Optimal group strategyproof cost sharing: budget-balance vs. efficiency

Ruben Juarez
Department of Economics, University of Hawaii
2424 Maile Way, Sauders Hall 542, Honolulu, HI 96822 (email: rubenj@hawaii.edu)

First version: November 2007. This version: October 2008.

Summary. Units of good are produced at some non-negative cost. A mechanism elicits the willingness to pay of the agents for getting one unit of the good, allocates some goods to some agents, and covers the cost by charging those agents.

We prove there is no mechanism that is simultaneously group strategyproof and efficient, even if it can have a budget surplus. In spite of this negative result, we compute second-best optimal mechanisms for increasing and decreasing average cost functions using the worst absolute surplus loss measure. This measure is particularly useful when the social planner who chooses the mechanism has no information about the types of agents who are participating in the economy.

When the cost function has decreasing average cost, the average cost mechanism is optimal even among the mechanisms allowing for a budget surplus. On the other hand, when the cost function has increasing average cost, an optimal mechanism generates a budget surplus and charges all agents (that are being served) equally. It reduces the efficiency loss by up to a half with respect to the optimal budget-balanced mechanism.

1 Introduction

Units of a nontransferable, indivisible and homogeneous good (or service) are available at some non-negative cost. Agents are interested in consuming at most one unit of that good and are characterized by their private valuation for it (which we call their utility). A mechanism elicits these utilities from the agents, allocates some goods to some agents, and covers the cost by charging the agents who are served.

These traditional mechanisms find several applications depending on the shape of the cost function. The most discussed problem in the literature is the case of decreasing marginal cost, also referred as economies to scale or natural

monopolies. Applications include the production of cars, pharmaceutical goods, software, railways, telecommunications, electricity, water and mail delivery. The canonical example in network economics is the network facility location problem with a single facility, where there is an homogeneous connection cost of the agents to the facility and a fixed cost of opening the facility.

On the other hand, increasing marginal cost finds applications in the exploitation of natural resources (e.g. oil, natural gas or fisheries). Another interesting application is the scheduling of jobs, where the disutility of the agents is the waiting time until being served (see Cres and Moulin[2001] and Juarez[2006] for discussions). The management of queues in networks, for instance the Internet, is the canonical example.

One downside of the traditional analysis of this problem (and more generally of mechanism design) is that it usually assumes the designer of the mechanism knows the (probability distribution of the) type of agents that are participating in the game, and based on this distribution the designer chooses the mechanism to implement. For instance, traditional decision theory computes the optimal decision based on the beliefs of the mechanism designer (e.g. Myerson [1981]). However, there is little discussion in the literature on how to form these initial beliefs, assuming the designer can form them at all.

We consider the problem of designing mechanisms in a setting in which the designer does not have enough information about the (potential) participants to be able to form a prior belief about the distribution of their possible types. For instance, it may be that we are dealing with agents in a large network like the Internet, where the variation of the agent types is huge; or that doing a market survey would be costly and time consuming. For these problems, it is usually the case that traditional (Bayesian) techniques do not work well. Therefore, different mechanisms and techniques should be proposed.

This paper looks at two strong properties that are meaningful in the prior-free cost sharing model: group strategyproofness and a worst-case measure. On one hand, group strategyproofness (GSP) rules out coordinated misreports under any possible information context. In particular this property is robust because it works whether the information on individual characteristics is private or not. That is, GSP works whether a small or large group of agents can coordinate misreports.

On the other hand, we use the worst absolute surplus loss measure (*wal*), that is the supremum of the difference between the efficient surplus and the surplus of the mechanism, where the supremum is taken over all utility profiles. This measure has been used recently in the literature as a second-best efficiency measure in similar cost sharing models.[1] This measure is prior-free because it does not depend on the (potential) distribution of valuations.

Summary of the results

Efficiency of the mechanism requires that the surplus-maximizing set of

[1]See Moulin and Shenker[2001] and Juarez[2007] for applications of the *wal*-measure. See Moulin[2005] for an application of the *best relative gain*, a similar worst-case measure.

agents is being served at every utility profile. Proposition 1 shows that there is no mechanism that is efficient and GSP, even if it can have a budget surplus. This impossibility holds except in the trivial case of an additive cost function (see below).[2] In spite of this negative result, we compute *second-best optimal* GSP mechanisms for two orthogonal classes of cost functions using *wal*.

A classic mechanism in this economy is the average cost mechanism (AC). It is the mechanism in which all agents ex-ante pay the same price. That is, every agent simultaneously decides to buy or not buy a unit of good. If k agents decide to buy, they all pay the average cost of producing k units.

If the cost function has decreasing marginal cost,[3] then the average cost mechanism achieves the smallest worst absolute surplus loss (henceforth wal-optimum) among the GSP mechanisms that treat equal agents equally (Proposition 2). This result is reminiscent of Theorem 1 in Moulin and Shenker[2001]. They proved that the Shapley value mechanism, a more general mechanism defined for non-symmetric submodular cost functions, is wal-optimum among the GSP and budget-balanced mechanisms. When the cost function is symmetric, our result goes further by proving the optimality of this mechanism among a bigger class of cost functions even when the mechanisms are not budget-balanced.

On the other hand, if the cost function has increasing marginal cost, we propose the sequential average cost mechanism (SAC). Under SAC, the agents "sequentially" pay the same price. That is, given an arbitrary order of the agents, say $i_n, i_{n-1}, \ldots, i_1$, first it computes the agent $i_{\tilde{k}}$ in the smallest position whose utility is bigger than the average cost of producing $\tilde{k}$ units (if there is no such agent then none is served). Then every agent in $\{i_{\tilde{k}}, \ldots, i_1\}$ is offered a unit of good at this price. The SAC mechanism is wal-optimal among the GSP mechanisms (Proposition 3). Moreover, when the cost function has increasing marginal cost, SAC is not budget-balanced but it reduces the worst absolute surplus loss by up to a half with respect to the optimal budget-balanced mechanism (Proposition 4).

Related Literature

This work, specifically the worst-case measure used above is related to the large and growing literature in computer science of the worst-case scenario. In particular, the recent literature of the price of anarchy, introduced to measure the effects of selfish routing in a congested network (Koutsoupias and Papadimitriou [1999], Roughgarden and Tardos [2002], Roughgarden [2003]). More specifically, it relates with the rich applications of these measures to network economic problems (e.g. Johari and Tsilikis [2004], Johari et al. [2005], Roughgarden et al.[2006, 2007]), assignments of goods or bads (Moulin[2007]),

[2]This result contrasts with the classic imposibility to meet simultaneously strategyproofness, budget-balance and efficiency (Vickrey[1961], Clarke[1971] and Groves[1973]) because it persists even by allowing a budget surplus on the mechanism. Notice the VCG mechanisms are strategyproof and efficient but do not balance the budget.

[3]In fact, we prove it for the more general class of cost functions with decreasing average cost.

and cost-sharing (Moulin[2006], Juarez[2006, 2007a]).

The design of GSP cost-sharing mechanisms was first discussed by Moulin [1999]. The paper characterizes cross-monotonic mechanisms by GSP, budget-balance, voluntary participation, nonnegative transfers and strong consumer sovereignty. Juarez[2007] also characterizes cross-monotonic mechanisms by GSP and a weak continuity condition. On the other hand, Moulin and Shenker [2001] evaluate the trade-offs between efficiency and budget-balance. The Shapley value cross-monotonic mechanism (the analog to the AC mechanism for non-symmetric cost functions) is characterized there by GSP, budget-balance, voluntary participation, nonnegative transfers and strong consumer sovereignty. Roughgarden et al.[2008a] shows that this mechanism is also optimal among the class of strategyproof and 'weakly-monotonic' mechanisms using the worst 'relative' gain measure.

Roughgarden et al.[2006a, 2006b], Pa'l et al.[2003] and Immorlica et al.[2005] consider cross-monotonic mechanisms when the cost function is not submodular. Roughgarden et al.[2006] uses submodular cross-monotonic mechanisms to approximate budget-balance when the actual cost function is not submodular. Immorlica et al.[2005] shows that new cross-monotonic mechanisms emerge when consumer sovereignty is relaxed.

Very little has been said in the literature on mechanisms that generate cost functions that are not submodular. Sequential mechanisms similar to SAC are discussed by Moulin[1999] who imposes budget-balance for a supermodular cost function, and by Juarez[2007] who characterizes sequential mechanism by GSP and a weak continuity condition. However, there is no result in the literature concerning the efficiency of sequential mechanisms and cost functions with increasing marginal cost like the one discussed here.

2 The model

For a vector x, $x \in \mathbb{R}^N$, we denote by x_S the sum of the S-coordinates of x, $x_S = \sum_{i \in S} x_i$. Let $\delta_i : 2^N \to \{0, 1\}$ the classic indicator function, that is $\delta_i(T) = 1$ if $i \in T$, and 0 otherwise.

A problem consists of a marginal cost function $c_1, c_2, \ldots, c_n$, where c_i is the marginal cost of producing good i. $C(i) = c_1 + \cdots + c_i$ is the total cost of producing the first i units. The cost function C is submodular (supermodular) whenever $c_1 \geq c_2 \geq \cdots \geq c_n$ ($c_1 \leq c_2 \leq \cdots \leq c_n$). The average cost function is $ac(k) = \frac{C(k)}{k}$. Clearly, if the cost function is submodular (supermodular) then the average cost function is decreasing (increasing), but the converse is not true.

There are a finite number of agents $N = \{1, 2, \ldots, n\}$. Every agent has a utility (willingness to pay) for getting one unit of good. Let u, $u \in \mathbb{R}^N_+$, the vector of those utilities. Therefore, if agent i gets a unit paying by x_i, his net utility is $u_i - x_i$. If he does not get a unit, his net utility is zero.

Definition 1 *A mechanism $\xi = (S, \varphi)$ is a pair of functions $S : \mathbb{R}^N_+ \to 2^N$ and $\varphi : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ such that for each utility profile u :*

i. $\varphi(u)_N \geq C(|S(u)|)$

ii. $\varphi_i(u) = 0$ *if* $i \notin S(u)$

iii. $u_i \geq \varphi_i(u)$ *if* $i \in S(u)$.

A mechanism assigns to every report of utilities, units of good to some agents and cost shares to those agents. Condition i states that the mechanism covers the cost of providing service. Condition ii requires that the agents who are not served should pay nothing. Condition iii requires individual rationality, that is ex-post utility is not smaller than ex-ante utility.

The net-utility of agent i in the mechanism $\xi = (S, \varphi)$, denoted by NU_i^ξ, is $NU_i^\xi(u) = \delta_i(S(u))(u_i - \varphi_i(u))$. Let $NU^\xi(u)$ the vector of such net utilities.

Definition 2 (Group strategyproofness) *We say the mechanisms $\xi = (S, \varphi)$ is group strategyproof if for all $T \subseteq N$ and all utility profiles u and u' such that $u'_{N \setminus T} = u_{N \setminus T}$, it cannot be that $NU_i^\xi(u) \leq (u_i - \varphi_i(u'))\delta_i(S(u'))$ for all $i \in T$ and strict for at least one agent.*

Group strategyproofness (GSP) rules out coordinated misreports of any group of agents. That is, if a group of agents misreport, and the net-utility of an agent in the group strictly increases, then the net-utility of another agent in the group should strictly decrease.

3 GSP and Efficiency

The surplus of the mechanism ξ at a utility profile u is the sum of the net utilities of all agents $\sigma^\xi(u) = \sum_{i \in N} NU_i^\xi(u)$. The efficient surplus at u is $eff(u) = \max_{S \subseteq N} u(S) - C(S)$. A mechanism is efficient if it serves the group of agents that generates $eff(u)$ at any utility profile u.[4]

We say the cost function is not additive if $C(S \cup T) \neq C(S) + C(T)$ for some S, T such that $S \cap T = \emptyset$. Notice that non-additivity implies that the marginal cost function is not constant.

Proposition 1 *If the cost function is not additive, there is no GSP mechanism that is efficient.*

If the cost function is additive, then the fixed cost mechanism where agent i is offered a unit of good at price $C(i)$ independent of the valuations of the other agents is efficient and GSP.

[4]The usual definition of efficient mechanism requires that the efficient mechanism gives the efficient surplus at any utility profile. Our definition is more general, since we only require for the surplus maximizer set of agents to be served.

4 Optimal Mechanisms

Given that efficiency and GSP are not compatible for non-trivial cost functions, we find second-best optimal mechanisms that are GSP. These mechanisms have the smallest surplus loss relative to the efficient allocation. Two measures have been used recently to make this comparison. The first measure is the worst relative gain (Moulin[2005]), that is the infimum of the ratio of the surplus of the mechanism and the efficient surplus, where the infimum is taken over all utility profiles. With this measure, any mechanism that is group strategyproof has zero worst relative gain. Hence the measure is not informative.[5]

Definition 3 *Given the agents in N and the cost function C, the worst absolute surplus loss of ξ is:*

$$wal(N, C, \xi) = sup_{u \in \mathbb{R}^N_+} eff(u) - \sigma^\xi(u)$$

This measure is informative, that is it is finite for any mechanisms that allocates units to the agents with high utility independent of the profile of the other agents.[6]

4.1 Decreasing average cost

Definition 4 (Average cost mechanism) *Given a cost function C with decreasing average cost, the equilibrium of the average cost mechanism (AC) is computed as the Nash equilibrium of the game where every agent decides to buy or not buy a unit of good. If s agents decided to buy, each of them gets a unit of good at a price equal to $ac(s) = \frac{C(s)}{s}$.*

The average cost mechanism is clearly budget-balanced for the cost C. In particular, it is a feasible mechanism for C. Notice that multiplicity of AC-equilibria is possible; however, all equilibria are welfare equivalent.

Given that the cost function is symmetric, a normative requirement on the mechanisms is to treat equal agents equally (ETE). That is, for any utility profile u and any two agents i and j that report the same valuation for the good u_i, u_j such that $u_i = u_j$, then either they are simultaneously served at the same price ($i, j \in S(u)$ and $\varphi_i(u) = \varphi_j(u)$) or they are simultaneously not served $i, j \notin S(u)$.

Proposition 2 *If the cost function has decreasing average cost, then the AC mechanism:*

[5] To see this, consider a GSP mechanism and assume the cost function has increasing average cost (this is proved similarly for submodular case, and other non-additive cost functions). By GSP, there is a finite payment for coalition N, denote it by x^N. Let the utility profile be the vector $u = x^N + \epsilon \cdot 1_N$, for some $\epsilon > 0$. At this profile, the surplus is $n\epsilon$. Notice the efficient surplus is positive at this profile because $x_i > C(i)$ for some i. Hence as ϵ goes to zero, the surplus of the mechanism goes to zero, but the surplus of the efficient mechanism remains bounded below by a positive number. Hence the worst relative gain is zero.

[6] This property is called consumer sovereignty and was introduced by Moulin[1999]. The proof of this claim can be found in Lemma 1 of Juarez[2006].

i. Pareto dominates any other GSP mechanism that treats equal agents equally,

ii. is wal-optimum among the GSP and ETE mechanisms,

iii. has a loss equal to $wal(n, C, AC) = ac(1) + ac(2) + \cdots + ac(n) - C(n)$

Part i singles out AC as the only efficient choice among the feasible GSP mechanisms that treat equal agents equally. When the cost function is submodular, Moulin and Shenker[2001] proved a proposition similar to part ii but imposing an additional budget-balanced restriction to the mechanisms. This proposition shows the same result holds even when we allow a budget-surplus.

Notice part ii guarantees that AC has a strictly smaller worst absolute surplus loss with respect to any other GSP and ETE mechanism, therefore is not implied by part i.

4.2 Increasing average cost

If the cost function has increasing average cost, then a multiplicity of AC-equilibria that are not welfare equivalent is possible (see below). Moreover, AC is not even strategyproof. In particular, it is not GSP. To see this, consider the profile $u = (ac(2) - \epsilon, ac(2) - \epsilon, 0, 0, \ldots, 0)$ for $0 < \epsilon < \frac{c_2 - c_1}{2}$. At equilibrium, only one agent can be served; without loss of generality we assume $S^{AC}(u) = \{1\}$ and $\varphi_1^{AC}(u) = c_1$. Consider $\tilde{u} = (ac(2) - \epsilon, ac(2) + \epsilon, 0, 0, \ldots, 0)$. The AC equilibrium serves only one agent at price c_1. Clearly $S(\tilde{u}) \neq \{2\}$, otherwise agent 2 profits by misreporting $\tilde{u}_2 = ac(2) + \epsilon$ when the true profile is u. Thus $S(\tilde{u}) = \{1\}$. Hence agent 1 profits by misreporting $ac(2) - \epsilon$ when the true profile is $(ac(2) + \epsilon, ac(2) + \epsilon, 0, 0, \ldots, 0)$.

Definition 5 (Sequential average-cost mechanism) *Given cost function C with increasing average cost, and an arbitrary order of the agents $i_n, i_{n-1}, \ldots, i_1$, the equilibrium of the sequential average-cost mechanism (SAC) is computed as follows. Let k^* the largest index such that $u_{i_{k^*}} > ac(k^*)$. Then every agent i_l such that $l \leq k^*$ and $u_{i_l} > ac(k^*)$ gets a unit of good at price $ac(k^*)$.*

Clearly SAC is feasible for the cost function C. Contrary to AC, SAC is not budget-balanced. For instance if u is such that $u_{i_n} > ac(n)$ and $u_{i_l} < ac(n)$ for $l < n$, then only agent i_n is served at price $ac(n)$, and $ac(n) > c_1$. However, proposition 4 below shows that even though SAC is not budget-balanced, it has the same worst absolute surplus loss as the AC mechanism.

The SAC mechanism does not treat equal agents equally. However, it meets the *equal share property (ESP)*, which states that all agents that get service pay the same. That is, if $S(u) = S^*$ then $\varphi_i(u) = \varphi_j(u)$ for all $i, j \in S^*$. In the class of GSP mechanisms, ESP is weaker than ETE (see Juarez[2007]).

Proposition 3 *If the cost function has increasing marginal cost, then the SAC mechanism:*

i. is wal−optimal over all GSP mechanisms,

ii. Pareto dominates any other GSP, ESP and wal−optimal mechanism,

iii. $wal(n, C, SAC) = \max_k k[\frac{c_{k+1}+\cdots+c_n}{n}] - (n-k)[\frac{c_1+\cdots+c_k}{n}]$.

4.2.1 Welfare comparison between budget-balanced and budget-surplus mechanisms

The *incremental marginal cost mechanism (INC)* sequentially offers the agents a unit of good at a price equal to marginal cost. That is, for an arbitrary order of the agents, say $1, \ldots, n$ we offer agent 1 a unit of good at price c_1. Agent 2 is offered a unit of good at price c_2 if 1 accepts, or at price c_1 if 1 did not accept. And similarly for the following agents. Only the incremental mechanisms are GSP, budget-balanced and consumer-sovereign when the marginal cost function is increasing (see Moulin[1999]).

Proposition 4 *i. The incremental marginal cost mechanism has a loss equal to:*

$$wal(n, C, INC) = (c_{[\frac{n+1}{2}]} + \cdots + c_n) - (c_1 + \cdots + c_{[\frac{n-1}{2}]}).$$

ii. $1 \leq \frac{wal(n,C,INC)}{wal(n,C,SAC)} \leq 2$ *for all* n *and all* C.

iii. If marginal cost is linear, $c_i = a \cdot i$, *then* $\frac{wal(n,C,INC)}{wal(n,C,SAC)} = 2$ *for all* n.

In particular, part *iii* shows that the loss of a GSP and budget-balanced mechanism can be cut by up to half by allowing a budget surplus.

When the marginal cost is increasing, it is not difficult to see that the worst absolute surplus loss of AC equals exactly $wal(n, C, SAC)$ (see lemma 2 in Juarez[2006] for details). Hence by implementing SAC we gain GSP and only lose budget-balance.

References

[1] Clarke E. H.: Multipart pricing of public goods. Public Choice, 11:1733, 1971.

[2] Groves T.: Incentives in teams. Econometrica, 41:617631, 1973.

[3] Juarez R.: The worst absolute surplus loss in the problem of commons: Random Priority vs. Average Cost. Forthcoming Economic Theory (2006)

[4] Juarez R.: Group strategyproof cost sharing: The role of indifferences. Mimeo Rice University (2007)

[5] Juarez R.: Group strategyproof cost sharing: Mechanisms without indifferences. Mimeo Rice University (2007)

[6] Moulin H.: Incremental Cost Sharing: characterization by coalitional strategy-proofness. Social Choice and Welfare **16**, 279-320 (1999)

[7] Moulin H., Shenker S.: Strategyproof sharing of submodular costs: budget balance versus efficiency. Economic Theory **18**, 511-533 (2001)

[8] Moulin H.: The price of anarchy of serial cost sharing and other methods. Forthcoming Economic Theory (2007)

[9] Roughgarden T.: Sundararajan M. Approximately efficient cost-sharing mechanisms. Mimeo Stanford (2006)

[10] Roughgarden T.: Sundararajan M. New TradeOffs in Cost Sharing Mechanisms. STOC (2006)

[11] Vickrey W.: Counterspeculation, auctions, and competitive sealed tenders. Journal of Finance, 16:837, 1961.

5 Appendix: Proofs

Proof of Proposition 1.

We prove a stronger result. There is no mechanism that simultaneously meets GSP and serves the efficient set of agents.

Step 1. Assume $N = \{1, 2\}$ and $C(12) < C(1) + C(2)$.

Consider the profiles $u = (C(12) - C(2) + 2\epsilon, C(2) - \epsilon)$ and $\widetilde{u} = (C(1) - \epsilon, C(12) - C(1) + 2\epsilon)$. For small ϵ, 1 and 2 are not feasible at u because $C(12) - C(2) + 2\epsilon < C(1)$ and $C(2) - \epsilon < C(2)$. Thus by efficiency $S(u) = 12$. Similarly, $S(\widetilde{u}) = 12$. Let $x = \varphi(u)$ and $y = \varphi(\widetilde{u})$. Notice $x \neq y$, to see this $x_1 \leq C(12) - C(2) + 2\epsilon$ by voluntary participation of agent 1 at u and $y_1 \geq C(1) - 2\epsilon$ by VP and feasibility at $\widetilde{u}$. For small ϵ $C(12) - C(2) + 2\epsilon < C(1) - 2\epsilon$, hence $x_1 < y_1$.

Finally, by efficiency $S(C(1) - \epsilon, C(2) - \epsilon) = 12$. By SP, $\varphi_1(C(1) - \epsilon, C(2) - \epsilon) = x_1$. On the other hand, by GSP $\varphi_2(C(1) - \epsilon, C(2) - \epsilon) = x_2$. Indeed, if $\varphi_2(C(1) - \epsilon, C(2) - \epsilon) < x_2$ then agent 1 helps 2 by misreporting $C(1) - \epsilon$ when the true profile is u. On the other hand, if $\varphi_2(C(1) - \epsilon, C(2) - \epsilon) > x_2$, then 1 helps 2 by misreporting u_1 when the true profile is $(C(1) - \epsilon, C(2) - \epsilon)$. This contradicts GSP. Hence $\varphi(C(1) - \epsilon, C(2) - \epsilon) = x$. Similarly, $\varphi(C(1) - \epsilon, C(2) - \epsilon) = y$. This is a contradiction because $x \neq y$.

Step 2. Assume $N = \{1, 2\}$ and $C(12) > C(1) + C(2)$.

By feasibility $S(C(1) + \epsilon, C(2) + \epsilon) \neq 12$, assume w.l.g. that $S(C(1) + \epsilon, C(2) + \epsilon) = 1$. By efficiency, $S(C(1) + \epsilon, C(2) + 2\epsilon) = 2$ and $\varphi_2(C(1) + \epsilon, C(2) + 2\epsilon) = C(2)$.

Also by efficiency $S(0, C(2) + 2\epsilon) = 2$ and by GSP $\varphi_2(C(1) + \epsilon, C(2) + 2\epsilon) = \varphi_2(0, C(2) + 2\epsilon)$ (if one is smaller then agent 1 can help 2). Since $S(0, C(2) + 2\epsilon) = 2$ for all $\epsilon > 0$ then by SP and feasibility $\varphi_2(0, C(2) + 2\epsilon) = C(2)$. Thus $\varphi_2(C(1) + \epsilon, C(2) + 2\epsilon) = C(2)$. Hence by SP, $2 \in S(C(1) + \epsilon, C(2) + \epsilon)$. This is a contradiction.

Step 3. Assume $n > 2$.

Because the cost function is not additive, there are $i, j \in N$, $S \subset N \setminus \{i, j\}$ such that

$$C(S \cup i) + C(S \cup j) \neq C(S \cup i, j) + C(S). \tag{1}$$

Let $\bar{u}$ a utility profile such that $\bar{u}_s > C(N)$ for all $s \in S$ and $u_k = 0$ if $k \notin S \cup \{i, j\}$. By efficiency, the agents in S should be served and any agent not in $S \cup \{i, j\}$ should not be served. Agents i and j may or may not be served.

Consider the set of utility profiles $U = \{u \mid u_{[N \setminus i,j]} = \bar{u}_{[N \setminus i,j]}\}$.

Thus $S \subseteq S(u) \subseteq S \cup \{i, j\}$ for all $u \in U$.

Restricting the mechanism to U, it defines a GSP mechanism for agents i and j. By equation 1, the cost function is not additive, hence by steps 1 and 2 the mechanism is not efficient at U.

Proof of Proposition 2.

Part i.

By Juarez[2007] any GSP and ETE mechanism is cross-monotonic and meet the ESP.

Let (S, φ) be a GSP and ETE mechanism. By Juarez[2007] proposition 1, (S, φ) is cross-monotonic and the set of payments meet the equal share property.

Consider a utility profile u and assume $S(u) = S^*$. By feasibility $u_i \geq \varphi_i(u) \geq ac(S^*)$ for all $i \in S^*$.

Hence at the average cost equilibrium, $S^* \subseteq S^{AC}(u)$. Since the average cost is decreasing, then $\varphi_i^{AC}(u) \leq \varphi_i(u)$ for all $i \in S^*$.

Part ii.

Consider a mechanism $\xi = (S, \varphi)$ as in part i above.

Because ξ is Pareto dominated by AC then $wal(n, C, AC) \leq wal(n, C, \xi)$. We now prove the strict inequality.

If ξ is not welfare equivalent to AC, then there is a utility profile u such that $\varphi_i(u) > AC(S^*)$ for some $i \in S^*$, $S^* = S(u)$.

Let $i \in S^*$, $\epsilon > 0$, and consider the utility profile $\tilde{u}(\epsilon)$ such that $\tilde{u}_{-S^*}(\epsilon) = (ac(|S^*| + 1) - \epsilon, \ldots, ac(n) - \epsilon)$, and $\tilde{u}_{S^*}(\epsilon) = (ac(1) - \epsilon, ac(2) - \epsilon, \ldots, ac(|S^*| - 1) - \epsilon, ac(|S^*|) + \delta)$ where $\delta = \frac{\varphi_i(u) - AC(S^*)}{2} > 0$.

First notice $S(\tilde{u}(\epsilon)) = \emptyset$. To see this, clearly $S(\tilde{u}(\epsilon)) \neq N$ because the payment of all agent should be at least $ac(n)$, so this is not feasible for the agent with utility $ac(n) - \epsilon$. By cross-monotonicity, this agent is not served. Similarly, the agent with utility equal to $ac(|S^*| - 1) - \epsilon$ is not served. Continuing this way, $S(\tilde{u}(\epsilon)) \cap (N \setminus S^*) = \emptyset$.

Also notice S^* is not feasible at $\tilde{u}(\epsilon)$ because $ac(|S^*|) + \delta < \varphi_i(u)$. Hence this agent is not served. Continuing this way, $S(\tilde{u}(\epsilon)) = \emptyset$.

Hence $wal(N, C, \xi) \geq eff(\tilde{u}(\epsilon))$. Clearly, $eff(\tilde{u}(\epsilon)) = ac(1) + \cdots + ac(n) + \delta - (n-1)\epsilon - C(n)$ because $ac(i) \geq c_i$ for all i.

Hence as ϵ goes to zero, $wal(N, C, \xi) \geq ac(1) + \cdots + ac(n) + \delta - C(n)$. By part iii below, $wal(N, C, \xi) > ac(1) + \cdots + ac(n) - C(n) = wal(N, C, AC)$.

Part iii.

Consider the profile $u(\epsilon) = (ac(1) - \epsilon, ac(2) - \epsilon, \dots, ac(n) - \epsilon)$ for $\epsilon > 0$. Clearly, $S^{AC}(u(\epsilon)) = \emptyset$. Thus $wal(N, C, AC) \geq eff(\tilde{u}(\epsilon))$. Clearly, $eff(\tilde{u}(\epsilon)) = ac(1) + \cdots + ac(n) - (n-1)\epsilon - C(n)$ because $ac(i) \geq c_i$ for all i. Hence, as ϵ goes to zero, $wal(N, C, AC) \geq ac(1) + \cdots + ac(n) - C(n)$.

Consider a utility profile u such that $S(u) = \emptyset$. Up to reordering the vector, assume $u_1 \geq u_2 \geq \cdots \geq u_n$. Then, clearly $u \leq (ac(1), ac(2), \dots, ac(n))$. Since the efficient surplus is increasing in the utility profile, then

$$eff(u) - \sigma^{\xi}(u) \leq ac(1) + \cdots + ac(n) - C(n).$$

Next assume, $S(u) = S^* \neq \emptyset$. Because the average cost is decreasing, then $ac(i) \geq c_i$ for all i, then the efficient surplus serve at least the agents in S^*. Thus we can reduce the utility of the agents in S^* up to $ac(S^*)$ without affecting the loss. That is:

$$eff(u) - \sigma^{\xi}(u) = eff(ac(S^*)1_{S^*}, u_{-S^*}) - \sigma^{\xi}(ac(S^*)1_{S^*}, u_{-S^*}) = eff(ac(S^*)1_{S^*}, u_{-S^*})$$

Up to renaming the agents, assume $u_{-S^*} = (u_{S^*+1}, u_{S^*+2}, \dots, u_n)$, $u_{S^*+1} \geq u_{S^*+2} \geq \cdots \geq u_n$. Clearly, $u_{S^*+1} < ac(S^* + 1)$, $u_{S^*+2} < ac(S^* + 2)$, $\dots$ $u_n < ac(n)$.

Hence $eff(ac(S^*)1_{S^*}, u_{-S^*}) \leq eff(ac(1), ac(2), \dots, ac(n)) = ac(1) + \cdots + ac(n) - C(n)$.

Hence $wal(n, C, AC) = ac(1) + \cdots + ac(n) - C(n)$.

Proof of Proposition 3.

Step 1. $wal(n, C, SAC) = \max_{1 \leq k \leq n} k \cdot ac(n) - (c_1 + \cdots + c_k)$

Let $x^{SAC} = (ac(n), \dots, ac(n))$, and consider the fixed cost mechanism ξ that offers to every agent a unit of good at price $ac(n)$, independent of the other agents report. I claim ξ has a worst absolute surplus loss equal to $\max_{1 \leq k \leq n} k \cdot ac(n) - (c_1 + \cdots + c_k)$.

Indeed, let u a utility profile. Assume $u_{[S]} >> x^{SAC}_{[S]}$, $u_{[T]} << x^{SAC}_{[T]}$ and $u_{[N \setminus (S \cup T)]} = x^{SAC}_{[N \setminus (S \cup T)]}$.

Let $\epsilon > 0$ such that $\epsilon <| u_i - x^{SAC}_i |$ for all $i \in S \cup T$. Let

$$u^{\epsilon} = (x^{SAC}_{[S]} + \epsilon 1_{[S]}, x^{SAC}_{[T]} - \epsilon 1_{[T]}, u_{[N \setminus (S \cup T)]})$$

where $1_N = (1, \dots, 1)$.

Then, $eff(u^{\epsilon}) - \sigma^{\xi}(u^{\epsilon}) \geq eff(u) - \sigma^{\xi}(u)$. To see this, let $i \in S$, then $x^{SAC}_i - \epsilon < u_i$. Since $i \in S(u)$, then by replacing u_i by $x^{SAC}_i - \epsilon$ the efficient surplus decreases at most by $u_i - (x^{SAC}_i - \epsilon)$ while $\sigma^{\xi}(u)$ decreases exactly by $u_i - (x^{SAC}_i - \epsilon)$. On the other hand, if $i \in T$, then $x^{SAC}_i - \epsilon > u_i$. Thus by replacing u_i by $x^{SAC}_i - \epsilon$ the efficient surplus increases at most by $x^{SAC}_i - \epsilon - u_i$ while $\sigma^{\xi}(u)$ does not increase. The agents outside $S \cup T$ do not affect the surplus.

Finally, by the same argument $eff(u^\epsilon) - \sigma^\xi(u^\epsilon)$ is not decreasing as ϵ approaches to zero. Clearly, $\lim_{\epsilon\to 0} \sigma^\xi(u^\epsilon) = 0$ and $\lim_{\epsilon\to 0} eff(u^\epsilon) = max_S |S|ac(n) - C(S)$.

Hence,

$$wal(n, C, \xi) = \max_{S\subseteq N} |S|ac(n) - C(S) = \max_k k \cdot ac(n) - C(k).$$

On the other hand, by choosing the utility profile $u = (ac(n)+\epsilon, \ldots, ac(n)+\epsilon)$ and letting ϵ tend to zero, $wal(n, C, SAC) \geq \max_k k \cdot ac(n) - C(k)$.

Since SAC Pareto dominates ξ, then $wal(n, C, SAC) = wal(n, C, \xi)$.

Step 2. $wal(n, C, SAC) \leq wal(n, C, \xi)$ for any feasible mechanism ξ.

Consider a mechanism $\xi = (S, \varphi)$ with finite worst absolute surplus loss. Then for every agent i, there is x^i large such that i is served if $u_i > x^i$ independent of other agent's reports (i.e. the mechanism meets consumer sovereignty). Indeed, if this does not occur, then we can find a collection of utility profiles $u^1, u^2, \ldots$ such that $u_i^k \to \infty$ and $i \notin S(u^k)$ for all k. Since $i \in eff(u^k)$ for all k such that $u_i^k > C(N)$, then $eff(u^k) - \sigma^\xi(u^k) \to \infty$ as $k \to \infty$.

Given this, consider a utility profile $\tilde{u}$ such that $\tilde{u}_i > x^i$ for all i. Then, $S(\tilde{u}) = N$. We can assume without loss of generality that all agents are getting positive net utility.

Assume $\varphi(\tilde{u}) = x^N$. Thus, $\tilde{u}_i > x_i^N$ for all $i \in N$. Let $\epsilon > 0$, by SP $i \in S(x_i^N + \epsilon, u_{-i})$ and $\varphi_i(x_i^N + \epsilon, u_{-i}) = x_i^N$.

On the other hand, by GSP $j \in S(x_i^N + \epsilon, u_{-i})$ and $\varphi_j(x_i^N + \epsilon, u_{-i}) = x_j^N$ for all $j \neq i$. To see this, if $j \notin S(x_i^N + \epsilon, u_{-i})$ or $\varphi_j(x_i^N + \epsilon, u_{-i}) > x_j^N$ then agent i helps j by misreporting u_i when the true profile is $(x_i^N + \epsilon, u_{-i})$. On the other hand, if $\varphi_j(x_i^N + \epsilon, u_{-i}) < x_j^N$ then i helps j by misreporting $x_i^N + \epsilon$ when the true profile is u.

Hence, by changing one agent at a time, $S(x^N + \epsilon 1_N) = N$ and $\varphi(x^N + \epsilon 1_N) = x^N$.

Notice $\sigma^\xi(x^N + \epsilon 1_N) = n\epsilon$. On the other hand, $eff(x^N + \epsilon 1_N) = \max_S x_S^N - C(S) + \mid S \mid \epsilon$. Thus,

$$lim_{\epsilon\to 0} eff(x^N + \epsilon 1_N) - \sigma^\xi(x^N + \epsilon 1_N) = \max_S x_S^N - C(S).$$

Hence, $wal(n, C, \xi) \geq \max_S x_S^N - C(S)$.

Finally, since $x_N^N \geq C(N)$ then $\max_S x_S^N - C(S) \geq \max_S |S| \frac{ac(n)}{n} - C(S)$.

Along with step 1, this proves step 2.

Part iii.

This part follows immediately from step 1 in part i, and noticing:

$$\max_{1\leq k\leq n} k \cdot ac(n) - (c_1 + \cdots + c_k) = \max_k k[\frac{c_{k+1} + \cdots + c_n}{n}] - (n-k)[\frac{c_1 + \cdots + c_k}{n}].$$

Proof of Proposition 4

Part i.

Because the cost function is supermodular, the loss will be given at x^N. At this point, $x^N = (c_1, \ldots, c_n)$. The surplus of the mechanism at $u = x^N + \epsilon \cdot 1_N$ is $n \cdot \epsilon$. Hence

$$wal(n, C, INC) = \max_S C(S) - C(\mid S \mid) = (c_{[\frac{n+1}{2}]} + \cdots + c_n) - (c_1 + \cdots + c_{[\frac{n-1}{2}]}).$$

Part ii.

$1 \leq \frac{wal(n,C,INC)}{wal(n,C,SAC)}$ is obvious by optimality of $wal(n, C, SAC)$.

$\frac{wal(n,C,INC)}{wal(n,C,SAC)} \leq 2$ follows immediately by choosing $k = [\frac{n}{2}]$ in the equation of proposition 3, part *iii*.

Part iii.

For $c_i = i$,

$\max_k k[\frac{c_{k+1}+\cdots+c_n}{n}] - (n-k)[\frac{c_1+\cdots+c_k}{n}] = \frac{k\left(1/2\,(n+1)^2 - 1/2\,n - 1/2\,(k+1)^2 + 1/2\,k\right)}{n} - \frac{(n-k)\left(1/2\,(k+1)^2 - 1/2\,k - 1/2\right)}{n}$

This has a maximum at $k = \frac{n}{2}$ (Maple computations). By substituting, we get that the loss equals $\frac{n^2}{8}$.

On the other hand, it is easy to check, from part *i*, that $wal(n, C, INC) = \frac{n^2}{4}$.