] \\
&\geq 0.
\end{aligned}$$

and the proof is complete.

10. Bibliography

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