Group strategyproof cost sharing

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Summary. Every agent reports his willingness to pay for one unit of good. A mechanism allocates goods and cost shares to some agents. We characterize the group strategyproof (GSP) mechanisms under two alternative continuity conditions interpreted as tie-breaking rules. With the maximalist rule (MAX) an indifferent agent is always served. With the minimalist rule (MIN) an indifferent agent does not get a unit of good.

GSP and MAX characterize the cross-monotonic mechanisms. These mechanisms are appropriate whenever symmetry is required. On the other hand, GSP and MIN characterize the sequential mechanisms. These mechanisms are appropriate whenever there is scarcity of the good.

Our results are independent of an underlying cost function; they unify and strengthen earlier results for particular classes of cost functions.

Keywords: *Cost sharing, Mechanism design, Group strategyproof, Tie-breaking rule.*

JEL classification: C72, D44, D71, D82.

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1 Introduction

Units of a nontransferable, indivisible and homogeneous good (or service) are available at some non-negative cost. Agents are interested to consume at most one unit of that good and are characterized by their valuation for it (which we call their utility). We look for mechanisms that elicit these utilities from the agents, allocate some goods to some agents and charge some money only to the agents who are served.

These mechanisms have been widely explored in the cost-sharing literature (see below). The canonical example is sharing the cost of providing some optional service to geographically dispersed agents (e.g. Internet), where the cost function is not necessarily symmetric. Another example is auctions where the seller has multiple copies of a good.

When agents have private information about their utility, incentive compatibility of the mechanism, here interpreted as strategyproofness (SP), is an issue. The mechanisms that satisfy SP are the "auction" type mechanisms. That is, every agent is offered to buy a unit of good at a price that depends exclusively on the reports of the other agents.

A familiar strengthening of SP is group strategyproofness (GSP). This property rules out coordinated misreports of any group of agents. GSP is particularly interesting in settings where the designer of the mechanism has little information about the types of agents participating in the economy, for instance when the designer is dealing with agents in a large network like the internet. In these settings, it is usually the case that agents have the ability to coordinate misreports, and hence increase their net-utility. GSP is a robust property that rules out coordinated misreports under any possible information context. In particular, it works whether the information on individual characteristics is private or not.

For a SP mechanism, whether or not the agents who are offered a price equal to their valuation are served is of no consequence. Not so for GSP mechanisms. GSP is clearly violated if such an agent can be "bossy," i.e. affect the welfare of another agent without altering his own.[1] For instance, consider the mechanism that offers to the agents in $\{1, 2\}$, following the order $1 \succ 2$, the first unit at price p and the second unit at price p', $p' > p$. Assume agent's 1 utility for a unit of good equals exactly p and agent's 2 utility is strictly bigger that p, then GSP requires agent 1 not to be served. Otherwise, agent 1 can help agent 2 by reporting a utility below p. Whereby agent 2 is offered the cheaper price p.

This paper characterizes the GSP mechanisms under two continuity conditions, interpreted as tie-breaking rules. With the maximalist tie-breaking rule (MAX), an agent who is indifferent between getting or not getting a unit of good will always get a unit of good. With the minimalist rule (MIN), the indifferent agents never get a unit of good.

The mechanisms that satisfy GSP and MAX are the cross-monotonic mechanisms (Theorem 1), where unlike in the above example the price offered to an agent weakly decreases as more agents are served. Specifically, for any subset of agents S consider a vector of nonnegative payments $x^S \in [0, \infty]^N$ that are zero for all agents not in S. A collection of payments is cross-monotonic if the payments are weakly inclusion decreasing. Given a cross-monotonic

[1] In some contexts, GSP is equivalent to the combination of SP and non-bossiness: Papai[2000, 2001], Ehlers et al.[2003], Svensson et al.[2002]. In our context, a similar equivalence holds by imposing two alternative non-bossy conditions, see Mutuswami[2005].

collection of payments, we construct the mechanism as follows. For a report of utilities allocate S^* at cost x^{S^*}, where S^* is the largest coalition of agents such that everyone in S^* is willing to pay x^{S^*} to get service –this coalition exists by cross-monotonicity of the payments.

The mechanisms that satisfy GSP and MIN are the sequential mechanisms (Theorem 2). Loosely speaking, consider any binary tree of size n such that to every node is attached exactly one agent and any path from the root to a terminal node goes through all agents exactly once. At every decision node we also attach a nonnegative price. Given this tree, we construct the mechanism as follows. First we offer service to the root agent at the price attached to his node. We proceed on the right branch from the root if service is purchased and on the left branch if it is not. The key restriction on prices is that for any two nodes to which the same agent is attached, the price on the rightist node is not smaller than that on the leftist node.[2]

Surprisingly, the (welfarewise) intersection of sequential and cross-monotonic mechanisms is almost empty. It contains only the fixed cost mechanisms (Corollary 1), offering to each agent a price completely independent of the reports.

An important property of cross-monotonic mechanisms is to allow equal treatment of equals, which no other GSP mechanism does (Proposition 1). On the negative side, when there are only k units of good available, $k < n$, cross-monotonic mechanisms must exclude $n-k$ agents from the mechanism, that is they will never be served at any profile (see section 6.3). By contrast, not all sequential mechanisms exclude agents ex-ante. In fact, only the priority mechanisms, where agents are offered sequentially a unit of good at a fixed price until someone accepts the offer, meet GSP and allocate at most one unit of good at any profile (Proposition 2).

We do not make an actual cost function part of the definition of a mechanism. That is, we place no constraint on the total cost shares collected from the agents who are served. Thus our characterization results of GSP mechanisms are entirely orthogonal to budget balance and other feasibility requirements (such as bounds on the budget surplus or deficit). Naturally, one of the first questions we ask about the class of mechanisms identified in theorems 1 and 2 is when can they be chosen so as to cover exactly a given cost function. In examples 3 and 10 we answer these questions under a weak symmetry assumption. In this way, we recover most mechanisms identified in the earlier literature.

2 Related literature

There is some interesting literature in the design of GSP mechanisms for assignment problems of heterogeneous goods when money is not available (Ehlers[2002], Ehlers et al.[2003], Papai [2000, 2001] and Svensson et al.[2002]). Unfortunately, this literature usually characterizes mechanisms with poor equity properties (e.g. dictatorial mechanisms). By contrast, the class of GSP mechanism when money is available is very rich (see below).

The design of GSP cost sharing mechanisms for heterogeneous goods was first discussed by Moulin[1999] and Moulin and Shenker[2001]. When the cost function is submodular

[2]See definition 9 for precise conditions.

(concave), cross-monotonic mechanisms are characterized by GSP, budget balance, voluntary participation, nonnegative transfers and strong consumer sovereignty.[3] Roughgarden et al.[2006a, 2006b], Pa'l et al.[2003] and Immorlica et al.[2005] consider cross-monotonic mechanisms when the cost function is not submodular. Roughgarden et al.[2006] uses submodular cross-monotonic mechanisms to approximate budget balance when the actual cost function is not submodular. Immorlica et al.[2005] shows that new cross-monotonic mechanisms emerge when consumer sovereignty is relaxed.

The sequential mechanisms of our Theorem 2 are discussed by Moulin[1999] who imposes budget balance for a supermodular (convex) cost function. Theorem 1 there asserts wrongly that all GSP mechanisms meeting budget balance, voluntary participation, nonnegative transfers and strong consumer sovereignty charge successively marginal cost following an independent ordering of the agents. We correct this erroneous statement in example 9.

Roughgarden et al.[2007] uncovers a very clever class of weakly GSP mechanisms that are neither cross-monotonic nor sequential (see also Devanur et al.[2005]). This class contains sequential and cross-monotonic mechanisms, as well as hybrid mechanisms. They apply these mechanisms to the vertex cover and Steiner tree cost sharing problems to improve the efficiency of algorithms derived from cross-monotonic mechanisms. A closely related paper is the companion paper Juarez[2007b] developing a model where indifferences are ruled out. For instance, agents report an irrational number and payments are rational. It turns out that the class of GSP mechanisms becomes very large. In particular, it contains mechanisms very different to cross-monotonic and sequential mechanisms (and also those discussed by Roughgarden et al.[2007]). Juarez[2007b] provides three equivalent characterizations of the GSP mechanism in this economy, two of which are generalizations of the cross-monotonic and sequential mechanisms discussed in this paper.

When a cost function is specified, an important question is to evaluate the trade-offs between efficiency and budget balance. Moulin and Shenker[2001] discuss this issue for budget balanced cross-monotonic mechanisms when the underlying cost function is submodular. In particular, they find that the cross-monotonic Shapley value mechanism, where the payment of a coalition equals its stand alone cost, minimizes the worst absolute surplus loss.[4] Juarez[2007a] analyzes similar trade-offs for supermodular cost functions. Contrary to the submodular case, one can construct optimal sequential mechanisms that cuts the efficiency loss by half with respect to the optimal budget balanced mechanism.

Finally a result by Goldberg et al.[2004] on fixed cost mechanisms is closely related to our Corollary 1. It characterizes these mechanisms under a strengthening of GSP, where agents can coalitionally manipulate by misreporting, transferring goods and money between them.

[3]Strong consumer sovereignty says that every agent has reports such that he gets (or does not get) a unit of good irrespective of other people reports.

[4]See also Juarez[2006] for a comparison of average cost and random priority using this measure. Moulin[2007] uses a similar measure to compare the serial, incremental and average cost methods.

3 The model

For a vector x, $x \in \mathbb{R}^M$, we denote by $x_{[S]}$ the projection of x over $S \subset M$. x_S represents the sum of the $S-$coordinates of x, $x_S = \sum_{i \in S} x_i$. When there is no confusion we denote the projection $x_{[S]}$ simply as x_S. Let 1_M be the unitarian vector in $\mathbb{R}^M$, that is $1_M = (1, 1, \ldots, 1)$.

There is a finite number of agents $N = \{1, 2, \ldots, n\}$. Every agent has a utility (willingness to pay) for getting one unit of good. Let u, $u \in \mathbb{R}^N_+$, the vector of those utilities. Therefore, if agent i gets a unit paying x_i, his net utility is $u_i - x_i$. If he does not get a unit his net utility is zero.

Definition 1 *A mechanism (S, φ) allocates to every vector of utilities u a coalition of agents who get goods $S(u) \subseteq N$ and the cost shares (payments) $\varphi(u) \in \mathbb{R}^N$.*

Therefore, the net utility of agent i in the mechanism, denoted by NU_i, is $NU_i(u) = \delta_i(S(u))(u_i - \varphi_i(u))$.[5] Let $NU(u)$ be the vector of such net utilities. Notice two different mechanisms may be welfarewise equivalent, that is their net utilities at any profile be equal.

We restrict our attention to mechanisms that satisfy two familiar feasibility constraints.

- **Nonnegative Transfers (NNT):** $\varphi(u) \in \mathbb{R}^N_+$.
- **Individual Rationality (Voluntary participation (VP)):** $\varphi_i(u) \leq u_i \delta_i(S(u))$.

Nonnegative transfers requires all cost shares to be positive or zero. This is a common assumption when no transfers between agents are allowed and we do not want to subsidize any of them.

On the other hand, individual rationality implies that all agents enter the mechanism voluntarily. That is, the ex-post net utility of the agents is never smaller than their ex-ante net utility. Because we are assuming nonnegative transfer, individual rationality implies the agents with zero utility should pay nothing. However, they may get a unit for free. This is a basic equity condition protecting individual rights.

We want to characterize the mechanisms that are group strategyproof. That is, any misreport of a group of agents does not decrease their net utility and strictly increases the net utility of some agent.

- **Group strategyproof (GSP):** For all $S \subset N$, and all utility profiles u and u' such that $u'_{N \setminus S} = u_{N \setminus S}$, it cannot be that $NU_i(u) \leq (u_i - \varphi_i(u'))\delta_i(S(u'))$ for all $i \in S$ and strict inequality for at least one $j \in S$.

We define next our two systematic continuity conditions. Similar continuity conditions have been used in other models, for instance Deb and Razzolini[1999]. These are tractability conditions that allow us to get closed-form mechanisms. Nevertheless, these conditions can be easily interpreted, see below.

[5] δ is the classic delta function, $\delta_i(T) = 1$ if $i \in T$, and 0 otherwise.

- **Upper continuity (Maximalist tie-breaking rule (MAX)):** For any i, $u_{-i} \in \mathbb{R}_+^{N\setminus i}$, and $u_i^1 \geq u_i^2 \geq \cdots \to u_i^*$ such that $i \in S(u_i^k, u_{-i})$ for all k, then $i \in S(u_i^*, u_{-i})$.

- **Lower continuity (Minimalist tie-breaking rule (MIN)):** For any i, $u_{-i} \in \mathbb{R}_+^{N\setminus i}$, and $u_i^1 \leq u_i^2 \leq \cdots \to u_i^*$ such that $i \notin S(u_i^k, u_{-i})$ for all k, then $i \notin S(u_i^*, u_{-i})$.

In the space of strategyproof mechanisms, upper and lower continuity can be interpreted as tie-breaking rules. Upper continuity (MAX) serves the agents who are indifferent between getting or not getting a unit of good, whereas lower continuity (MIN) does not serve the indifferent agents. To see this, consider a SP mechanism. Then, there exist arbitrary pricing functions $f_i : \mathbb{R}_+^{N\setminus i} \to [0, \infty]$ for $i = 1, \ldots, n$, such that at the utility profile u, agent i is offered a unit of good at price $f_i(u_{-i})$. That is, if $u_i > f_i(u_{-i})$ then i is served at price $f_i(u_{-i})$; if $u_i < f_i(u_{-i})$ then i is not served and pays nothing; and if $u_i = f_i(u_{-i})$ then i may get a unit of good at this price or may not get it. Under MAX, if $u_i = f_i(u_{-i})$ then the agent gets a unit of good at price $f_i(u_{-i})$. On the other hand, under MIN, if $u_i = f_i(u_{-i})$ then i does not get a unit of good and pays nothing.

Finally, our model is equivalent to the reduced model where agents have utility bounded above by a positive value L. A price equal to ∞, $f_i(u_{-i}) = \infty$, is reinterpreted in the new model as a price bigger than L. That is, agent i is offered a unit of good at a price above his maximum utility.

4 Cross-monotonic mechanisms and MAX

Definition 2 *A cross-monotonic set of cost shares (payments) assigns to every coalition $S \subseteq N$ a vector $x^S \in [0, \infty]^N$ such that $x^S_{[N\setminus S]} = 0$ and moreover*

$$\textit{If } S \subseteq T \textit{ then } x^S_{[S]} \geq x^T_{[S]}.$$

We denote by χ^N a cross-monotonic set of cost shares, $\chi^N = \{x^S \mid S \subseteq N\}$.

We interpret x^S as the payment when the agents in S, and only them, are served. Therefore, by NNT and VP it should be zero for the agents outside S.

The key feature is that payments should not increase as coalition increases. This implies that for every utility profile u the set of feasible coalitions, $F(u) = \{S \in 2^N \mid x^S \leq u\}$, has a maximum element with respect to the inclusion $\subseteq$. To see this, notice if $S, T \in F(u)$ then by cross-monotonicity $S \cup T \in F(u)$.

Definition 3 *Given a cross-monotonic set of cost shares χ^N, we define a cross-monotonic mechanism (S, φ) as follows. For every utility profile u, $S(u)$ is the maximum feasible coalition at u and $\varphi(u) = x^{S(u)}$.*

Theorem 1 *A mechanism satisfies GSP and MAX if and only if it is cross-monotonic.*

In an economy without indifferences, cross-monotonic mechanisms are also characterized by GSP and monotonicity in size, that is if $u \leq \tilde{u}$ then $S(u) \subseteq S(\tilde{u})$. See Juarez[2007b] for details.

Given a cross-monotonic set of cost shares χ^N, we can also implement the truthful outcome of the cross-monotonic mechanism by playing the following demand game proposed by Moulin[1999]. We offer agents in N units of good at price x^N. If all of they accept it, then everyone is served at prices x^N. If only agents in S accept, then we remove agents in $N \setminus S$ from the game and offer agents in S units of good at price x^S. Continue similarly until all of the agents in a coalition accepted or every agent in N was removed from the game.

Example 1 (Geometric description of cross-monotonic mechanisms for $n = 1, 2$) *The one agent mechanisms can be described by a constant x, $x \in [0, \infty]$. The agent gets a unit and pays x if his utility is bigger than or equal to x. He does not get a unit and pays nothing otherwise.*

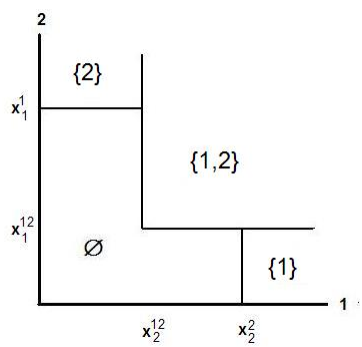

Figure 1: Generic form of 2-agent cross-monotonic mechanisms.

The two agent mechanisms should be generated by a cross-monotonic set of cost shares. Thus $0 \leq x_1^{\{1,2\}} \leq x_1^1$ and $0 \leq x_2^{\{1,2\}} \leq x_2^2$ (see figure 1).

By MAX, the level set of $\{1, 2\}$ is closed. The borders between the level sets of $\{1\}$ and $\emptyset$, and $\{2\}$ and $\emptyset$, should belong to the $\{1\}$ and $\{2\}$ respectively.

As is well know from previous literature, if the actual cost of the service C is submodular with respect to coalitions, we can choose a cross-monotonic mechanism to cover this cost exactly. For instance, we can choose the cross-monotonic set of cost shares χ^N where the payments of the agents in S are given by the egalitarian solution $x_i^S = \frac{C(S)}{|S|}$ for all $i \in S$. We can alternatively choose the payments of those agents given by the Shapley value or the Dutta-Ray egalitarian solution (Dutta and Ray[1999]) on the stand alone cost function.

Definition 4 *We say a mechanism satisfies strong consumer sovereignty (SCS) if every agent i has utility profiles $\bar{u}_i$ and $\tilde{u}_i$ such that for any profile of the other agents u_{-i}, $i \notin S(\bar{u}_i, u_{-i})$ and $i \in S(\tilde{u}_i, u_{-i})$.*

Moulin[1999] proved that, in the space of submodular cost functions, any mechanism that is budget balanced, SCS and GSP should be implemented as a cross-monotonic mechanism for a set of cross-monotonic and budget-balanced cost shares. The result we propose is more general. We show that cross-monotonic mechanisms emerge simply from the combination of GSP and MAX. However, as shown in example 3, this does not imply the cost sharing function defined by $C(S) = \sum_{i \in S} x_i^S$ is submodular. Hence we capture Moulin's mechanisms and a few more.

Example 2 *Immorlica et al.[2005] proposes an example where exactly one agent pays a positive amount when a coalition of agents is served. This example relaxes the SCS condition on Moulin[1999] result (see above), therefore is not captured by Moulin's mechanisms. However, it is captured by our class of cross-monotonic mechanisms. For a submodular cost function, order the agent arbitrary, say $i_1 \succ i_2 \succ \cdots \succ i_n$. Offer the agents, following this order, a unit of good at the cost of himself and the agents after him. The mechanism ends when someone accepts the offer or when we have made an offer to every agent. That is, agent i_1 will be offer a unit at price $C(i_1, \ldots, i_n)$. If he accepts, the mechanism ends there. If he rejects, we offer agent i_2 a unit of good at price $C(i_2, \ldots, i_n)$, and so on. The cross-monotonic set of cost shares that implements this mechanism is $x_{i^*}^S = C(D_{i^*})$ and $x_j^S = 0$ for all $j \neq i^*$, where i^* is the maximal element in S and D_i^* is the set of agents dominated by i^* with $\succ$ (including him).*

Definition 5 *We say the mechanism (S, φ) meets the equal share property (ESP) if every agent in the coalition that is getting service pays the same. That is, if $\varphi_i(u) = \varphi_j(u)$ for all $i, j \in S(u)$.*

Example 3 *Consider any cost function $C : 2^N \to \mathbb{R}_+$ such that its average cost function AC, $AC(S) = \frac{C(S)}{|S|}$, is not increasing as coalition increases.*

$x_i^S = AC(S)$ if $i \in S$, $x_i^S = 0$ if $i \notin S$, defines a cross-monotonic set of cost shares that covers the cost exactly and meets the ESP.

It is easy to see that the monotonicity of AC does not imply the concavity of C. Hence, there are ESP cross-monotonic set of cost shares whose associated cost function is not concave.

Finally, notice that a ESP cross-monotonic set of cost shares covers exactly the cost of C if and only if its average cost AC is not increasing.

In general, if the cross-monotonic set of cost shares χ^N does not meet the ESP, then the cost function C exactly covered by χ^N may not be easy to describe. See Sprumont[1990] and Norde et al.[2002] for characterizations of these cost functions in simple cases.

5 Sequential mechanisms and MIN

Definition 6 *A sequential tree is a binary tree of length n such that:*

i. at every node there is exactly one agent in N and a price in $[0, \infty]$.

ii. Every path from the root to a terminal node contains all agents in N exactly once.

Definition 7 (Sequential mechanisms) *Given a sequential tree we construct a sequential mechanisms as follows:*

We offer the agent in the root of the tree a unit of good at the price of his node. If his utility is strictly bigger than the offered price, then we allocate him a unit at this price and go right on the tree. If his utility is smaller than or equal to the offered price then we do not allocate him a unit and go left on the tree. We continue similarly with the following agent until we reach the end of the tree.

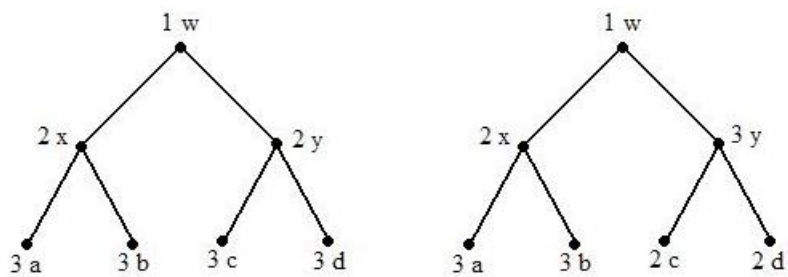

Figure 2: Sequential trees for three agents. (a) Agents follow order 1,2,3. (b) Agents 2 and 3 follow different orders depending on whether agent 1 is going right or left.

Example 4 *In figure 2 we show the only two possible (up to renaming the agents) sequential trees for the agents in $N = \{1, 2, 3\}$. Every node contains a number and a letter. The number represent the agent in this node. The letter represent a prices in $[0, \infty]$.*

Consider the sequential tree of figure 2(a) and the mechanism (S, φ) that it implements. If the utility profile u is such that $u_1 > w, u_2 > y$ and $u_3 \leq d$ then the outcome is $S(u) = \{1, 2\}$ and $\varphi(u) = (w, y, 0)$.

On the other hand, if $\tilde{u}$ is such that $\tilde{u}_1 \leq w, \tilde{u}_2 > x$ and $\tilde{u}_3 \leq b$ then $S(\tilde{u}) = \{2\}$ and $\varphi(\tilde{u}) = (0, x, 0)$.

Sequential mechanisms are not group strategyproof. For instance, consider the mechanism generated by the sequential tree of figure 2(a). If $y < x$, then when the true utility profile is such that $u_1 = w$ and $u_2 > y$, agent 1 can help agent 2 by reporting a utility bigger than w, whereby agent 2 is offered a unit at a cheaper price. However, these mechanisms are weakly group strategyproof, that is if a coalition of agents successfully misreports, then at least one agent in this coalition is indifferent. Definition 9 gives the exact conditions under which sequential mechanisms are GSP.

Given a sequential tree, consider any path in the tree and a non terminal node ζ in this path. We say ζ is leftist (rightist) on this path if the edge in the path that follows ζ is a left (right) edge. For instance, the path $[1w, 2y, 3c]$ in figure 2(a) contains one rightist node and one leftist node. $1w$ is rightist and $2y$ is leftist.

One useful path is from a node to the root of the tree. We denote by $P_0(\zeta)$ this path starting at node ζ. For instance, in figure 2(a), $P_0(3c) = [1w, 2y, 3c]$, $P_0(3d) = [1w, 2y, 3d]$ and $P_0(2x) = [1w, 2x]$.

Notice the intersection of two paths to the root of the tree is also path to the root of the tree. We use $\sqcap$ to denote it. For instance, in figure 2(a), $P_0(3c) \sqcap P_0(3d) = [1w, 2y]$. Notice this intersection may also lead to the degenerated path that contains only the root of the tree, for instance $P_0(2x) \sqcap P_0(2y) = [1w]$.

Definition 8 *Let ζ and ζ' two nodes in a sequential tree. We say the node ζ is on the left of ζ' if the terminal node of $P_0(\zeta) \sqcap P_0(\zeta')$ is leftist on $P_0(\zeta)$ and rightist on $P_0(\zeta')$.*

For instance, in figure 2(a), $P_0(3c) = [1w, 2y, 3c]$, $P_0(3d) = [1w, 2y, 3d]$. Since $2y$ is leftist in $[1w, 2y, 3c]$ and rightist in $[1w, 2y, 3d]$, then $3c$ is on the left of $3d$.

Finally, if T is a path and i is an agent in this path, $i \in T$, then we denote by x_i^T the price of agent i in T.

Definition 9 (Feasible sequential tree) *Consider a sequential tree and any two nodes ζ and ζ' with a common agent k such that ζ is on the left of ζ'. Also, assume every rightist node in $P_0(\zeta)$ or $P_0(\zeta')$ has finite price. Let L the maximal sub-path of $P_0(\zeta)$ that does not intersect $P_0(\zeta')$, that is $L = P_0(\zeta) \backslash (P_0(\zeta) \sqcap P_0(\zeta'))$. Similarly, let $R = P_0(\zeta') \backslash (P_0(\zeta) \sqcap P_0(\zeta'))$.*

We say that a sequential tree is feasible if for any two nodes ζ and ζ' as above, whenever the prices of agent k are such that $x_k^L > x_k^R$, there exist nodes $\tilde{\zeta} \in L$ and $\bar{\zeta} \in R$ that contain the same agent i and:

(a) $\tilde{\zeta}$ is leftist in L and $\bar{\zeta}$ is rightist in R and $x_i^L < x_i^R$, or

(b) $\tilde{\zeta}$ is rightist in L and $\bar{\zeta}$ is leftist in R and $x_i^L \geq x_i^R$.

We say a sequential mechanism is feasible if it is implemented by a feasible sequential tree.

Notice a sufficient condition to guarantee a feasible sequential tree is that for any two nodes with the same agent, the price on the leftist node is not bigger than the price on the rightist node. This condition is necessary when there are at most three agents (see examples 5, 6 and 7). Example 8 shows this is not true when there are more than three agents.

Theorem 2 *A mechanism is GSP and MIN if and only if it is a feasible sequential mechanism.*

Consider a feasible sequential mechanism and assume agent i^* is in the root of its feasible sequential tree. Consider the leftist (rightist) sequential mechanism for $N \setminus i^*$ agents, generated by the feasible sequential subtree where agent i^* is leftist (rightist). Then, the outcome of this leftist mechanism should Pareto dominate the outcome of the rightist mechanism at any profile of $N \setminus i^*$ agents. That is, for any profile of $u_{N \setminus i^*}$ agents, any agent in $N \setminus i^*$ should be better off without agent i^* than with agent i^*. To see this, assume at this profile agent $j \in N \setminus i^*$ is strictly better off with the rightist mechanism. Then, when the utility of agent i^* equals his offered price, $u_{i^*} = x_i^*$, by MIN we should allocate with leftist mechanism and i^* is not served. Thus agent i^* can help agent j^*. He can increase his utility profile, he will be served at a price equal to his valuation and agent j^* will be better off.

Example 5 (Geometric description of feasible sequential mechanisms for $n = 1, 2$) *The one agent mechanisms are easy to describe. Given $x_1 \in [0, \infty]$, agent 1 gets a unit of good at price x_1 if and only if $u_1 > x_1$.*

A two agents mechanism such that 2 has priority over 1, is shown in figure 3. Agent 2 gets a unit of good at price x^2 if and only if $u_2 > x^2$. If 2 gets a unit of good, then agent 1 gets a unit of good at price d^1 if $u_1 > d^1$. On the other hand, if agent 2 did not get a unit of good, then agent 1 gets a unit of good at price d^2 if $u_1 > d^2$. By feasibility of the tree $d^2 \leq d^1$.

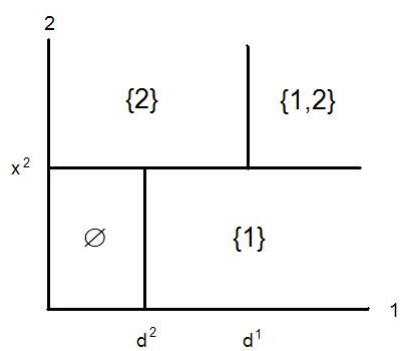

Figure 3: Generic form of 2-agent feasible sequential mechanisms.

Example 6 *Assume there are three agents. Figure 2 shows sequential trees for three agents. Every node contains an agent from $\{1, 2, 3\}$ and a nonnegative price.*

On figure 2(a), a feasible sequential tree (assuming finite prices) implies: $x \leq y$, $a \leq b \leq d$ and $a \leq c \leq d$. Also, if $x < y$ then $b \leq c$.

To see this, consider nodes $2x$ and $2y$. Since they are consecutive nodes, their paths to the root of the tree only differ in $2x$ and $2y$ respectively. Then, conditions (a) and (b) cannot be satisfied. Hence $x \leq y$.

Similarly, $a \leq b$ and $c \leq d$ are satisfied by comparing nodes $3a$ and $3b$, and $3c$ and $3d$ respectively.

On the other hand, by comparing nodes $3a$ and $3c$, conditions (a) and (b) are not satisfied because $2x$ and $2y$ are both leftist. Hence $a \leq c$. Similarly $b \leq d$.

Now consider the nodes $3b$ *and* $3c$*. If* $x < y$*, then condition (a) is not satisfied because* $2y$ *is not rightist. Condition (b) is not satisfied because* $x < y$*. Therefore it cannot be that* $b > c$*. Hence* $x < y$ *and* $a \leq b \leq c \leq d$*.*

Finally, assume $x = y$*. From the argument given above,* $a \leq b \leq d$ *and* $a \leq c \leq d$*.*

If $b \leq c$ *then for every two nodes with the same agent, the price on the leftist node is smaller than the price on the rightist node.*

On the other hand, if $b > c$ *then because agents 1 and 2 have priority, we can exchange their order on the tree. This will look like figure 4. With this order, for every two nodes with the same agent, the price on the leftist node is smaller than the price on the rightist node.*

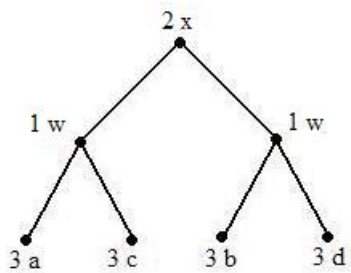

Figure 4: Three agents sequential tree such that the positions of agents 1 and 2 can be switched without affecting the final outcome.

Example 7 *Now consider the figure 2(b). Then feasibility of the tree (assuming finite prices) requires that* $a \leq b \leq y$ *and* $x \leq c \leq d$*. That is for every two nodes with the same agent, the price on the leftist node is smaller than the price on the rightist node.*

To see this, by comparing nodes $3a$ *and* $3b$*, and* $2c$ *and* $2d$*, we get (similarly to example above) that* $a \leq b$ *and* $c \leq d$ *respectively.*

Now we compare nodes $3b$ *and* $3y$*. Then there is no common agent in their path to the root, thus conditions (a) and (b) cannot be satisfied. Hence* $b \leq y$*. That is,* $a \leq b \leq y$*.*

Similarly, by comparing nodes $2x$ *and* $2c$*,* $x \leq c$*. Hence* $x \leq c \leq d$*.*

Example 8 *Consider the mechanism generated by the sequential tree of figure 5 (agents are in the rectangles). For every two nodes with the same agent, the price on the leftist node is not bigger than the price on the rightist node, except for nodes (4 10) and (4 9). At these nodes, their paths to the root contain the common agent 2. This agent meets condition (b). Therefore this tree is feasible.*

However, the price on the leftist node (4 10) is bigger than that on the rightist node (4 9).

Since agents 1 and 2 have priority, we can also exchange their positions and leave agent agent 2 in the root. If this is the case, node (3 8) is on the left of (3 7).

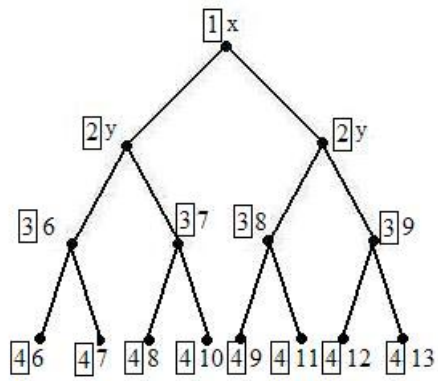

Figure 5: Four agents feasible sequential tree such that for every two nodes with the same agent, the price of the rightist node may not be smaller than the price of the leftist node.

Sequential mechanisms are related to the incremental cost mechanisms of Moulin[1999]. That is, consider a supermodular (convex) cost function and a tree as above. Start with the agent i_1 in the root and offer him a unit of good at price $C(i_1)$. If he buys, continue with the agent i_2 on the right of the tree and offer him a unit of good at price $C(i_1, i_2) - C(i_1)$. If i_1 did not buy, then offer the agent on the left of the tree, k_2, a unit of good at price $C(k_2)$. Proceed similarly with the following agents until you reach the end of the tree.

Theorem 1 in Moulin[1999] suggests that incremental cost mechanisms are GSP mechanisms when the cost function is supermodular. However, this is not true, as shown on next example.

Example 9 *Consider the supermodular cost function:*

$$C(i) = 1, C(1,2) = 3, C(1,3) = 5, C(2,3) = 6, C(1,2,3) = 15.$$

By choosing the ordering $1 \succ 2 \succ 3$, *the cost shares are as follows:*
$x^{\{1,2,3\}} = (1,2,12), x^{\{1,2\}} = (1,2,0), x^{\{1,3\}} = (1,0,4), x^{\{2,3\}} = (0,1,5), x^{\{i\}} = 1_i$.

When the utility profile is $u = (1, 1.5, 4.5)$ *there are two options depending on whether 1 decides to get or not get a unit. If agent 1 gets a unit, then 2 does not get a unit and 3 gets a unit. Thus* $\{1,3\}$ *is served and the cost shares are* $(1,0,4)$. *If agent 1 does not get a unit, then 3 does not get a unit. Thus* $\{2\}$ *is served and the cost shares are* $(0,1,0)$. *Given that 1 is indifferent between getting and not getting a unit, he may help 2 or 3. Thus the mechanism cannot be* GSP. *The reason is clear by our analysis. The leftist mechanism without agent 1 does not Pareto dominate the rightist mechanism at the utility profile* u.

What is important from Moulin[1999] is that incremental cost mechanism may not be fully GSP, but they are GSP except when agents are indifferent between getting and not getting a unit of good. Thus the mistake is very tiny.

Whenever the supermodular cost function and the ordering of the agents give a sequential mechanism that is feasible, it must be captured by a sequential mechanism discussed above.

On the other hand, given a feasible sequential mechanism, the associated budget balance cost function —the cost of S defined as the sum of the payments on coalition S— may not be supermodular (see example below). So these mechanisms capture even more mechanism that those generated by the incremental cost mechanisms.

Example 10 (Feasible sequential mechanisms that meet ESP) *Consider an arbitrary order of the agents, assume without loss of generality that $1 \succ 2 \succ \cdots \succ n$, and arbitrary prices $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. Given this order and prices, construct the cost function as follows:*

$$C(S) = \mid S \mid \max_{k \in S} a_k. \tag{1}$$

For this cost function, there is a feasible sequential mechanism that covers its cost exactly and meets ESP. To see this, construct a sequential tree following linearly the order $\succ$. The price of a node ζ is a_k, where k is the rightist agent in $P_0(\zeta)$ with the lowest index.

In figure 6 we illustrate this feasible sequential mechanism for five agents. The agents in every coalition that contains agent 1 pay a_1. The agents in every coalition that contains agent 2 but not 1 pay a_2. The agents in every coalition that contains agent 3 but neither 1 or 2 pay a_3, etc.

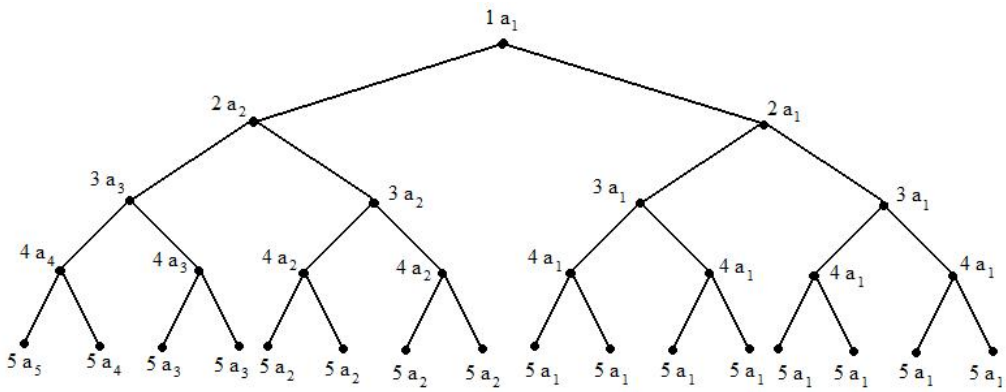

Figure 6: Five agents feasible sequential mechanism that meets ESP.

Clearly, this mechanism meets ESP. This tree is feasible because for every two nodes with the same agent, the price on the leftist node is not smaller than the price on the rightist node. Thus the mechanism is a feasible sequential mechanisms that covers the cost C exactly.

It is also clear that any feasible sequential mechanism that meets ESP should be of this form. Hence, the class of cost functions whose cost is covered exactly by an ESP feasible sequential mechanism are those described above.

Notice these cost functions may not be supermodular. We can easily find prices that meet the next inequality:

$$C(1,3) + C(2,3) = 2a_1 + 2a_2 > 3a_1 + a_3 = C(1,2,3) + C(3).$$

Finally, this class of cost function is smaller than the class of cost functions presented in example 3. There, any cost function with non-increasing average cost is achievable by a ESP and cross-monotonic mechanism. On the other hand, the class of cost function given by equation 1 have non-decreasing average cost. Moreover, the average cost is constant for coalitions that contain agent 1; or for coalitions that do not contain 1 but contain 2, etc.

6 Comparison between cross-monotonic and sequential mechanisms

6.1 The intersection of cross-monotonic and sequential mechanisms

Although the intersection of MAX and MIN is empty by definition, there is a small class of mechanisms that are welfare equivalent to both a sequential and a cross-monotonic mechanism.

Definition 10 *Given $x_1, \ldots, x_n \in [0, \infty]$, the corresponding fixed cost mechanism offers to agent i a unit of good at price x_i. Indifferences are broken arbitrarily. That is, for the utility profile u, agent i is guaranteed a unit at price x_i if $u_i > x_i$. Agent i does not get a unit if $u_i < x_i$. At $u_i = x_i$ he may or may not get a unit.*

Corollary 1 *A mechanism is welfare equivalent to a cross-monotonic and a feasible sequential mechanism if and only if it is a fixed cost mechanism.*

This result shows that the behavior of indifferences have a big impact on the class of GSP mechanism. But one can argue that indifferences are rare event, so that a better model is one where the domain of utilities and the class of mechanisms preclude indifferences. On such domain, the class of GSP mechanisms will contain many more mechanisms than the sequential and cross-monotonic mechanisms. Juarez[2007b] analyzes such domain and characterizes the corresponding GSP mechanisms.

6.2 Equal treatment of equal agents

Definition 11 *We say a mechanism satisfies equal treatment of equals (ETE) if for any u such that $u_i = u_j$, $i \in S(u)$ then $j \in S(u)$ and $\varphi_i(u) = \varphi_j(u)$.*

Proposition 1 *A mechanism meets GSP and ETE if and only if it is welfare equivalent to a cross-monotonic mechanism that meets ESP.*

This proposition not only talks in favor of cross-monotonic mechanisms as GSP mechanisms meeting the basic equity requirement of ETE. It also shows the incompatibility of GSP and fairness for any other mechanism that is not welfare equivalent to a cross-monotonic. In particular, it rules out sequential mechanisms and also those GSP mechanisms discussed by Juarez[2007b] and Roughgarden[2007].

6.3 Limited number of goods

When a social planner or seller has (can produce) less than n units of good, it is impossible to meet simultaneously ETE and GSP.[6] This is easy to check by looking at the utility profiles of the form $(x, \ldots, x)$, $x > 0$. By ETE, $S(x, \ldots, x) = \emptyset$ for all x. Hence, by proposition 1 above and taking into account that the smallest cost share in a cross-monotonic mechanism is achieved when serving N, the mechanism should not allocate any unit at all.

Moreover, when there is scarcity of the good, cross-monotonic mechanisms exclude ex-ante some agents from the mechanism. That is, if only k units of good are available, $k < n$, then any cross-monotonic mechanism is such that $n - k$ agents are not served at any profile. To see this, notice coalition N never gets service, therefore the cost shares of N should have at least one coordinate equal to ∞. Thus the agent i with such coordinate never participates in the game because his smallest payment is achieved when serving N. We remove this agent from the game and proceed similarly with the remaining coalition $N \setminus i$, until we have removed at least $n - k$ agents.

On the other hand, there are many sequential mechanisms that do not ex-ante exclude any agent. If $k \geq 2$, some easy combination of sequential and cross-monotonic mechanisms can be constructed.

Definition 12 *Given an arbitrary order of the agents $i_1, \ldots, i_n$ and arbitrary prices (some of them may be infinity) $x^1, x^2, \ldots, x^n$, we define a priority mechanism as follows: Start with agent i_1 and offer him a unit of good at price x^1. If he buys the mechanism stops there. If he does not buy, then continue with agent i_2 and offer him a unit of good at price x^2. Continue similarly until some agent buy or we offered a unit to all agents.*

Notice priority mechanisms are feasible sequential mechanisms for the feasible sequential tree such that agents are ordered linearly following the order $i_1, \ldots, i_n$; only the most leftist branch of the tree has prices equal to $(x^1, x^2, \ldots, x^n)$ and any other node has a price equal to ∞.

Proposition 2 *Suppose a mechanism is GSP and allocates at most one unit of good at any profile, then the mechanism is welfare equivalent to a priority mechanism.*

Notice this proposition is independent of the tie-breaking rule. In particular, it shows that when there is only one unit of good, feasible sequential mechanisms are the only GSP mechanisms that do not exclude ex-ante any agent.

[6] Except by the trivial mechanism that does not serve anyone at any profile.

7 Conclusions

This paper characterizes the GSP mechanisms under two alternative continuity conditions. On one hand, cross-monotonic mechanisms are characterized by GSP and MAX. These mechanisms are very useful when symmetry is required. However, they are very inefficient when there is scarcity of the good.

On the other hand, sequential mechanisms are characterized by GSP and MIN. These mechanisms are appropriate when there is scarcity of the good, for instance when there is only one unit of good available. Unfortunately, deterministic sequential mechanisms fail standard equity conditions like symmetry.

Group strategyproof mechanisms without any of the two alternative continuity conditions can be easily constructed, for instance some priority compositions of sequential an cross monotonic mechanism are GSP (see Juarez[2007b] or Roughgarden[2007]). However, the full characterization of GSP mechanisms in this economy is an open question.

Finally, GSP mechanisms without VP and NNT can be easily constructed by making the following transformation. Fix a GSP mechanism (S, φ) meeting VP and NNT, and a vector $v \in \mathbb{R}^N$. Let $(\tilde{S}, \tilde{\varphi})$ be a mechanism defined as $(\tilde{S}(u), \tilde{\varphi}(u)) = (S(u), \varphi(u) + v)$ for all $u \in R^N_+$.

Clearly $(\tilde{S}, \tilde{\varphi})$ meets GSP. If $v >> 0$ then $(\tilde{S}, \tilde{\varphi})$ meets NNT but does not meet VP. On the other hand, if $v << 0$ then $(\tilde{S}, \tilde{\varphi})$ meets VP but does not meet NNT. If v contains negative and positive coordinates simultaneously, then the mechanism is neither VP nor NNT.

Proofs

Proof of Theorem 2.

Feasible sequential mechanisms meet MIN and GSP.

Feasible sequential mechanisms trivially meet MIN.

We prove by contradiction these mechanisms meet GSP. Assume coalition $\widetilde{S}$ profitably misreports $\widetilde{u}_{\widetilde{S}}$ at the true profile u. Let $k \in \widetilde{S}$ be an agent who strictly increases his net utility by misreporting. Let ζ and ζ' be the nodes that contain agent k in the paths that generate $S(u)$ and $S(\widetilde{u}_{\widetilde{S}}, u_{-\widetilde{S}})$ respectively.

First notice ζ is on the left of ζ'. To see this, let i^* be the agent in the terminal node of $P_0(\zeta) \sqcap P_0(\zeta')$. Then, in order to move from $P_0(\zeta)$ to $P_0(\zeta')$, agent i^* misreports. If i^* is rightist in $P_0(\zeta)$ then by MIN his net utility is positive, so he will never agree to move to $P_0(\zeta')$ because he is not served there.

Let L and R be as in definition 9. Since agent k strictly increases his net utility, then $x_k^L > x_k^R$. Assume condition (a) of feasibility is satisfied. That is, there exist nodes $\widetilde{\zeta}$ and $\bar{\zeta}$ that contain the same agent i such that $\widetilde{\zeta}$ is leftist in L, $\bar{\zeta}$ is rightist in R and $x_i^L < x_i^R$. Since $\widetilde{\zeta}$ is leftist in L then $u_i \leq x_i^L < x_i^R$. Thus, for the path $P_0(\zeta')$ to realize, $i \in \widetilde{S}$ and $\widetilde{u}_i > x_i^R$. Hence the net utility of agent i is negative when he misreports because $u_i < x_i^R$. This is a contradiction.

On the other hand, assume condition (b) of feasibility is satisfied. That is, there exist nodes $\widetilde{\zeta}$ and $\bar{\zeta}$ that contain the same agent i such that $\widetilde{\zeta}$ is rightist in L, $\bar{\zeta}$ is leftist in R and $x_i^L \geq x_i^R$. Given that $\widetilde{\zeta}$ is rightist in L, $u_i > x_i^L \geq x_i^R$. Thus, for the path $P_0(\zeta')$ to realize, $i \in \widetilde{S}$ and $\widetilde{u}_i \leq x_i^R$. Hence, the net utility of agent i strictly decreases from $u_i - x_i^L$ to zero when he misreports. This is a contradiction.

Any GSP and MIN mechanism is a feasible sequential mechanism.

Let (S, φ) a mechanism that meets GSP and MIN. Steps 1, 2 and 3 are three preliminary properties of (S, φ). Steps 4 and 5 prove (S, φ) is a sequential mechanism. Step 6 proves it is a feasible sequential mechanism.

Step 1. If $S(u) = S^*$ and $\varphi(u) = \varphi^*$, then for all $\widetilde{u}$ such that $\widetilde{u}_{[S^*]} >> \varphi_{[S^*]}$ and $\widetilde{u}_{[N \setminus S^*]} \leq u_{[N \setminus S^*]}$, $S(\widetilde{u}) = S^*$ and $\varphi(\widetilde{u}) = \varphi^*$.

Proof.

First notice that by MIN, an agent gets positive net utility if and only if he is served.

Let $i \in S^*$. Then $S(\widetilde{u}_i, u_{-i}) = S^*$ and $\varphi(\widetilde{u}_i, u_{-i}) = \varphi^*$. To see this, if $i \notin S(\widetilde{u}_i, u_{-i})$ or $\varphi_i(\widetilde{u}_i, u_{-i}) > \varphi_i^*$, then agent i misreports u_i when the true profile is $(\widetilde{u}_i, u_{-i})$, which contradicts SP. On the other hand, if $i \in S(\widetilde{u}_i, u_{-i})$ and $\varphi_i(\widetilde{u}_i, u_{-i}) < \varphi_i^*$, then agent i misreports $\widetilde{u}_i$ when the true profile is u, which also contradicts SP. Therefore, $i \in S(\widetilde{u}_i, u_{-i})$ and $\varphi_i(\widetilde{u}_i, u_{-i}) = \varphi_i^*$.

Let j, $j \neq i$. If $NU_j(\widetilde{u}_i, u_{-i}) > NU_j(u)$, then agent i helps j by misreporting $\widetilde{u}_i$ when the true profile is u. This contradicts GSP. The case $NU_j(\widetilde{u}_i, u_{-i}) < NU_j(u)$ is analogous. Thus $NU_j(\widetilde{u}_i, u_{-i}) = NU_j(u)$ for all $j \neq i$. Therefore, by MIN $S(\widetilde{u}_i, u_{-i}) = S^*$ and $\varphi(\widetilde{u}_i, u_{-i}) = \varphi^*$.

By applying the previous argument to each agent in S^*, we have that $S(\widetilde{u}_{S^*}, u_{-S^*}) = S^*$ and $\varphi(\widetilde{u}_{S^*}, u_{-S^*}) = \varphi^*$.

Let $j \notin S^*$. Then $S(\widetilde{u}_{S^*\cup j}, u_{-S^*\cup j}) = S^*$ and $\varphi(\widetilde{u}_{S^*\cup j}, u_{-S^*\cup j}) = \varphi^*$. First notice that $j \notin S(\widetilde{u}_{S^*\cup j}, u_{-S^*\cup j})$, otherwise by voluntary participation

$$\varphi_j(\widetilde{u}_{S^*\cup j}, u_{-S^*\cup j}) < \widetilde{u}_j \leq u_j.$$

Thus agent j misreports $\widetilde{u}_j$ when true profile is $(\widetilde{u}_{S^*}, u_{-S^*})$. This contradicts SP.

On the other hand, if $NU_k(\widetilde{u}_{S^*\cup j}, u_{-(S^*\cup j)}) < NU_k(\widetilde{u}_{S^*}, u_{-S^*})$ for some $k \neq j$, then agent j helps k by reporting $\widetilde{u}_j$ when true profile is $(\widetilde{u}_{S^*}, u_{-S^*})$, this contradicts GSP. Similarly, by GSP $NU_k(\widetilde{u}_{S^*\cup j}, u_{-(S^*\cup j)}) > NU_k(\widetilde{u}_{S^*}, u_{-S^*})$ cannot occur. Thus $NU_k(\widetilde{u}_{S^*\cup j}, u_{-(S^*\cup j)}) = NU_k(\widetilde{u}_{S^*}, u_{-S^*})$ for all $k \neq j$. Hence, by MIN $S(\widetilde{u}_{S^*\cup j}, u_{-(S^*\cup j)}) = S^*$ and $\varphi(\widetilde{u}_{S^*\cup j}, u_{-S^*\cup j}) = \varphi^*$.

By repeatedly using the previous argument to every agent in $N \setminus S^*$, we have that $S(\widetilde{u}) = S^*$ and $\varphi(\widetilde{u}) = \varphi^*$.

Step 2. If $S(u) = S(\widetilde{u})$ then $\varphi(u) = \varphi(\widetilde{u})$.

Proof.

Let $S^* = S(u) = S(\widetilde{u})$, $\bar{v}_{[S]} = \max(\widetilde{u}_{[S]}, u_{[S]})$ and $\bar{v}_{[N\setminus S]} = \min(\widetilde{u}_{[N\setminus S]}, u_{[N\setminus S]})$ (where max and min are taken coordinate by coordinate).

By step 1, comparing $\bar{v}$ and u, $S(\bar{v}) = S^*$ and $\varphi(\bar{v}) = \varphi(u)$. Similarly, comparing $\bar{v}$ and $\widetilde{u}$, $\varphi(\bar{v}) = \varphi(\widetilde{u})$.

By step 2, there exist at most one vector of payments for every coalition. Let x^{S^*} be the payment of coalition S^* when S^* is served at some profile.

Step 3. Let u be such that $S(u) = S^*$ and $\varphi(u) = \varphi^*$. Then for every $i \in S^*$ and $u_i^* \leq \varphi_i^*$, $S^*\backslash i \subseteq S(u_i^*, u_{-i})$ and $\varphi_{S^*\backslash i}(u_i^*, u_{-i}) \leq \varphi^*_{S^*\backslash i}$.

Proof.

First notice that for every $j \in S^* \setminus i^*$, $j \in S(\varphi_i^*, u_{-i})$ and $\varphi_j(\varphi_i^*, u_{-i}) \leq \varphi_j^*$. Indeed, by MIN the net utility of agent j at u is positive. If $j \notin S(\varphi_i^*, u_{-i})$ or $\varphi_j(\varphi_i^*, u_{-i}) > \varphi_j^*$ then agent i can help j by misreporting u_i when the true profile is (φ_i^*, u_{-i}) : By MIN, agent i is not being served at the profile (φ_i^*, u_{-i}), thus he is indifferent between misreporting u_i and getting a unit at price φ_i^*, or truly reporting φ_i^* and not getting a unit, whereby agent j is better of at u. This contradicts GSP.

Finally, since $i \notin S(\varphi_i^*, u_{-i})$ and by step 1, $S(u_i^*, u_{-i}) = S(\varphi_i^*, u_{-i})$ and $\varphi(u_i^*, u_{-i}) = \varphi(\varphi_i^*, u_{-i})$ for all $u_i^* \leq \varphi_i^*$.

Step 3.1 If $S(u) = S^*$, then for any T, $T \subset S^*$, there exist $\widetilde{u}$ such that $S(\widetilde{u}) = T$ and $x^T_{[T]} \leq x^{S^*}_{[T]}$.

Proof.

Let $\bar{u} = (u_{S^*}, 0_{-S^*})$. By step 1, $S(\bar{u}) = S^*$ and $\varphi(\bar{u}) = x^{S^*}$. Let $i \in S^*$. By MIN $i \notin S(x_i^{S^*}, \bar{u}_{-i})$. By step 3, $S^* \setminus i \subset S(x_i^{S^*}, \bar{u}_{-i})$. Since the utilities of agents outside S^* are zero, then by MIN $S^* \setminus i = S(x_i^{S^*}, \bar{u}_{-i})$. Thus by step 3, $x^{S\setminus i^*}_{[S\setminus i^*]} \leq x^S_{[S\setminus i^*]}$. Finally, to check the claim we repeatedly apply the above argument to every agent in $S^* \setminus T$.

Step 4. Assume there is u^* such that $S(u^*) = N$. Then, there is an agent to whom is offered a unit of good at a price that is independent of the utilities of the other agents (we say this agent has priority).

We prove this by induction in the size of N.

If $N = \{1\}$ then the GSP and MIN mechanisms are clearly fixed cost mechanisms. That is, there is a fixed price x, $x \in [0, \infty]$ such that if $u_1 > x$ then 1 is served at price x. If $u_1 \leq x$ then he is not served.

For the induction hypothesis, assume that for any GSP and MIN mechanism for $n-1$ agents there is an agent who has priority. Let (S, φ) be a mechanism for the agents in $N = \{1, \ldots, n\}$.

For every j, consider the utility profiles where agent j has zero utility, that is

$$U^j = \{u \in \mathbb{R}^N_+ \mid u_j = 0\}.$$

By MIN, agent j is not being served at any profile of U^j. Thus, the restriction of (S, φ) to U^j defines a MIN and GSP mechanism for the agents in $N \setminus j$. Let $\rho^j = \{x^S \mid j \notin S\}$ be the set of payments in this mechanism. Notice because N is being served, then by step 3.1 every coalition $S \subset N$ is being served. In particular ρ^j contains a payment for every group of agents that does not contain agent j. Also, notice that by step 2 if $x^T \in \rho^j$ and $\tilde{x}^T \in \rho^k$ then $x^T = \tilde{x}^T$.

Finally by step 3.1 payments are nondecreasing as coalition increases. That is, if $S \subset T$ then $x^S_S \leq x^T_S$.

By the induction hypothesis, on ρ^1 there is an agent i_1 who has priority. The monotonicity of the payments implies $x^{N\setminus 1}_{i_1} = x^{i_1}_{i_1}$. Similarly, there is an agent who has priority on ρ^{i_1}. Call this agent i_2, thus $x^{N\setminus i_1}_{i_2} = x^{i_2}_{i_2}$. We continue this procedure until we reach a cycle. Without loss of generality, we assume the cycle is $i_1, i_2, \ldots, i_k$. This means i_{j+1} has priority on ρ^{i_j} for $j = 1, \ldots, k-1$, and i_1 has priority on ρ^{i_k}.

Case 1. The cycle has size less than n, that is $k < n$.

Let $\bar{v}_{[N\setminus\{i_1,i_2,\ldots,i_k\}]}$ be such that $\bar{v}_{[N\setminus\{i_1,i_2,\ldots,i_k\}]} >> x^N_{[N\setminus\{i_1,i_2,\ldots,i_k\}]}$.

Consider the profiles

$$U = \{u \in \mathbb{R}^N_+ \mid u_{[N\setminus\{i_1,i_2,\ldots,i_k\}]} = \bar{v}_{[N\setminus\{i_1,i_2,\ldots,i_k\}]}\}.$$

Notice that for every $u \in U$, $N\setminus\{i_1, i_2, \ldots, i_k\} \subset S(u)$. Indeed, consider $(\widetilde{u}_{\{i_1,i_2,\ldots,i_k\}}, u_{-\{i_1,i_2,\ldots,i_k\}})$ such that $\widetilde{u}_{\{i_1,i_2,\ldots,i_k\}} >> x^N_{[i_1,i_2,\ldots,i_k]}$. By step 1, $S(\widetilde{u}_{\{i_1,i_2,\ldots,i_k\}}, u_{-\{i_1,i_2,\ldots,i_k\}}) = N$. By steps 1 and 3, $N\setminus\{i_1\} \subseteq S(u_{i_1}, \widetilde{u}_{\{i_2,\ldots,i_k\}}, u_{-\{i_1,i_2,\ldots,i_k\}})$. Similarly, $N\setminus\{i_1, i_2\} \subseteq S(u_{i_1,i_2}, \widetilde{u}_{\{i_3,\ldots,i_k\}}, u_{-\{i_1,i_2,\ldots,i_k\}})$. Continuing this way, $N \setminus \{i_1, i_2, \ldots, i_k\} \subseteq S(u_{\{i_1,i_2,\ldots,i_k\}}, u_{-\{i_1,i_2,\ldots,i_k\}})$.

By step 3.1, for every coalition T such that $N \setminus \{i_1, i_2, \ldots, i_k\} \subset T$, there is $\widetilde{u} \in U$ such that $S(\widetilde{u}) = T$. This is clear because coalition N is being served at some profile of U, so we can reduce (one agent at a time) the utility of the agents not in T to zero.

Clearly, the mechanism restricted to U defines a GSP mechanism for the agents in $\{i_1, i_2, \ldots, i_k\}$. By the induction hypothesis, there is an agent who has priority, say i_1. Thus, $x_{i_1}^{N\setminus\{i_2,\ldots,i_k\}} = x_{i_1}^N$. On the other hand, because i_1 has priority on ρ^{i_k}, $x_{i_1}^{i_1} = x_{i_1}^{N\setminus\{i_2,\ldots,i_k\}}$. Therefore, $x_{i_1}^N = x_{i_1}^{i_1}$. Hence by the monotonicity of the payments $x_{i_1}^T = x_{i_1}^S$ for all $S, T \subseteq N$ such that $i_1 \in S, T$.

Finally, we prove agent i_1 has priority. Assume there is u such that $u_{i_1} > x_{i_1}^{i_1}$ but $i_1 \notin S(u)$. Consider the profile $(u_{i_1}, \tilde{u}_{-i_1})$ where $\tilde{u}_{-i_1} >> \max(x_{-i_1}^N, u_{-i_1})$ and max is taken coordinate by coordinate. By step 1, $S(u_{i_1}, \tilde{u}_{-i_1}) = N$. By step 1 and 3, $N\setminus\{i_2\} \subseteq S(u_{i_1}, u_{i_2}, \tilde{u}_{-i_1,i_2})$. Similarly, by steps 1 and 3, $N \setminus \{i_2, i_3\} \subseteq S(u_{i_1}, u_{i_2}, u_{i_3}, \tilde{u}_{-i_1,i_2,i_3})$. Continuing this way, $\{i_1\} \subseteq S(u)$. This is a contradiction.

Case 2. The cycle has size n, that is $k = n$.

Without loss of generality, assume agent 2 has priority over $N \setminus 1$, agent 3 has priority over $N \setminus 2, \ldots,$ etc. Thus,

$$x_2^2 = x_2^{N\setminus 1}, \ldots x_3^3 = x_3^{N\setminus 2}, \ldots x_1^1 = x_1^{N\setminus n}. \tag{2}$$

Also, assume to get a contradiction, that there is no agent who has priority. That is,

$$x_2^{N\setminus 1} < x_2^N, x_3^{N\setminus 2} < x_3^N, \ldots, x_1^{N\setminus n} < x_1^N.$$

Let u^* be such that $S(u^*) = N$. By MIN, $u^* >> x^N$.

By step 3, $2 \in S(x_1^N, u_{-1}^*)$ and 2 pays x_2^2, $x_2^2 < x_2^N$, because he has priority on ρ^1. Also by step 3, $2 \in S(x_{1,3}^N, u_{-1,3}^*)$ and 2 pays not more than x_2^2. Continuing similarly, $2 \in S(x_{-2}^N, u_2^*)$ and 2 pays not more than x_2^2. By step 1, $2 \in S(x^N)$ because $u_2^* > x_2^N > x_2^2$.

Finally, since everything is symmetric, $S(x^N) = N$. This contradicts MIN.

Step 5. Assume there is no u such that $S(u) = N$. If the mechanism is not trivial ($S(u) \neq \emptyset$ for some u), there is an agent who has finite priority. That is, there is an agent i^* and a payment x^*, $0 \leq x^* < \infty$, such that $i^* \in S(u)$ for all u such that $u_{i^*} > x^*$.

First notice there is a group of agents S^* who has priority. That is, for all $\widetilde{u}$ such that $\widetilde{u}_{[S^*]} \geq x_{[S^*]}^{S^*}$, $S(\widetilde{u}) = S^*$. To see this, consider $\widetilde{u}$ such that $\widetilde{u} >> x^T$ for all possible payments x^T, $x^T = \varphi(v)$ for some v (we know by step 2 that there is at most one vector of payments for every coalition, thus it is feasible to choose such $\widetilde{u}$). Let S^* be such that $S(\widetilde{u}) = S^*$. Notice that, for any i, $i \notin S^*$, $S(\widetilde{u}_{-i}, \bar{v}_i) = S^*$ for all $\bar{v}_i$. Indeed, if $\bar{v}_i \leq \widetilde{u}_i$ then by step 1 $i \notin S(\widetilde{u}_{-i}, \bar{v}_i)$. On the other hand, if $\bar{v}_i > \widetilde{u}_i$, then $i \notin S(\widetilde{u}_{-i}, \bar{v}_i)$. This is easy to see by contradiction, assume $i \in S(\widetilde{u}_{-i}, \bar{v}_i)$, then by the choice of $\widetilde{u}$, $\varphi_i(\widetilde{u}_{-i}, \bar{v}_i) < \widetilde{u}_i < \bar{v}_i$. Therefore, by step 1, $i \in S(\widetilde{u})$, which is a contradiction.

Hence, $S(\widetilde{u}_{-i}, \bar{v}_i) = S^*$ for all $\bar{v}_i$. Thus, by changing the utilities of the agents in $N \setminus S^*$ one at a time, $S(\widetilde{u}_{S^*}, u_{-S^*}) = S^*$. Hence by step 1, $S(\bar{u}_{S^*}, u_{-S^*}) = S^*$ for all $\bar{u}_{S^*} \geq x^{S^*}_{[S^*]}$ and all u_{-S^*}.

We now prove step 5 by induction. For $n = 1$, if $S(u) \neq 1$ for all u then clearly the mechanism is trivial ($S(u) = \emptyset$ for all u). So the claim is true.

For the induction hypothesis, assume the claim is true for any mechanism of $n-1$ agents. We prove it for any mechanism of n agents.

Let S^* be defined as above and $j \notin S^*$. Consider the restriction of the mechanism to $U^j = \{u \in \mathbb{R}^N_+ \mid u_j = 0\}$. Then this restriction is a GSP and MIN mechanism for the agents in $N \setminus j$. By induction and step 4, there is an agent i^* who has (finite) priority for the agents $N \setminus j$. Clearly $i^* \in S^*$, otherwise his payment is dependent on the agents in S^*.

We now prove by contradiction that for any profile u_{-i^*}, i^* has priority. Assume there is u such that $f_{i^*}(u_{-i^*}) \neq x^{S^*}_{i^*}$, where $f_{i^*}(u_{-i^*})$ is the price of a unit of good that the mechanism makes to agent i^* when the utilities of the other agents are u_{-i^*} (recall this function exists because the mechanism meets SP, VP and NNT). Let $u_{i^*} = \tilde{u}_{i^*}$, a utility bigger than all possible payments for agent i^*, in particular $u_{i^*} > x^{S^*}_{i^*}$. First notice that $j \in S(u)$, otherwise, by step 1 $S(u) = S(0, u_{-j})$ and $\varphi(u) = \varphi(0, u_{-j})$. Thus i^* is served at u at a price equal to $x^{S^*}_{i^*}$, which contradictions our assumptions. Hence $j \in S(u)$. By step 3, $f_{i^*}(u_{-i^*}) > x^{S^*}_{i^*}$.

Let $k \in S^* \setminus i^*$ and $\bar{u}_k > \max(u_k, x^{S^*}_k)$, then $f_{i^*}(\bar{u}_k, u_{-k,i^*}) \geq x^{S^*}_{i^*}$. Indeed, if $k \in S(u)$, then by step 1 $S(\bar{u}_k, u_{-k}) = S(u)$ and $f_{i^*}(\bar{u}_k, u_{-k,i^*}) = f_{i^*}(u_k, u_{-k,i^*}) > x^{S^*}_{i^*}$. On the other hand, if $k \notin S(u)$ and $k \notin S(\bar{u}_k, u_{-k})$, then by step 1 $S(\bar{u}_k, u_{-k}) = S(u)$ and $f_{i^*}(\bar{u}_k, u_{-k,i^*}) = f_{i^*}(u_k, u_{-k,i^*}) > x^{S^*}_{i^*}$. Finally, if $k \notin S(u)$ and $k \in S(\bar{u}_k, u_{-k})$, then by step 3 $f_{i^*}(\bar{u}_k, u_{-i^*,k}) \geq f_{i^*}(u_k, u_{-i^*,k}) > x^{S^*}_{i^*}$.

By repeatedly using the above argument to every agent in $S^* \setminus i^*$ we conclude that $f_{i^*}(u_{-S^*}, \bar{u}_{S^* \setminus i^*}) > x^{S^*}_{i^*}$ for some $(u_{i^*}, \bar{u}_{S^* \setminus i^*}) \geq x^{S^*}_{[S^*]}$. This contradicts the priority of coalition S^*.

Steps 4 and 5 showed that for any GSP and MIN mechanism there exists an agent whose payment is independent of the other agent's utilities. By induction, this clearly implies the mechanism is sequential.

Step 6. The mechanism is implemented by a feasible sequential tree.

Proof.

Given a sequential mechanism (definition 7) that meets GSP and MIN, we show by contradiction that this mechanism is feasible (as in definition 9).

Assume the sequential tree that implements this mechanism is not feasible. Let ζ and ζ' be two achievable[7] nodes that contain the same agent k such that $x^L_k > x^R_k$, and for every two nodes $\tilde{\zeta} \in L$ and $\bar{\zeta} \in R$ that contains the same agent i, one of the next conditions hold:

1. $\tilde{\zeta}$ is leftist in L, $\bar{\zeta}$ is rightist in R and $x^L_i \geq x^R_i$.
2. $\tilde{\zeta}$ is rightist in L and $\bar{\zeta}$ is leftist in R and $x^L_i < x^R_i$.

[7] That is, all rightist agents in their paths to the root of the tree have finite prices

3. $\tilde{\zeta}$ and $\bar{\zeta}$ are leftist in L and R.

4. $\tilde{\zeta}$ and $\bar{\zeta}$ are rightist in L and R.

Let i^* be the agent in the terminal node of $P_0(\zeta) \sqcap P_0(\zeta')$. Fix a utility profile u such that:

a. u_{i^*} equal the price of his node.

b. u_k such that $x_k^L > u_k > x_k^R$

c. $u_i = x_i^R$ if condition 1 holds.

d. $u_i = \frac{x_i^L + x_i^R}{2}$ if condition 2 holds.

e. $u_i = 0$ if condition 3 holds.

f. u_i such that $u_i > \max(x_i^L, x_i^R)$ if condition 4 holds.

g. If j is unique rightist agent in $(P_0(\zeta) \sqcup P_0(\zeta')) \setminus (L \sqcap R)$ then u_j is bigger than the price of its node.

h. If j is unique leftist agent in $(P_0(\zeta) \sqcup P_0(\zeta')) \setminus (L \sqcap R)$ then $u_j = 0$.

i. Any other agent has zero utility.

First notice the profile u realizes the path $P_0(\zeta)$.

If an agent is leftist in $P_0(\zeta)$ then either his utility equals to zero, or condition 1 is satisfied, or he is i^*. If his utility equals to zero, by MIN he is not served. If condition 1 is satisfied then $u_i = x_i^R \leq x_i^L$ so he is not served. If he is i^*, then his utility equal the price of his node, so he is not served.

On the other hand, if an agent is rightist in $P_0(\zeta)$ then his utility is bigger than the price of his node. To see this, if condition 2 is satisfied then by part d he is served. If condition 4 is satisfied, then by part f he is served. The remaining rightist agents are served by part g.

Let T be the common agents who meet condition 1 and $S = T \cup \{i^*, k\}$. We now check that when the true profile is u, coalition S can profitably misreport. First notice all agents in S are not being served at u, so they get zero net utility.

Let $\widetilde{u}_S$ be such that:

- $\widetilde{u}_i > u_i$ if $i \in T \cup \{i^*\}$.
- $\widetilde{u}_k = u_k$

Then at the profile $(\widetilde{u}_S, u_{-S})$ the path $P_0(\zeta')$ realizes. Indeed, an agent j whose node is in $(P_0(\zeta) \sqcup P_0(\zeta')) \setminus (L \sqcap R)$ is obviously served if rightist and not served of leftist. If i meets condition 2, then $u_i = \frac{x_i^R + x_i^L}{2} < x_i^R$, so he is not served. If i meets condition 3, then by e his utility equals zero, thus he is not served. If i meets condition 4, then by f his utility is bigger than x_i^R, thus he is served. Also, i^* is rightist and he is served at a price equal to his true valuation u_i, thus his net utility is zero. If $i \in T$, that is $i \in L \sqcap R$ is rightist in R, then he is being served at a price equal to his valuation because $\widetilde{u}_i > u_i = x_i^R$, thus his net utility is zero. Finally, agent k is being served at a price x_k^R, $u_k > x_k^R$. Hence his net utility increases by misreporting.

Proof of Theorem 1.

Cross-monotonic mechanisms meet MAX and GSP.

Cross-monotonic mechanisms clearly meet MAX.

We prove by contradiction that these mechanisms meet GSP. Consider the cross-monotonic mechanism generated by the cross monotonic set of cost shares $\{x^S \mid S \subseteq N\}$.

Consider the offer function $f_i(u_{-i})$, the price agent i should pay to get a unit of good when the utilities of the remaining agents are u_{-i}. That is, $f_i(u_{-i}) = x_i^{S^*}$ where S^* is the maximal feasible coalition at (∞, u_{-i}). By cross-monotonicity of the cost shares and the definition of f_i, the offer function does not increase when u_{-i} increases. That is, if $v_{-i} \geq u_{-i}$ then $f_i(v_{-i}) \leq f_i(u_{-i})$.

Furthermore, the set of offer functions $f_1, \ldots, f_n$ generate precisely the mechanism (S, φ). That is, $S(u) = S^*$ if and only if $u_i \geq f_i(u_{-i})$ for all $i \in S^*$ and $u_j < f_j(u_{-j})$ for all $j \notin S^*$. Indeed, to prove the only if part, assume $S(u) = S^*$. Let $i \in S^*$, since S^* is the maximal feasible coalition at u, then by cross monotonicity S^* is the maximal feasible coalition at (∞, u_{-i}), thus $f_i(u_{-i}) = x_i^{S^*} \leq u_i$. Let $j \notin S^*$ and T the maximal feasible coalition at (∞, u_{-j}). To get a contradiction, assume that $u_j \geq x_j^T = f_j(u_{-j})$. Then T is feasible at u, thus $T \subseteq S^*$. Furthermore, since $j \in T$, then $j \in S^*$, which is a contradiction. We now prove the if part. Let T be the maximal feasible coalition at (∞, u_{-i}) and assume $u_i \geq f_i(u_{-i}) = x_i^T$. Then $i \in T$ and T is feasible at u. Thus $T \subseteq S(u)$, hence $i \in S(u)$. On the other hand, let $\widetilde{T}$ be the maximal feasible coalition at (∞, u_{-j}) and assume $u_j < f_j(u_{-j}) = x_j^{\widetilde{T}}$. To get a contradiction, assume that $j \in S(u)$. Then by monotonicity, $S(u) \subseteq \widetilde{T}$. Thus $u_j \geq x_j^{S(u)} \geq x_j^{\widetilde{T}}$, which contradicts our initial assumptions.

Assume coalition $\widetilde{S}$ profitably misreports $\widetilde{u}_{\widetilde{S}}$ when the true profile is u. Let $\bar{v}_{\widetilde{S}} = \max(u_{\widetilde{S}}, \widetilde{u}_{\widetilde{S}})$, where max is taken coordinate by coordinate. Because the offer function does not increase, coalition $\widetilde{S}$ also profits from misreporting $\bar{v}_{\widetilde{S}}$ when the true profile is u.

By monotonicity of the offer function, $S(u) \subseteq S(\bar{v}_{\widetilde{S}}, u_{-\widetilde{S}})$. Since coalition $\widetilde{S}$ profits from misreporting, then $S(u) \subsetneq S(\bar{v}_{\widetilde{S}}, u_{-\widetilde{S}})$. Since $S(\bar{v}_{\widetilde{S}}, u_{-\widetilde{S}})$ is not feasible at u, then there is an agent i such that $u_i < x_i^{S(\bar{v}_{\widetilde{S}}, u_{-\widetilde{S}})}$. Clearly $i \notin \widetilde{S}$ would contradict voluntary participation. Thus $i \in \widetilde{S}$, hence i is worse off by misreporting, which is a contradiction.

Any mechanism that is MAX and GSP is cross-monotonic.

Let (S, φ) be a mechanism that meets these properties. Recall that $f_i(u_{-i})$ is the price agent i should pay to get a unit of good when the utilities of the remaining agents are u_{-i}. Also, $NU_i(u)$ denotes the net utility of agent i at the profile u.

The proof of this part is divided in four steps. Steps 1 and 2 are very similar to step 1 and 2 in the proof of Theorem 2. However, step 1 involves more details because MAX does not imply that an agent is served if and only if his net utility is positive.

Step 0.[Monotonicity] $f_j(\widetilde{u}_i, u_{-ij}) \leq f_j(u_i, u_{-ij})$ for all $\widetilde{u}_i > u_i$.

Proof.

We prove this by contradiction. Suppose $f_j(\widetilde{u}_i, u_{-ij}) > f_j(u_i, u_{-ij})$. Let $\bar{v}_j$ be such that $f_j(\widetilde{u}_i, u_{-ij}) > \bar{v}_j > f_j(u_i, u_{-ij})$.

Case 1. $f_i(\bar{v}_j, u_{-ij}) > \widetilde{u}_i$.

By SP, agent i is not served at the profiles $(\widetilde{u}_i, \bar{v}_j, u_{-ij})$ and $(u_i, \bar{v}_j, u_{-ij})$ because $f_i(\bar{v}_j, u_{-ij}) > \widetilde{u}_i > u_i$. Hence when the true utility profile is $(\widetilde{u}_i, \bar{v}_j, u_{-ij})$, agent i can help j by misreporting u_i. This contradicts GSP.

Case 2. $f_i(\bar{v}_j, u_{-ij}) \leq u_i$.

By SP and MAX, agent i is served at the profiles $(\widetilde{u}_i, \bar{v}_j, u_{-ij})$ and $(u_i, \bar{v}_j, u_{-ij})$ because $f_i(\bar{v}_j, u_{-ij}) \leq u_i < \widetilde{u}_i$. Hence, similarly to case 1, when the true utility profile is $(\widetilde{u}_i, \bar{v}_j, u_{-ij})$, agent i can help j by misreporting u_i. This also contradicts GSP.

Case 3. $u_i < f_i(\bar{v}_j, u_{-ij}) \leq \widetilde{u}_i$.

Let $\widehat{u}_i = f_i(\bar{v}_j, u_{-ij})$. By SP and MAX, agent i is being served at price $\widehat{u}_i$ at the profiles $(\widetilde{u}_i, \bar{v}_j, u_{-ij})$ and $(\widehat{u}_i, \bar{v}_j, u_{-ij})$. Thus, by GSP $f_j(\widehat{u}_i, u_{-ij}) \geq \bar{v}_j$. To see this, assume $f_j(\widehat{u}_i, u_{-ij}) < \bar{v}_j$. Then, when the true profile is $(\widetilde{u}_i, \bar{v}_j, u_{-ij})$, agent i helps j by misreporting $\widehat{u}_i$. This contradicts GSP.

Hence, at the true profile $(\widehat{u}_i, \bar{v}_j, u_{-ij})$, agents i and j get zero net utility because $f_j(\widehat{u}_i, u_{-ij}) \geq \bar{v}_j$ and $\widehat{u}_i = f_i(\bar{v}_j, u_{-ij})$. Thus agent i helps j by reporting u_i : Agent i is not served at the misreport because $u_i < f_i(\bar{v}_j, u_{-ij})$, however agent j is better off because $\bar{v}_j > f_j(u_i, u_{-ij})$. This contradicts GSP.

Step 1. If $S(u) = S^*$ and $\varphi(u) = \varphi^*$ then for all $\widetilde{u}$ such that $\widetilde{u}_{[S^*]} \geq \varphi_{[S^*]}$ and $\widetilde{u}_{[N \setminus S^*]} \leq u_{[N \setminus S^*]}$, $S(\widetilde{u}) = S^*$ and $\varphi(\widetilde{u}) = \varphi^*$.

Proof.

We prove step 1 in steps 1.1 and 1.2.

Step 1.1. Let $i \in S^*$ and $\widetilde{u}_i \geq f_i(u_{-i}) = \varphi_i^*$. We will prove that $S(\widetilde{u}_i, u_{-i}) = S^*$ and $\varphi(\widetilde{u}_i, u_{-i}) = \varphi^*$.

First, notice that by SP and MAX, $i \in S(\widetilde{u}_i, u_{-i})$ and $\varphi_i(\widetilde{u}_i, u_{-i}) = \varphi_i^*$.

Second, notice $NU_j(\widetilde{u}_i, u_{-i}) = NU_j(u)$ for all $j \neq i$. To see this, if $NU_j(\widetilde{u}_i, u_{-i}) > NU_j(u)$, then when the true profile is u, agent i helps j by reporting $\widetilde{u}_i$. This contradicts GSP. Similarly, if $NU_j(\widetilde{u}_i, u_{-i}) < NU_j(u)$ then agent i helps j by misreporting $\widetilde{u}_i$ when the true utility profile is u.

Third, notice if $j \in S^* \setminus i$ and $NU_j(\widetilde{u}_i, u_{-i}) = NU_j(u) > 0$ then $j \in S(\widetilde{u}_i, u_{-i})$ and $\varphi_j(\widetilde{u}_i, u_{-i}) = \varphi_j^*$.

Finally, to get a contradiction, assume $S(\widetilde{u}_i, u_{-i}) \neq S^*$. Then, there is an agent j such that $NU_j(\widetilde{u}_i, u_{-i}) = NU_j(u) = 0$ and either (A.1.) $j \in S^*$ but $j \notin S(\widetilde{u}_i, u_{-i})$ or (A.2.) $j \notin S^*$ but $j \in S(\widetilde{u}_i, u_{-i})$. We show next that these situations cannot occur.

Case A.1. Assume $NU_j(\widetilde{u}_i, u_{-i}) = NU_j(u) = 0$, $j \in S^*$ but $j \notin S(\widetilde{u}_i, u_{-i})$.

Since $j \notin S(\widetilde{u}_i, u_{-i})$, by SP and MAX $f_j(\widetilde{u}_i, u_{-ij}) > u_j = f_j(u_{-j})$. Thus, by step 0, $u_i > \widetilde{u}_i \geq \varphi_i^*$.

Let $\bar{v}_j$ be such that $\bar{v}_j > u_j$. By step 0,

$$f_i(\bar{v}_j, u_{-ij}) \leq f_i(u_j, u_{-ij}) = \varphi_i^* \leq \widetilde{u}_i < u_i.$$

Therefore, when the true profile is $(\widetilde{u}_i, \bar{v}_j, u_{-ij})$, agent i can help j by misreporting u_i : Agent i is served in both profiles at price $f_i(\bar{v}_j, u_{-ij})$, however agent j is offered a unit at the cheaper price $f_j(u_{-j})$ when i misreports. This contradicts GSP.

Case A.2. Assume $NU_j(\widetilde{u}_i, u_{-i}) = NU_j(u) = 0$, $j \notin S^*$ but $j \in S(\widetilde{u}_i, u_{-i})$.

By SP and MAX, $f_j(\widetilde{u}_i, u_{-ij}) = u_j > f_j(u_{-j})$. So, we are in exactly in the previous case but switching the role of $\widetilde{u}_i$ and u_i. Thus, this case cannot occur.

By repeatedly using step 1.1 to every agent in S^* we have that $S(\widetilde{u}_{S^*}, u_{-S^*}) = S^*$ and $\varphi(\widetilde{u}_{S^*}, u_{-S^*}) = \varphi^*$.

Step 1.2. Let $j \notin S^*$ be such that $\tilde{u}_j < u_j$. Then $S(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = S^*$ and $\varphi(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = \varphi^*$.

Since $\widetilde{u}_j < u_j < f_j(\widetilde{u}_{S^*}, u_{-S^* \cup j})$, then by SP $j \notin S(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^* \cup j})$. Similarly to step 1.1, by GSP $NU_k(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = NU_k(\widetilde{u}_{S^*}, u_{-S^*})$ for all $k \neq j$.

Assume $S(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) \neq S^*$. Clearly, if $NU_k(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = NU_k(\widetilde{u}_{S^*}, u_{-S^*}) > 0$ for some $k \neq j$, then $k \in S^*$, $k \in S(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j})$ and $\varphi_k(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = \varphi_k^*$.

Thus, there is k such that $NU_k(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^* \cup j}) = NU_k(\widetilde{u}_{S^*}, u_{-S^*}) = 0$ and either (B.1) $k \in S^*$ but $k \notin S(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^* \cup j})$; or (B.2) $k \notin S^*$ but $k \in S(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^* \cup j})$. We show next these cases cannot occur.

Case B.1. $NU_k(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^* \cup j}) = NU_k(\widetilde{u}_{S^*}, u_{-S^*}) = 0$, $k \in S^*$ and $k \notin S(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^* \cup j})$.

By SP and MAX,

$$f_k(\widetilde{u}_j, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}) > \widetilde{u}_k = f_k(u_j, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}). \tag{3}$$

Let $\bar{v}_k$ be such that $\bar{v}_k > \widetilde{u}_k$. By monotonicity

$$f_j(\bar{v}_k, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}) \leq f_j(\widetilde{u}_k, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}).$$

First we assume that

$$f_j(\widetilde{u}_k, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}) \geq f_j(\bar{v}_k, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}) > u_j > \widetilde{u}_j.$$

Then, when the true profile is $(\bar{v}_k, \widetilde{u}_j, \widetilde{u}_{S^*\setminus k}, u_{-S^*\cup j})$, agent j can help agent k by misreporting u_j : Agent j does not get a unit in either profile, however by equation 3 agent k gets a unit at the cheaper price $f_k(u_j, \widetilde{u}_{S^*\setminus k}, u_{-S^*\cup j})$ when j misreports. This contradicts GSP.

On the other hand, we now assume

$$f_j(\bar{v}_k, \widetilde{u}_{S^*\setminus k}, u_{-S^*\cup j}) \leq u_j < f_j(\widetilde{u}_k, \widetilde{u}_{S^*\setminus k}, u_{-S^*\cup j}). \tag{4}$$

Let $\bar{v}_j$ be such that $\bar{v}_j > u_j$. By step 1,

$$f_k(\bar{v}_j, \widetilde{u}_{S^*\setminus k}, u_{-S^*\cup j}) \leq f_k(u_j, \widetilde{u}_{S^*\setminus k}, u_{-S^*\cup j}) = \widetilde{u}_k < \bar{v}_k. \tag{5}$$

Thus when true profile is $(\widetilde{u}_k, \bar{v}_j, \widetilde{u}_{S^*\setminus k}, u_{-S^*\cup j})$, agent k helps j by misreporting $\bar{v}_k$: By equation 5, agent k is served at a price $f_k(\bar{v}_j, \widetilde{u}_{S^*\setminus k}, u_{-S^*\cup j})$ in either profile; however by equation 4 agent j is served at the cheaper price $f_j(\bar{v}_k, \widetilde{u}_{S^*\setminus k}, u_{-S^*\cup j})$ when k misreports. This contradicts GSP.

Hence, if $k \in S^*$ then $k \in S(\widetilde{u}_{S^*\cup j}, u_{-S^*\cup j})$ and $\varphi_k(\widetilde{u}_{S^*\cup j}, u_{-S^*\cup j}) = \varphi_k(\widetilde{u}_{S^*}, u_{-S^*})$.

Case B.2. $NU_k(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^*\cup j}) = NU_k(\widetilde{u}_{S^*}, u_{-S^*}) = 0$, $k \notin S^*$ and $k \in S(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^*\cup j})$.

By SP and MAX,

$$f_k(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^*\cup j,k}) = u_k < f_k(u_j, \widetilde{u}_{S^*}, u_{-S^*\cup j,k}).$$

However, this contradicts monotonicity because $\widetilde{u}_j < u_j$.

By repeating step 1.2 to every agent in $N \setminus S^*$, $S(\widetilde{u}) = S^*$ and $\varphi(\widetilde{u}) = \varphi^*$.

Step 2. If $S(u) = S(\widetilde{u})$ then $\varphi(u) = \varphi(\widetilde{u})$.

Proof.

Let $S^* = S(u) = S(\widetilde{u})$, $\bar{v}_{[S]} = \max(\widetilde{u}_{[S]}, u_{[S]})$ and $\bar{v}_{[N\setminus S]} = \min(\widetilde{u}_{[N\setminus S]}, u_{[N\setminus S]})$ (where max and min are taken coordinate by coordinate).

By step 1, comparing $\bar{v}$ and u, $S(\bar{v}) = S^*$ and $\varphi(\bar{v}) = \varphi(u)$. Similarly, comparing $\bar{v}$ and $\widetilde{u}$, $\varphi(\bar{v}) = \varphi(\widetilde{u})$. Hence $\varphi(u) = \varphi(\widetilde{u})$.

Step 3.

In this final step we prove the theorem by induction on the number of agents. The base of induction is the case $n = 1$. The mechanisms are easy to construct. Given $x \in [0, \infty]$, if $u_1 \geq x$ then $(S, \varphi)(u_1) = (1, x)$. On the other hand, if $u_1 < x$ then $(S, \varphi)(u_1) = (\emptyset, 0)$. These mechanisms are clearly cross-monotonic.

For the induction hypothesis, assume that any GSP and MAX mechanism for k agents, $k < n$, is cross-monotonic. We prove this for the $n-$agent case. Let (S, φ) be a GSP and MAX mechanism defined for the agents $N = \{1, \ldots, n\}$.

Case 1. Assume there is a utility profile u^* such that $S(u^*) = N$.

Let $x^N = \varphi(u^*)$. By step 1, for all $\widetilde{u} \geq x^N$, $S(\widetilde{u}) = N$ and $\varphi(\widetilde{u}) = x^N$.

For every agent $j \in N$, consider the set of utility profiles such that $u_j = 0$, that is

$$U^j = \{u \in \mathbb{R}^N_+ \mid u_j = 0\}.$$

By induction, there is a cross-monotonic mechanism (S^j, φ^j) for $N \setminus j$ agents defined on U^j. Thus, $S^j(v) = S(v, 0) \cap (N \setminus j)$ and $\varphi^j(v) = \varphi(v, 0)_{[N\setminus j]}$ for all $v \in \mathbb{R}_+^{N\setminus j}$. Let $\tilde{\rho}^j$ be the cross-monotonic set of cost shares that defines this mechanisms.

Let S^*, $S^* \subseteq N \setminus j$, be the maximal coalition that is served by (S^j, φ^j) under any utility profile of $N \setminus j$ agents (by cross-monotonicity this coalition exists).

For every $T \subseteq N \setminus j$ consider the vector of cost shares y^T as follows:

$y^T = x^T$ if $T \subseteq S^*$ and $x^T \in \tilde{\rho}^j$.

$y_i^T = \infty$ if $i \in T \setminus S^*$ and $T \nsubseteq S^*$.

$y_i^T = x_i^{S^*}$ if $i \in S^* \cap T$ and $T \nsubseteq S^*$; where $x^{S^*} \in \tilde{\rho}^j$.

Let $\bar{\rho}^j$ be the set of these cost shares.

Clearly, if $S^* = N \setminus j$, then $\bar{\rho}^j = \tilde{\rho}^j$. If $S^* \neq N \setminus j$, this may not be true, however it generates the same cross-monotonic mechanism (S^j, φ^j) (see below).

First, we show that $\bar{\rho}^j$ is a cross-monotonic set of cost shares. Indeed, consider $L \subset M$ and $k \in L$. If $k \notin S^*$ then $y_k^M = y_k^L = \infty$. Now assume $k \in S^*$. If $L \subset M \subseteq S^*$ then $y_k^M = x_k^M \leq x_k^L = y_k^L$ where $x^M, x^L \in \tilde{\rho}^j$. If $M \nsubseteq S^*$ then $y_k^M = x_k^{S^*} \leq y_k^L$.

Next, we show that $\bar{\rho}^j$ generates the mechanism (S^j, φ^j). Indeed, let v be a utility profile for $N \setminus j$ agents. Then $S^j(v) \subseteq S^*$ by definition of S^*. Clearly, $y^T \in \bar{\rho}^j$ is not feasible at v for any $T \nsubseteq S^*$ because $y_k^T = \infty$ for $k \in T \setminus S^*$. Moreover, $\bar{\rho}^j$ coincides with $\widetilde{\rho}^j$ for any subset in 2^{S^*}, and $S^j(v)$ is the maximal feasible coalition in $\widetilde{\rho}^j$ for the utility profile v. Hence, $S^j(v)$ is the maximal feasible coalition in $\bar{\rho}^j$ for the utility profile v.

Let ρ^j be the embedding of $\bar{\rho}^j$ into U^j by adding a $j-$th coordinate equal to zero.

We define the cost share of coalition T, $T \subsetneq N$ as

$$\tilde{x}^T = \max_{\{x^T \in \rho^j | j \in N \setminus T\}} x^T,$$

where max is taken coordinate by coordinate. The cost share of coalition N is simply x^N. Let ρ^* be the set that contains these cost shares.

We first check that if $S(u) = \bar{S} \neq N$ for some u, then $\varphi(u) = \tilde{x}^{\bar{S}}$. Indeed, by step 1 for any $j \in N \setminus \bar{S}$, $\varphi(u) = \varphi(0, u_{-j}) = x^{\bar{S}}$ where $x^{\bar{S}} \in \rho^j$. Thus for any $i, j \in N \setminus \bar{S}$, $x^{\bar{S}} = \varphi(u) = y^{\bar{S}}$ where $x^{\bar{S}} \in \rho^j$ and $y^{\bar{S}} \in \rho^i$. Furthermore, $\tilde{x}^{\bar{S}} = x^{\bar{S}}$ where $x^{\bar{S}} \in \rho^j$. Hence $\varphi(u) = \tilde{x}^{\bar{S}}$.

We now show ρ^* is a cross-monotonic set of cost shares. Let $S \subset T \subsetneq N$ and $k \in S$. First notice that $x_k^S \geq x_k^T$ holds for any $i \in N \setminus T$, $x^S, x^T \in \rho^i$ by cross-monotonicity on ρ^i. By taking max on both sides of the inequality and maximizing over all agent in $N \setminus T$, $\tilde{x}_k^S \geq \tilde{x}_k^T$ for all $k \in S$.

We now check that $x_i^N \leq \tilde{x}_i^{N\setminus j}$ for all $j \in N$, $i \in N \setminus j$ and $\tilde{x}_i^{N\setminus j} \in \rho^*$. Let S^* be the maximal coalition that is served at (S^j, φ^j) under any u_{-j}. By the choice of $\bar{\rho}^j$, if $i \in N \setminus (j \cup S^*)$, then $\tilde{x}_i^{N\setminus j} = \infty > x_i^N$. On the other hand, if $i \in S^*$, then $\tilde{x}_i^{N\setminus j} = \tilde{x}_i^{S^*}$. To prove the above claim by contradiction, assume there is $i \in S^*$ such that $\tilde{x}_i^{S^*} < x_i^N$. Let $u \in U^j$ such that $\varphi(u) = \tilde{x}^{S^*}$, and $\widetilde{u} = (x_i^N, \max(x_{-i}^N, u_{-i}))$. Since $\widetilde{u} \geq x^N$, then $S(\widetilde{u}) = N$ holds by step 1. Thus $x_i^N = f_i(\widetilde{u}_{-i})$. On the other hand, by step 0, $x_i^N = f_i(\widetilde{u}_{-i}) \leq f_i(u_{-i}) = \varphi_i(u) = \tilde{x}_i^{S^*}$. This is a contradiction.

In particular, cross-monotonicity implies that agent i cannot be served if his utility is smaller than x_i^N. Hence, the mechanism (S, φ) satisfies:

- If $u \geq x^N$ then $S(u) = N$ and $\varphi(u) = x^N$.
- If for some i, $u_i < x_i^N$ then $i \notin S(u)$. Thus, by step 1 $(S, \varphi)(u) = (S, \varphi)(0, u_{-i}) = (S^i(u_{[N\setminus i]}), \tilde{x}^{S^i(u_{[N\setminus i]})})$.

Finally, we check (S, φ) is the cross-monotonic mechanism generated by ρ^*. If $u \geq x^N$, then $S(u) = N$ and obviously N is the maximal feasible coalition in ρ^*. Assume u is such that $u_i < x_i^N$ for some agent i. Let $S^* = S(u)$. By cross-monotonicity, no coalition that contains agent i is feasible for u. On the other hand, since (S^i, φ^i) is cross-monotonic, then $S^* = S^i(u_{[N\setminus i]})$ is the maximal feasible coalition in ρ^i and payments are $x^{S^*} \in \rho^i$. Hence S^* is the maximal feasible coalition in ρ^* because $x^{S^*} = \widetilde{x}^{S^*} \in \rho^*$, and $y^T \geq x^T$ for all $x^T \in \rho^i$ and $y^T \in \rho^*$.

Case 2. Assume there is no u^* such that $S(u^*) = N$.

We will show there is $j \in N$ such that $j \notin S(\widetilde{u})$ for all $\widetilde{u}$. We prove this by contradiction. Assume for any j there is u^j such that $j \in S(u^j)$. Let $\bar{v} = \max(u^1, \ldots, u^n)$ where max is taken coordinate by coordinate. By step 0, at $\bar{v}$ every agent j is offered a unit of good at price not bigger than u_j^j, thus $j \in S(\bar{v})$ for all $j \in N$. This is a contradiction.

Since there is an agent who is not serviced at any profile, say agent j^*, then by step 1 $(S, \varphi)(u) = (S, \varphi)(u_{-j^*}, 0)$ for all u. Hence by induction the mechanism is cross-monotonic.

Proof of Corollary 1.

If the mechanism meets GSP and MIN (MAX), then for every agent i his payment does not decrease (increase) when coalition increases.

Therefore, in order to have a common point at every coalition, it must be that $x_i^N = x_i^i$ for all i. Hence, the cost share of agent i is fixed.

Proof of Proposition 1.

By ETE, $S(x \cdot 1_N)$ serve N or $\emptyset$.

First notice that $S(x \cdot 1_N) = \emptyset$ for all $x > 0$ implies the mechanism is welfare equivalent to the trivial mechanism where no agent is served at any profile. To see this, assume $NU_k(u) > 0$ for some agent k at some utility profile u. Let $u^{max} = \max(u_1, \ldots, u_n) \cdot 1_N$. Then, $S(u^{max}) = \emptyset$. Thus, when the true profile is u^{max}, agents in N help k by misreporting u : Agent k is strictly better off because he is getting a unit at a price below u_k, while any other agent j may or may not get a unit at a price less or equal to u_j. This contradicts GSP.

On the other hand, assume $S(x \cdot 1_N) = N$ for some $x > 0$ and $\varphi_i(x \cdot 1_N) = \varphi^*$ for all i. Notice we can assume w.l.g. that $x > \varphi^*$. Indeed, assume $x = \varphi^*$. Consider $\widetilde{x}$ such that $\widetilde{x} > x$. By GSP and ESP, $S(\widetilde{x} \cdot 1_N) = N$ and $\varphi_i(\widetilde{x} \cdot 1_N) = \varphi^*$. Indeed, if $\varphi_i(\widetilde{x} \cdot 1_N) < \varphi^*$ then agents in N misreport $\widetilde{x} \cdot 1_N$ when the true profile is $x \cdot 1_N$, this contradicts GSP. Similarly, if $\varphi_i(\widetilde{x} \cdot 1_N) > \varphi^*$ then agents in N misreport $x \cdot 1_N$ when the true profile is $\widetilde{x} \cdot 1_N$, this also contradicts GSP.

By GSP, for all $u >> \varphi^* \cdot 1_N$, $S(u) = N$ and $\varphi_i(u) = \varphi^*$ for all i. To see this, let $v = x \cdot 1_N$. By SP, $1 \in S(v_{-1}, u_1)$ and $\varphi_1(v_{-1}, u_1) = \varphi^*$. Thus, by GSP, $S(v_{-1}, u_1) = N$ and $\varphi_i(v_{-1}, u_1) = \varphi^*$ for all i. Changing the profiles one agent at a time $S(u) = N$ and $\varphi_i(u) = \varphi^*$ for all i.

We now prove the proposition by induction in the number of agents. This is obvious when there is only one agent. Assume this is true for any number of agents less than n. We prove it for n agents.

Consider U^j the set of utility profiles where agent j has utility zero. By induction, the restriction of the mechanism to U^j is welfare equivalent to a ESP cross-monotonic mechanism of $N \setminus j$ agents. Let x^S be the payment of coalition S on U^j and $x^N = \varphi^* \cdot 1_N$. First notice $x_i^S \geq \varphi^*$ for all $S \subseteq N \setminus j$.

To see this, by cross-monotonicity we just need to check that $x_i^{N\setminus j} \geq \varphi^*$. Assume $x_i^{N\setminus j} < \varphi^*$.

Let $\epsilon > 0$ be such that $\varphi^* - \epsilon > x_i^{N\setminus j}$ and $u = (x^N + \epsilon 1_N)$. Then $S(u) = N$ and $\varphi(u) = x^N$. By SP, $i \notin S(\varphi^* - \epsilon, u_{-j})$. Thus by GSP $S(\varphi^* - \epsilon, u_{-j}) = N \setminus j$ and $\varphi(\varphi^* - \epsilon, u_{-j}) = x^{N\setminus j}$.

Since $u_k > x_k^{N\setminus j}$ for all $k \in N\setminus j$, then by GSP $S((\varphi^* - \epsilon)\cdot 1_N) = N\setminus j$ and $\varphi((\varphi^* - \epsilon)\cdot 1_N) = x^{N\setminus j}$. This contradicts ETE. Hence $x_i^S \geq \varphi^*$ for all $j \in N$, $S \subseteq N \setminus j$.

Thus the mechanism is clear. If $u \geq x^N$ then the mechanism is welfare equivalent to $S(u) = N$ and $\varphi(u) = x^N$. If $u_i < x^N$ then $i \notin S(u)$. Hence by GSP the mechanism is welfare equivalent to $S(u) = S(0, u_{-i})$ and $\varphi(u) = \varphi(0, u_{-i})$. Since the restriction to U^i is welfare equivalent to a cross-monotonic mechanism with cost-shares not smaller than x^N, then $S(u)$ is the biggest feasible coalition —notice this argument is very similar to the one given at the end of step 3 (case 1) on the proof of theorem 1, page 41.

Proof of Proposition 2.

First notice if agent i is not served at any profile, then by GSP $NU_k(u) = NU_k(\widetilde{u}_i, u_{-i})$ for all $k \neq i$, u and $\widetilde{u}_i$. Thus we can remove this agent from the mechanism without any welfare consequence.

We prove the proposition by contradiction. Assume without loss of generality that every agent in N is served in at least one profile and that there is no agent who has priority. Then for every agent i there exist profiles u^i and $\widetilde{u}^i$ such that $i \in S(u^i)$, $i \notin S(\widetilde{u}^i)$, $u_i^i, \widetilde{u}_i^i > \bar{x}_i$ where $\bar{x}_i = \varphi_i(u^i)$.

Let $\bar{v} >> \max_{k\in N}(u^k, \widetilde{u}^k)$ where max is taken coordinate by coordinate over all utility profiles u^k, $\widetilde{u}^k$.

By GSP, $S(\bar{v}) \neq \emptyset$, otherwise coalition N misreport u^1 when true profile is $\bar{v}$. Assume $S(\bar{v}) = i^*$. By GSP, $\varphi_{i^*}(\bar{v}) = \bar{x}_i$, otherwise coalition N misreport u^1 when true profile is $\bar{v}$ or vice-versa.

By SP, $k \notin S(\widetilde{u}_k^{i^*}, \bar{v}_{-k})$ for all $k \neq i^*$. Thus by GSP, $S(\widetilde{u}_k^{i^*}, \bar{v}_{-k}) = i^*$ and $\varphi_{i^*}(\widetilde{u}_k^{i^*}, \bar{v}_{-k}) = \bar{x}_{i^*}$. Changing the profiles one agent at a time, $S(\widetilde{u}_{-i^*}^{i^*}, \bar{v}_{i^*}) = i^*$ and $\varphi_{i^*}(\widetilde{u}_{-i^*}^{i^*}, \bar{v}_{i^*}) = \bar{x}_{i^*}$.

Since $\widetilde{u}_{i^*} > \bar{x}_{i^*}$ then by strategyproof $S(\widetilde{u}^{i^*}) = i^*$. This is a contradiction.

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