# Recursive equilibrium in stochastic OLG economies

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**Abstract**

We prove generic existence of recursive equilibrium for overlapping generations economies with uncertainty. Generic here means in a residual set of utilities and endowments. The result holds provided there is sufficient intragenerational household heterogeneity.

## 1 Introduction

The overlapping-generations (OLG) model, introduced first by Samuelson (1958), is one of the two major workhorses for macroeconomic and financial modeling of open-ended dynamic economies. Following developments in the study of two-period economies, the OLG model has been subsequently extended to cover stochastic economies with production and possibly incomplete financial markets. In such instances, the general notion of competitive equilibrium à la Arrow-Debreu has proven to be not very useful for applied, quantitative work, even in stationary Markovian environments. This is due, among other things, to the large dimensionality of the allocation and price sequences when histories of arbitrary length are allowed, which strains the ability of approximating solutions with present-day computers. It also strains the notion of rational expectations equilibrium because of the complexity of the forecasts involved in some of these equilibria.

An advance in the direction of simplifying computations was provided by Duffie et al. (1994), who give a general theorem for the existence of stationary Markov equilibria for OLG economies, with associated ergodic measure.[1] While these equilibria provide the analogue in stochastic economies of steady states or low-order cycles in deterministic economies, they are still too complicated to allow for computational work. Hence, the applied literature has focused on a notion of simple time-homogeneous Markov equilibrium, also known as recursive equilibrium.[2]

In a recursive equilibrium the state space is reduced to the exogenous shocks and the initial distribution of wealth for the agents –asset portfolios from the previous period, and capital and storage levels if production is considered. A recursive equilibrium can be thought of as a time-homogeneous Markov equilibrium that is based on a minimal state space.

However, no existence theorem is available for such recursive equilibria. In fact, Kubler and Polemarchakis (2004) provide two examples of nonexistence of recursive equilibrium in OLG exchange economies. The idea that recursive equilibria may not exist is based on the observation that when there are multiple temporary equilibria the continuation of an equilibrium may depend on past economic variables other than the wealth distribution. That is, the current wealth distribution may not be enough to summarize the information contained in past equilibrium prices and marginal utilities.

While this phenomenon may occur, we prove that it is nongeneric — under some qualifying condition. The argument follows three fundamental observations.

The first is that it is always possible to find competitive equilibria which are time-homogeneous Markov over a simple state space. This is the state space made of the current exogenous state, current wealth distribution, and of commodity prices and marginal utilities of income for all generations except for the first and the last ('newly born' and 'eldest').

The second observation is that in such equilibria prices and multipliers are typically a function of the current state and of the current wealth distribution, provided that there is sufficient dispersion of individual characteristics within each generation. We call these equilibria nonconfounding. The

[1] See e.g. also the earlier work by Spear (1985) and Spear and Srivastava (1986), and by Cass et al. (1992) and Gottardi (1996).

[2] Examples by now abund; see, e.g., Rios Rull (1996), Constantinides, Donaldson and Mehra (2002), Geanakoplos, Magill and Quinzii (2004), Storesletten, Telmer and Yaron (2004a, 2004b).

trick we use here is to abstract from the equilibrium equations and instead, using stationarity, focus on the optimization problem of an arbitrary individual born at any date-event. This trick allows one to bypass the infinite dimensional nature of the equilibrium set, and the fact that with overlapping generations there is an infinite number of individuals and an infinite number of market clearing equations, rendering direct genericity analysis quite problematic. A simple Markov equilibrium fails to be nonconfounding essentially when, at different prices for goods in a given state, an individual of a given type spends the same amount of money, and this for all types of individuals. However, typically this cannot be the case, as with *enough heterogeneity* in preferences and endowments there is almost always going to be an individual who will spend differently at different prices, no matter what prices are at all the remaining states. Since we need to check this for all admissible, and not just equilibrium, prices, the degree of heterogeneity we use must be large —it should be noted in passing that this is not at odds with the notion of price-taking behavior which is assumed in competitive models such as ours. The third and final step is to show that nonconfounding simple time-homogeneous Markov equilibria are indeed recursive equilibria.[3]

The notion of genericity we will use will rely on utility perturbations, and therefore will only be topological. In fact, due to the infinite dimension of the equilibrium set, we will not be able to establish local uniqueness of competitive equilibria, whether or not time-homogeneous. Without this prerequisite, the argument essentially showing some one-to-oneness property of prices will have to be made without knowing whether such prices are or not 'critical'. Therefore, we will resort to an argument reminiscent of Mas-Colell and Nachbar (1991), and we will show the existence of recursive equilibria for a *residual* or *nonmeager* subset of parameters, i.e., a set of stationary utilities and endowments which is dense and is the countable intersection of open and dense sets. This is a well-established notion of genericity for dynamic systems.

Our class of OLG economies has multiple goods, generations and types within each generation; it allows for complete or incomplete markets, short-

[3]In a previous paper (Citanna and Siconolfi 2006), we show that the examples constructed by Kubler and Polemarchakis (2004) are nonrobust in a stronger sense. In fact, we can show existence of recursive equilibria for an open and dense class of exchange economies where individuals live for two periods –'young' and 'old' age–, restrictions apply to their preferences when young, but we allow for any degree of heterogeneity. Our main idea there is to prove existence of recursive equilibria via short-memory equilibria.

or long-lived assets, in zero or positive net supply, but no outside money, as assets are real —namely, numéraire— and with nonzero payoffs. As it stands our result does not provide a positive answer to existence of recursive equilibrium in all economies that have already been used in applied work, where often preferences are CRRA and at times there is no intragenerational heterogeneity of individual characteristics. However, our result still suggests that the notion of recursive equilibrium is computationally useful as well as coherent as an exact concept, adding robustness to its interpretation and quantitative use.[4] The heterogeneity level we use in our theorem allows for the state space dimension in the recursive equilibrium to be drastically reduced. If we denote by $G+1$ the number of periods an individual lives, i.e., the number of generations present in an economy, this heterogeneity is proportional to the dimension of a tree of length $G+1$, and this is the minimal length of a truncated economy. More importantly, the idea of the generic existence of recursive equilibrium seems to promise useful in exploring existence also in economies with less heterogeneity, provided more powerful perturbation arguments can be constructed.

# 2 The model

We consider standard stochastic OLG economies where time and uncertainty are represented by date-events, and a tree structure $(\widetilde{S}, \prec)$. Here $\widetilde{S}$ is a set and $\prec$ is a precedence relation: $\prec$ is irreflexive and transitive, and partially orders the set $S$. If $s, s'$ are two elements of a chain in $\widetilde{S}$, with $s' \prec s$, and there is no other $s'' \in \widetilde{S}$ with $s' \prec s'' \prec s$, $s'$ is the (unique) immediate predecessor of $s$, also denoted $s_-$. Let $\{s\}_+ \subset \widetilde{S}$ denote the set of *immediate successors* of $s$. We assume that $\{s\}_+ \neq \emptyset$ and $\#\{s\}_+ < \infty$ for all $s \in \widetilde{S}$. Let $s_0$ be the root of the tree, i.e., the unique element of $\widetilde{S}$ with no predecessor.

A notion of time is imposed as follows. We denote by $t(s) = t$ the length of the chain between $s_0$ and $s$. Note that $t(s_0) = 0$. The tree is then partitioned into 'dates', or equivalence classes of length $t$, with $t = 0, 1, ...$ representing such dates. Time is discrete, the horizon is infinite, but at each date $t$, $S_t < \infty$ events or states can be realized, and $\widetilde{S} = \cup_t S_t$ is countable. Each element of $S_t$ is denoted by $s_t$. To each $s_t$, $\{s_t\}_+ \subset S_{t+1}$. We also define a *history* up to and including date $t$ as an array of states, one for each

[4] See Kubler and Schmedders (2005).

date $\tau \leq t$, and is denoted by $s^t = (s_0, s_1, ..., s_t) \in S_0 \times ... \times S_t$, the set of all such histories; we also write $s^t = (s^{t-1}, s_t)$. A *proper history* $s^t$ leading to a date-event $s_t$ is the (finite) sequence of date-events, with $s_{\tau-1} = s_{\tau-}$ for all $\tau = 1, .., t$. From now on, when we refer to date event as histories, we always mean proper histories.

We are going to consider trees which are generated by a (finite) set $S$, i.e., where $\{s\}_+ = S$ for all $s \in \widetilde{S}$; for all $t > 0$, $s^t = (s_0, s_1, ..., s_t)$ where $s_\tau \in S$, all $\tau \leq t$. That is, an underlying first–order Markov stochastic process generates the tree, with time–invariant transition $\pi(s_{t+1}|s_t)$, all $t$. We assume that $\pi(s_{t+1}|s_t) > 0$ for all $s_t, s_{t+1} \in S \times S$ (full support of the transition).

At each $s^t$, there are $C \geq 1$ physical commodities, and the demographic structure is that of an overlapping generations economy. At each $s^t$, $H \geq 1$ individuals are born living $G + 1 \geq 2$ periods, or generations, indexed by $a = 0, ..., G$, from the youngest $(a = 0)$ to the oldest $(a = G)$ age. The economy starts off at $s_0$ with $H$ individuals of each generation. This simple demographic structure can be generalized to any exogenous stochastic process which is a time-homogenous finite Markov chain.

The commodity space is the space of sequences $(\mathbb{R}^{(G+1)C}_{++})^{\widetilde{S}}$, $(G + 1)C$–dimensional vectors. The utility of the young agent $h$ born at $s^t$ is $U^{h,s^t}$ : $\mathbb{R}^{C(\sum_{a=0}^{G} S^{t+a})}_{++} \to \mathbb{R}$, which is time-separable and of the von Neumann - Morgenstern type:

$$U^{h,s^t}(x^{h0}(s^t), .., x^{hG}(s^{t+G}), ...) = \mathbb{E}_{s^t}\{\sum_{a=0}^{G} u^{ha}(x^{ha}(s^{t+a}))\}$$

where if $s_{t+a}$ is the current state for history $s^{t+a}$, and $s^{t+a} = (s^{t+a-1}, s_{t+a})$ for all $s_{t+a} \in \{s_{t+a-1}\}_+$. In addition, each utility $u^{ha} : \mathbb{R}^C_{++} \to \mathbb{R}$ for $a = 0, ..., G$ is smooth, differentially strictly increasing, and strictly concave. The utility $u^{ha}$ for $a > 0$ already includes a discount factor. Agent $h, s^t$ is also endowed with physical goods at all ages, i.e., $e^{ha}(s^{t+a})$, all $a$. We assume that utilities and endowments are stationary, in that $e^{ha}(s^{t+a}) = e^{ha}(s)$, with $s \in S$, and $e^{ha} : S \to \mathbb{R}^C_+$ , for all $a$. That is, the economy has a first-order Markovian structure.[5]

[5]For simplicity, and to strengthen our results, here we omit the possible state–dependence of utilities. In that case, stationarity would imply that $u^{ha}(x^{ha}(s^{t+a}), s^{t+a}) = u^{ha}(x^{ha}(s^{t+a}), s)$.

We also assume that $e_1^{ha}(s) > 0$ for all $s$, all $a$, all $h$, and that $\sum_{h,a} e_c^{ha}(s) > 0$ for $c > 1$, all $s$. Finally, we assume that $u^{ha}$ satisfies the boundary condition $\lim_{n\to\infty} x_{1,n}^{ha} = 0$, then $\lim_{n\to\infty} ||D_1 u^{ha}(x_n^{ha})||^{-1} x_n^{ha\prime} D_1 u^{ha}(x_n^{ha}) = 0$. These conditions simply state that good $c = 1$ is necessary to individuals of all ages, and that total resources in the economy are positive.

At each $s^t$, there are spot markets for the exchange of physical commodities. The price vector of the $C$ commodities at $s^t$ is $p(s^t) \in \mathbb{R}^C_{++}$. Commodity $c = 1$ is dubbed the numéraire commodity.

There are also $J \leq S$ one-period securities in zero net supply, paying in units of the numéraire commodity. Their prices are $q(s^t) \in \mathbb{R}^J$. Their payoffs at $s^t$ are given by an $S \times J$-dimensional matrix $Y(s^t) = Y$ for all $s^t$, with column rank $J$ –hence, there is no outside money. There is one asset, say asset $j = 1$, which has positive payoffs, $y_s^1 > 0$ for all $s \in S$, and $Y$ is in general position. Agents hold $m^{ha}(s^t) \in \mathbb{R}^J$ units of these assets when $a < G$, without loss of generality. We will discuss extensions to long-lived assets and positive net supply further below.

Economies will be triples $(e, \pi, u)$ of endowments, transition probabilities and utilities satisfying our assumptions. Endowments and probabilities lie in open subsets of Euclidean spaces. Utilities $u^{ha}$ are points in the space of $\mathcal{C}^\infty(\mathbb{R}^C_{++}, \mathbb{R})$ functions with the topology of $\mathcal{C}^2$–uniform convergence. Namely, they belong to the $G_\delta$ subset of such space consisting of functions satisfying our maintained assumptions. Then, we let $\Omega$ be the space of such endowments, probabilities and utilities, endowed with the product topology.

For each $s^t \in \widetilde{S}$, we let $\Xi(s^t) \subset \Xi$ be the set of endogenous variables at $s^t$, i.e., of admissible vectors $\xi(s^t) = (x(s^t), m(s^t), p(s^t), q(s^t))$. The sequence $\xi = \left(\xi(s^t), s^t \in \widetilde{S}\right) \in \times_{s^t \in \widetilde{S}} \Xi(s^t)$ represents a vector–valued stochastic process of consumption, asset holdings and prices for commodities and assets –adapted to $\widetilde{S}$. Given a tree $\widetilde{S}$ and a history $s^t \in \widetilde{S}$, we let $\widetilde{S}_{s^t}$ be the subtree starting at $s^t$, i.e., the set of $s^\tau$ with $\tau \geq t$ and $s^\tau = (s^t, s_{t+1}, ..., s_\tau)$. If $\xi \in \times_{s^t \in \widetilde{S}} \Xi(s^t)$, for any $s^t \in \widetilde{S}$, $\xi_{s^t} = \left(\xi(s^\tau), s^\tau \in \widetilde{S}_{s^t}\right) \in \times_{s^\tau \in \widetilde{S}_{s^t}} \Xi(s^\tau)$ is the process in the subtree starting at $s^t$, and we let $\xi_+(s^t) = (x^{ha}(s^t)_{a<G}, m^{ha}(s^t)_{a<G}, q(s^t), \xi_{s^{t+1}})$ be the *continuation* of the process at $s^t$.

## 2.1 Competitive equilibrium

Letting $m^{h(-1)}(s^t) \equiv 0$, $m^{hG}(s^t) \equiv 0$ for all $h, s^t$, a *competitive equilibrium* is a stochastic process $\xi$ such that:

(H) agents optimize given prices, i.e., for each $h, s^t$,

$$\begin{gathered}\max_{x^{ha}(s^{t+a})_{a=0}^{G}, m^{ha}(s^{t+a})_{a=0}^{G}} U^{h,s^t}(x^{h0}(s^t), .., x^{hG}(s^{t+G}), ...)\\ \text{s.t. } p(s^{t+a})[x^{ha}(s^{t+a}) - e^{ha}(s^{t+a})] + q(s^{t+a})m^{ha}(s^{t+a}) =\\ p_1(s^{t+a})y_{s_{t+a}}m^{h(a-1)}(s^{t+a-1}), \text{ for all } s^{t+a} = (s^{t+a-1}, s_{t+a}), \text{ all } a,\end{gathered}$$

and at $s^0$, for $\overline{a} > 0$ agents of age $\overline{a}$ optimize given prices and initial portfolios,

$$\begin{gathered}\max_{x^{ha}(s^{a-\overline{a}})_{a=\overline{a}}^{G}, m^{ha}(s^{a-\overline{a}})_{a=\overline{a}}^{G-1}} U^{h,s^0}(x^{h\overline{a}}(s^0), .., x^{hG}(s^{G-\overline{a}}), ...)\\ \text{s.t. } p(s^{a-\overline{a}})[x^{ha}(s^{a-\overline{a}}) - e^{ha}(s^{a-\overline{a}})] + q(s^{a-\overline{a}})m^{ha}(s^{a-\overline{a}}) =\\ p_1(s^{a-\overline{a}})y_{s_{a-\overline{a}}}m^{h(a-1)}(s^{a-\overline{a}-1}), \text{ for all } s^{a-\overline{a}} = (s^{a-\overline{a}-1}, s_{a-\overline{a}}), \text{ all } a \geq \overline{a},\end{gathered}$$

with $m^{h(\overline{a}-1)}(s_{0-}) = \overline{m}^{h(\overline{a}-1)}(s_{0-})$ given;

(M) markets clear, or $\sum_{h,a}[x^{ha}(s^t) - e^{ha}(s^t)] = 0$ and $\sum_{h,a} m^{ha}(s^t) = 0$, all $s^t \in \widetilde{S}$.

Notice that, by time and state separability of the agents' utility functions, we do not need to write separately the optimization problem of agents of age $a > 0$ for histories $s^t \neq s^0$, as they are implied by their maximization when young. A competitive equilibrium exists for these economies using a standard truncation argument (see Balasko and Shell (1980)), and combining it with the argument for existence of equilibrium in a two-period economy with incomplete markets and numéraire assets (Geanakoplos and Polemarchakis (1986) if assets are short-lived; if they were long-lived, existence can be established for truncated economies by using bounds on asset sales, with an argument à la Radner (1972). Let $E(\omega)$ be the set of competitive equilibria of an economy $\omega \in \Omega$.

As we explained in the Introduction, the focus of this paper will be on a special kind of competitive equilibrium, also known as recursive equilibrium, to which we now turn.

# 3 Recursive equilibrium

Let $W$ be a subset of $\mathbb{R}^{HG}$, with element $w = (..., w^{ha}, ...)_{h\in H, a>0}$. This is the space of initial financial wealth levels.

A *recursive equilibrium*[6] is the state space $S \times W$ together with time–invariant price functions $p(s, w), q(s, w)$, allocation functions $m^{ha}(s, w), x^{ha}(s, w)$

[6] See, e.g., Rios Rull (1996).

for all $a$, and transitions $T_a(s, w, s', w') = 0$, for $s, s' \in S$ and $w, w' \in W$, all $a < G$ such that:

(H') for each $h$, each $(s, w) \in S \times W$, individuals of age $\overline{a} \geq 0$ optimize given the price functions and $(T_a)_{G > a \geq \overline{a}}$, i.e.,

$$\begin{gathered} x^{ha}(s_a, w_a)_{a=\overline{a}}^{G}, m^{ha}(s_a, w_a)_{a=\overline{a}}^{G-1} \in \\ \arg\max_{x^{ha}(s_a, w_a) \in \mathbb{R}_+^C, m^{ha}(s_a, w_a) \in \mathbb{R}^J} U^h(x^{h\overline{a}}(s_{\overline{a}}, w_{\overline{a}}), ..., x^{hG}(s_G, w_G), ...) \\ \text{s.t. } p(s_a, w_a)[x^{ha}(s_a, w_a) - e^{hy}(s_a)] + q(s_a, w_a)m^{ha}(s_a, w_a) = \\ p_1(s_a, w_a)y_{s_a}m^{h(a-1)}(s_{a-1}, w_{a-1}), \text{ all } a \geq \overline{a} \end{gathered}$$

for all $(s_a, w_a)_{a=\overline{a}}^{G}$ s.t. $T_a(s_a, w_a, s_{a+1}, w_{a+1}) = 0$, for $a < G$, $(s_{\overline{a}}, w_{\overline{a}}) = (s, w)$, and with $m^{h(-1)}(s, w) \equiv 0$, $m^{hG}(s, w) \equiv 0$ all $(s, w) \in S \times W$;

(M') markets clear, or $\sum_{h,a}[x^{ha}(s, w) - e^{ha}(s)] = 0$ and $\sum_{h,a} m^{ha}(s, w) = 0$ for all $(s, w) \in S \times W$;

(R) the transitions $T_a$ are consistent with optimization (H') and with nature's moves, that is, given $(s_a, w_a) \in S \times W$, then $T_a(s_a, w_a, s_{a+1}, w_{a+1}) = 0$ implies

- $s_{a+1} \in \{s_a\}_+$;
- $w_{a+1}^{h(a+1)} = (w)_{s_{a+1}}^{h(a+1)} = y_{s_{a+1}} m^{ha}(s_a, w_a)$ all $a < G$.

Note that in this definition the equilibrium process starts from a point in $W$. In particular, a recursive equilibrium is a competitive equilibrium only if the initial distribution of asset portfolios, $\overline{m}(s_{0-})$, multiplied by $y_{s_0}$, is in $W$.

Consumption plans for any age $a$ only depend on the current state $s, w$; any more variability in consumption is not going to be optimal for the agent, given the equilibrium price functions and transition, separability and strict concavity of the utility function. This is why feasible plans are already restricted to functions $x^{ha}(s, w)$.

## 3.1 Generic existence of recursive equilibrium

We want to show the existence of these equilibria in a large class of economies. In order to accomplish this, the basic idea is to show that the initial wealth distribution in each period is typically a sufficient statistic of the memory of the economic system at least at some competitive equilibrium.

### 3.1.1 Existence of simple time-homogeneous Markov equilibria

We are going to first show that there are always competitive equilibria which are time-homogenous and Markov on the state space given by $p$, $(\lambda^{ha})_{h\in H,0<a<G}$, $w, s$, that is, the current exogenous state, current wealth distribution, current Lagrange multipliers for all individuals of all generations except for the eldest and the youngest, and current commodity prices. Denote by $E_M(\omega) \subset E(\omega)$ the set of these Markov equilibria.

**Lemma 1** *For all $\omega \in \Omega$, $E_M(\omega) \neq \varnothing$.*

**Proof.** We are going to construct one such equilibrium for each $\omega$. Start from a $\xi \in E(\omega)$ and take two date-events $s^t, s^{t\prime}$. Assume that the vectors $\xi(s^t)$ and $\xi(s^{t\prime})$ are such that

$p(s^t) = p(s^{t\prime})$,

$\lambda^{ha}(s^t) = \lambda^{ha}(s^{t\prime})$ for all $h$, all $0 < a < G$,

$w^{ha}(s^t) = y_{s_t} m^{h(a-1)}(s^{t-1}) = y_{s_t'} m^{h(a-1)}(s^{t-1\prime}) = w^{ha}(s^{t\prime})$ for all $h$, all $a > 0$,

and $s_t = s_t' = s$.

We show that we can replace the variables in $\xi_+(s^{t\prime})$ with the corresponding variables in $\xi_+(s^t)$ without altering the equilibrium, i.e., without falsifying any of the competitive equilibrium equations. This is obvious for $\xi(s^\tau)$ with either $\tau > t'$ and $s^\tau \notin \widetilde{S}_{s^{t\prime}}$, or $\tau \leq t'$ and $s^\tau \nleq s^{t\prime}$, since we have not touched these variables, or the equations where they appear. It is also obvious for $\xi(s^\tau)$ with $\tau > t'$ and $s^\tau \in \widetilde{S}_{s^{t\prime}}$, since $\xi_+(s^t)$ was an equilibrium to start with. As for $s^\tau$ with $\tau \leq t'$ and for $s^\tau \leq s^{t'}$, the only equations where variables in $\xi_+(s^{t\prime})$ appear are: the budget constraints at $(s^{t\prime})_- = s^{t-1\prime}$, where $m^{h(a-1)}(s^{t-1\prime})$ is, for $a > 0$, all $h$; the no arbitrage conditions also at $s^{t-1\prime}$, where $\lambda^{ha}(s^{t-1\prime}, s)$ appears; the first-order conditions to problem $(H)$, the budget constraints and the no arbitrage equations at $s^{t\prime}$, for all $h$, all $a$. By assumption, $\lambda^{ha}(s^{t\prime}) = \lambda^{ha}(s^t)$ for all $h$, all $0 < a < G$. Since $y_s w^{h(G-1)}(s^{t-1}) = y_s m^{h(G-1)}(s^{t-1\prime})$ for all $h$, and since $p(s^t) = p(s^{t\prime})$, the optimization problem for individuals of age $a = G$ at $s^t$ and $s^{t\prime}$ is the same. Given strict concavity of $u^{hG}$, it has a unique solution given commodity prices and initial wealth, $\left(x^{hG}(s^{t\prime}), \lambda^{hG}(s^{t\prime})\right) = \left(x^{hG}(s^t), \lambda^{hG}(s^t)\right)$ and all no arbitrage equations at $s^{t-1\prime}$ are still satisfied after the substitution. Wealth equality at successor state $s^{t\prime}$ is the only constraint that $m^{h(a-1)}(s^{t-1\prime})$ must satisfy for all $h$, all $a < G$, but this has been assumed to hold. Finally, FOCs, budget constraints and no arbitrage equations can simply be substituted without altering anything else in the equilibrium.

Next, we construct the process $\hat{\xi} \in \times_{s^t \in \widetilde{S}} \Xi(s^t)$ from $\xi$ recursively as follows. Use the natural order sum, $\sum_t S_t$, on $\widetilde{S}$. Let $s^t(n)$ be the node reached at step $n \geq 1$ of the construction algorithm. Let $\hat{\xi}_{n-1} \in \times_{s^t \in \widetilde{S}} \Xi(s^t)$ be given, with $\hat{\xi}_0 = \xi$. For any array $(p, (\lambda^{ha})_{h \in H, 0<a<G}, w, s)$ possible in $\hat{\xi}_{n-1}$, let

$$S_{n-1}(p, (\lambda^{ha})_{h \in H, 0<a<G}, w, s) =$$
$$\{s^t \in \widetilde{S} | p(s^t) = p, \lambda^{ha}(s^t) = \lambda^{ha}, h \in H, 0 < a < G, w(s^t) = w, s_t = s\}$$

be the corresponding equivalence class. Since under the order sum $\widetilde{S}$ is well-ordered, $\min S_{n-1}(p, (\lambda^{ha})_{h \in H, 0<a<G}, w, s) = s_{n-1}^{p,(\lambda^{ha})_{h \in H, 0<a<G}, w, s}$ is well defined, and $s_{n-1}^{p,(\lambda^{ha})_{h \in H, 0<a<G}, w, s} \leq s^t(n)$. Define $\hat{\xi}_n$ by assigning to $s^t(n)$ the continuation $\hat{\xi}_{n,+}(s^t(n)) = \hat{\xi}_{n-1,+}(s_{n-1}^{p,(\lambda^{ha})_{h \in H, 0<a<G}, w, s})$, and letting $\hat{\xi}(s^t(n))$ be the $s^t(n)$-th element in the vector $\hat{\xi}_{n+}(s^t(n))$, and then go on to step $n+1$. Note that each node $s^t$ is eventually assigned a value $\hat{\xi}(s^t)$ in at most $n$ many steps, with $s^t = s^t(n)$. Moreover, $\hat{\xi}$ is uniquely determined at each node by $s, w$, $(\lambda^{ha})_{h \in H, 0<a<G}$ and $p$: for each $s^t$, with $s_t = s$ and $\hat{\xi}(s^t)$ such that $\hat{p}(s^t) = p, \hat{\lambda}^{ha}(s^t) = \lambda^{ha}$, all $h \in H, 0 < a < G$, $w(s^t) = w$, there is a unique continuation $\hat{\xi}_+(s^t) = \hat{\xi}(p, (\lambda^{ha})_{h \in H, 0<a<G}, w, s)$. Since the process on $S$ is Markov, so is the one on $(p, (\lambda^{ha})_{h \in H, 0<a<G}, w, s)$. Since $\xi \in E(\omega)$, then also $\hat{\xi} \in E(\omega)$, i.e., $\hat{\xi} \in E_M(\omega)$ and we are done.■

Let $\lambda(s) = (\lambda^{ha}(s))_{h \in H, 0<a<G}$ be the collection of marginal utilities of the numéraire commodity for all individuals $h \in H$ and cohorts $a$, $a = 1, ..., G-1$. Equilibria in $E_M(\omega)$ do not have multiplicity problems in any continuation. In particular, we have excluded situations where even if $p(s^t), \lambda(s^t)$, $w(s^t)$ and $s_t$ are given at an arbitrary $s^t$, multiple equilibrium asset prices $q(s^t)$ and commodity price expectations $p(s^{t+1})$ are possible along the tree $\widetilde{S}$. Note that we do not know whether such selection process gives rise to continuous, or even measurable, transitions, but neither property will be used in our argument or is required in the definition of recursive equilibrium.

### 3.1.2 Nonconfounding simple time-homogeneous Markov equilibria

For the Markov equilibria in $E_M(\omega)$, a Markov state is defined by a current realization of $s \in S$, by commodity prices $p(s)$, by multipliers $\lambda(s)$, and by the initial wealth distribution $w(s)$. We want to show that the Markov equilibria which we constructed above, typically and under some qualifying condition on intragenerational heterogeneity, have the following injection property: if at $s$, two Markov states are given, $(p(s), \lambda(s), w(s); s)$ and $\left(\hat{p}(s), \hat{\lambda}(s), \hat{w}(s); s\right)$ with $w(s) = \hat{w}(s)$, then $(p, \lambda(s)) = (\hat{p}(s), \hat{\lambda}(s))$. We call Markov equilibria that satisfy this property *nonconfounding*, and denote the set of nonconfounding Markov equilibria $E_M^{NC}(\omega)$.

The qualifying condition on intragenerational heterogeneity used below is

**A1** $H > 2[(C-1)\sum_{a=0}^{G} S^a + J\sum_{a=0}^{G-1} S^a]$.

We aim at proving the following proposition.

**Proposition 2** *Under A1, $E_M(\omega) = E_M^{NC}(\omega)$ in a residual subset $\Omega^*$ of $\Omega$.*

To this end, we consider the demand functions of individuals of age $a = 0$ at some pair of states $(s_{01}, s_{02})$, with eventually $s_{01} = s_{02}$, for two sets of commodity prices $p(s_k^a)$, $s_k^a \in S^a$, $k = 1, 2$, and $a = 0, ..., G$, and of asset prices $q(s_k^a)$, $s_k^a \in S^a$, $k = 1, 2$, and $a = 0, ..., G-1$, where $s_k^0 = s_{0k}$ for $k = 1, 2$. We do not need to keep track of the asset prices $q(s_k^G)$, $s_k^G \in S^G$, $k = 1, 2$, since in the absence of arbitrage the old generation $G$ does not trade on the asset market.

Without loss of generality normalize commodity prices by setting

$$p_1(s_k^a) = 1, \text{ for all } s_k^a \in S^a, k = 1, 2,$$

To have a compact notation, let $(p, q)$ denote the entire collection of prices over the two finite trees. Thus, $p \in \mathbb{R}_{++}^{2(C-1)\sum_{a=0}^{G} S^a}$ and $q \in \mathbb{R}^{2J\sum_{a=0}^{G-1} S^a}$. Let $N_{(s_{01}, s_{02})}$ be the set of parameters $\omega$ and variables

$$z = (\left(x^{ha}(s_k^a), \lambda^{ha}(s_k^a), m^{ha}(s_k^a)\right)_{h, s_k^a, a, k}, p, q)$$

which satisfy the equations

$$
\begin{array}{cr}
Du^{ha}(x^{ha}(s_k^a)) - \lambda^{ha}(s_k^a)p(s_k^a) = 0, & (1, h; s_k^a) \\
-\lambda^{ha}(s_k^a)q(s_k^a) + \sum_s \pi(s|s^a)\lambda^{ha}(s_k^a, s)y_s = 0,\ a < G & (2, h; s_k^a) \\
p(s_k^a)[x^{ha}(s_k^a) - e^{ha}(s_k^a)] + q(s_k^a)m^{ha}(s_k^a) = & \\
y_s m^{h(a-1)}(s_k^{a-1}), \text{ for all } s_k^a = (s_k^{a-1}, s), \text{ all } a \geq 0 & (3, h; s_k^a)
\end{array}
$$

with $q(s_k^G) = m^{hG}(s_k^G) \equiv 0$, for all $s_k^G \in S^G$, and $m^{h(-1)}(s_k^{-1}) = 0$, for $k = 1, 2$ and

$$
\begin{array}{cl}
\sum_h x^{ha}(s_k^a) \leq M(\omega), & (i; k) \\
x^{ha}(s_k^a) \geq 0 & (ii; k)
\end{array}
\qquad \text{(IN)}
$$

Here $M(\omega) > 0$ is a real, with $M(\omega) = 2\max_s \sum_{ha} e^{ha}(s)$. Note that this system has $2[(C-1)\sum_{a=0}^{G} S^a + J\sum_{a=0}^{G-1} S^a]$ too many unknowns: $p(s_k^a)$, for $s_k^a \in S^a$, $a = 0, ..., G$, $k = 1, 2$, and $q(s_k^a)$, for $s_k^a \in S^a$, $a = 0, ..., G-1$, $k = 1, 2$.[7]

Let $s_1^a$ and $s_2^a$ be two arbitrary histories of length $a > 0$, with $s_k^a = (s_k^{a-1}, s)$ for some $s$, and all $a > 0$. For $k = 1, 2$, define

$$
\hat{p}(s_k^a) = \begin{cases} p(s_k^a) & \text{if } a = G \\ (\lambda^{ha}(s_k^a))_{h \in H} & \text{otherwise.} \end{cases}
$$

We are interested in the subset $N_{(s_{01}, s_{02})}(s_1^a, s_2^a, s)$ of $N_{(s_{01}, s_{02})}$ consisting of the solutions to equations $(1-3)$, inequalities $(IN)$, and the $H$ equations

$$
y_s(m^{h(a-1)}(s_1^{a-1}) - m^{h(a-1)}(s_2^{a-1})) = 0 \quad (4, h)
$$

for all $h$, with

$$
\|(\hat{p}(s_1^a) - \hat{p}(s_2^a)\| \neq 0 \qquad \text{(NC)}
$$

System $(1-4)$ has now $H - 2[(C-1)\sum_{a=0}^{G} S^a + J\sum_{a=0}^{G-1} S^a]$ too many equations, a positive number under A1.

Clearly any $\xi \in E_M(\omega)$ 'goes through' system $(1-3)$ and $(IN)$, i.e., $\xi$ intersects some $N_{(s_{01}, s_{02})}$. Moreover, if $\xi \in E_M(\omega)$ but $\xi \notin E_M^{NC}(\omega)$, then (4) must also hold when $(NC)$ holds for some pair of histories $(s_1^a, s_2^a)$ with current state $s$, or $\xi$ intersects also $N_{(s_{01}, s_{02})}(s_1^a, s_2^a, s)$. We are then going to show that in a residual set of parameters $\Omega_{(s_{01}, s_{02})}(s_1^a, s_2^a, s)$ the set

[7] For $G > 1$; when $G = 1$, no asset prices are unmatched, and the term $J\sum_{a=1}^{G-1} S^a$ drops.

$N_{(s_{01},s_{02})}(s_1^a, s_2^a, s)$ is empty. Repeating this across all finitely many pairs of initial states $(s_{01}, s_{02})$, histories $(s_1^a, s_2^a)$ and states $s$, we obtain a residual set

$$\Omega^* = \cap_{(s_{01},s_{02})} \cap_{a=0}^{G-1} \cap_{(s_1^{a-1},s_2^{a-1},s)} \Omega_{(s_{01},s_{02})}(s_1^a, s_2^a, s)$$

where if $(p(s), \lambda(s), w(s); s)$ and $\left(\hat{p}(s), \hat{\lambda}(s), \hat{w}(s); s\right)$ are given and $(p(s), \lambda(s))$ $\neq \left(\hat{p}(s), \hat{\lambda}(s)\right)$, then $w \neq \hat{w}$, i.e., $E_M(\omega) = E_M^{NC}(\omega)$ if $\omega \in \Omega^*$, under A1, proving Proposition 2.

To show that $N_{(s_{01},s_{02})}(s_1^a, s_2^a, s)$ is empty in a residual set $\Omega_{(s_{01},s_{02})}(s_1^a, s_2^a, s)$, we are going to restrict attention to sets $N^n_{(s_{01},s_{02})}(s_1^a, s_2^a, s)$ where $(1-4)$ and $(IN)$ hold, and moreover $z$ satisfies

$$\|\hat{p}(s_1^a) - \hat{p}(s_2^a)\| \geq 1/n, \qquad \text{(NC.n)}$$

for some positive integer $n$. Since $N_{(s_{01},s_{02})}(s_1^a, s_2^a, s) \subset \cup_n N^n_{(s_{01},s_{02})}(s_1^a, s_2^a, s)$, emptiness of $N^n_{(s_{01},s_{02})}(s_1^a, s_2^a, s)$ for all $n$ will imply that $N_{(s_{01},s_{02})}(s_1^a, s_2^a, s)$ is empty.

We are then left to showing the following proposition.

**Proposition 3** *Given any $(s_{01}, s_{02})$, $(s_1^a, s_2^a)$, $s$ and $n$, there is an open and dense set $\Omega^n_{(s_{01},s_{02})}(s_1^a, s_2^a, s) \subset \Omega$ such that $N^n_{(s_{10},s_{20})}(s_1^a, s_2^a, s)$ is empty.*

Setting

$$\Omega_{(s_{01},s_{02})}(s_1^a, s_2^a, s) = \cap_n \Omega^n_{(s_{01},s_{02})}(s_1^a, s_2^a, s)$$

will deliver the desired conclusion. For suppose not, and there is a $z \in N_{(s_{01},s_{02})}(s_1^a, s_2^a, s)$. Then there is an $\hat{n} > 0$ such that $(NC.\hat{n})$ holds true and (4) is satisfied. But this implies $N^{\hat{n}}_{(s_{01},s_{02})}(s_1^a, s_2^a, s) \neq \varnothing$, a contradiction. Hence, in a residual set of economies, if $\xi \in E_M(\omega)$ and $(p(s), \lambda(s)) \neq \left(\hat{p}(s), \hat{\lambda}(s)\right)$, then $\|w(s) - \hat{w}(s)\| \neq 0$, as desired. The remainder of the section is devoted to proving Proposition 3.

**Proof of Proposition 3** We are going to prove Proposition 3 in two separate parts.

**Openness** The set of endogenous variables and parameters $N^n_{(s_{01},s_{02})}(s^a_1, s^a_2, s)$ is closed in $N_{(s_{01},s_{02})}$, since it is the preimage of continuous functions and weak inequalities. If the projection of endogenous variables and parameters satisfying $(1-3)$, and $(IN)$, that is, $\Pr : N_{(s_{01},s_{02})} \to \Omega$, is proper, $\Omega \backslash \Omega^n_{(s_{01},s_{02})}(s^a_1, s^a_2, s)$, the set of parameters $\Pr(N^n_{(s_{01},s_{02})}(s^a_1, s^a_2, s))$ is closed. Clearly, its complement $\Omega^n_{(s_{01},s_{02})}(s^a_1, s^a_2, s)$ is an open set of parameters where either (4) does not hold or $\left\|(p(s^a_1), \lambda^{ha}(s^a_1))_{h\in H}) - (p(s^a_2), \lambda^{ha}(s^a_2))_{h\in H})\right\| < 1/n$, i.e., where $N^n_{(s_{01},s_{02})}(s^a_1, s^a_2, s)$ is empty.

**Lemma 4** *The projection* Pr *is proper.*

**Proof.** Consider a sequence $\{\omega_m\}^{+\infty}_{m=1} \subset \Omega$, with $\omega_m \to \omega \in \Omega$, and any associated sequence $\{z_m\}^{+\infty}_{m=1} \subset N_{s,\overline{s}}$. First, notice that $M(\omega_m) \to M(\omega) < +\infty$. Using $(IN.i; k - ii; k)$, for all $h$, $a$ and $s^a_k$ there is a subsequence $x^{ha}_m(s^a_k) \to x^{ha}_a(s^a_k)$ such that $M(\omega) \geq x^{ha}_a(s^a_k) \geq 0$. Since $u^{ha}$ satisfies the boundary condition, $x^{ha}(s^a_k) \gg 0$. Using convergence of $x^{ha}_m(s^a_k)$, the price normalization, $Du^{ha} \gg 0$ and equations $(1, h; s^a_k)$, we have that $\lambda^{ha}_m(s^a_k) \to \lambda^{ha}(s^a_k) \gg 0$. Then, again the same equations imply that $p_m(s^a_k) \to p(s^a_k) \gg 0$. Equations $(2, h; s^a_k)$ now imply that $(q, \hat{q})_m \to (q, \hat{q})$, and using equations $(3, h; s^a_k)$ and no redundancy, we obtain that $m^{ha}_m(s^a_k) \to m^{ha}(s^a_k)$ as well.■

**Density** We now use transversality, i.e., Sard's and the preimage theorems, to show that the set of economies where $N^n_{(s_{01},s_{02})}(s^a_1, s^a_2, s)$ is empty is dense. Indeed, we are going to show that the set $N^n_{(s_{01},s_{02})}(s^a_1, s^a_2, s)$ is a negative dimensional manifold for a dense subset of parameters $\Omega^n_{(s_{01},s_{02})}(s^a_1, s^a_2, s)$; put it differently, for a dense subset $\Omega^n_{(s_{01},s_{02})}(s^a_1, s^a_2, s)$, either (4) does not hold or $\left\|(p(s^a_1, s), \lambda^{ha}(s^a_1, s))_{h\in H}) - (p(s^a_2, s), \lambda^{ha}(s^a_2, s))_{h\in H})\right\| < 1/n$.

First observe that inequalities $(IN.ii; k)$ hold strictly when $z \in N^n_{(s_{01},s_{02})}(s^a_1, s^a_2, s)$. This follows from equations $(1, h; s^a_k)$, $k = 1, 2$, and the boundary condition on $u^{ha}$. Let $F(z, \omega) = 0$ represent system $(1-4)$. It is enough to show that $F(z, \omega) = 0$ has, typically, no solution in $\Omega$, disregarding the inequalities $\left\|(p(s^a_1), \lambda^{ha}(s^a_1))_{h\in H}) - (p(s^a_2), \lambda^{ha}(s^a_2))_{h\in H})\right\| \geq 1/n$ and $(IN.i; k)$ —i.e., assuming that they do not bind— since this will be *a fortiori* true if the inequalities are satisfied (note that if they are satisfied with equality, we are adding equations and so potentially reducing the dimensionality of $N^n_{(s_{01},s_{02})}(s^a_1, s^a_2, s)$ relative to $F^{-1}(0)$).

Hereafter, we work out the less cumbersome case of economies with $G = 1$. The argument for economies with $G > 1$ is left to the Appendix.

When $G = 1$, whether or not markets are complete, the argument must only show that equation $(4, h)$ cannot hold when $(NC.n)$ holds for the pair $\left(s_1^{G*}, s_2^{G*}\right)$. Since $G = 1$, this is just the statement that $\|p(s_{01}, s^*) - p(s_{02}, s^*)\| \geq 1/n$ for some $s^*$, the critical state. The set $N^n_{(s_{01},s_{02})}(s_1^a, s_2^a, s)$ is just $N^n_{(s_{01},s_{02})}(s^*)$, and $\Omega^n_{(s_{01},s_{02})}(s_1^a, s_2^a, s)$ is just $\Omega^n_{(s_{01},s_{02})}(s^*)$.

**Lemma 5** *$N^n_{(s_{01},s_{02})}(s^*)$ is empty on a dense subset $\Omega^n_{(s_{01},s_{02})}(s^*)$.*

**Proof.** Since $p(s_{01}, s^*) \neq p(s_{02}, s^*)$, there exists a perturbation of $e^{h1}(s^*)$ such that $p(s_{01}, s^*)\Delta e^{h1}(s^*) = 1$, while $p(s_{02}, s^*)\Delta e^{h1}(s^*) = 0$.

Pick any $h$. Consider the system $F^h(\cdot) = (f_1^h, f_2^h, W^h) = 0$, for $f_k^h$ representing the left-hand side of equations $(1, h; s_k^a) - (3, h; s_k^a)$ for a pair $(h, k)$, and $W^h(\cdot) = 0$ is $(4, h)$, i.e., $y_{s^*}(m^h(s_{01}) - m^h(s_{02}))$. By regularity of demand, $D_{x^{ha}(s_2^a),m^h(s_{02}),\lambda^{ha}(s_2^a)} f_2^h$ is invertible. Since $\Delta e^h$ does not affect the solution to the programming problem for $k = 2$, the problem boils down to showing that $D_{x^{ha}(s_1^a),m^h(s_{01}),\lambda^{ha}(s_1^a),\Delta e^h}(f_1^h, W^h)$ has full rank. Using standard notation, let

$$R = \begin{bmatrix} -q(s_{01}) \\ Y \end{bmatrix}, \text{ and } \Psi = \begin{bmatrix} p(s_{01}) & 0 & & 0 \\ 0 & p(s_{01},1) & & \\ & & \ddots & 0 \\ 0 & & 0 & p(s_{01},S) \end{bmatrix}$$

be the $(S{+}1)\times C(S{+}1)$-dimensional matrix of commodity prices. For the programming problem at $k = 1$, simplify the notation by letting $(x^h, \lambda^h, m^h) = \left(\left(x^{ha}(s_1^a), \lambda^h(s_1^a)\right)_{a=0,1}, m^h(s_{01})\right)$. The sequential budget constraint can then be rewritten as

$$\Psi(x^h - e^h) = Rm^h$$

and the no arbitrage conditions as

$$R^T\Pi\lambda^h = 0,$$

with $\Pi$ an $(S + 1) \times (S + 1)$ diagonal matrix with elements: 1, if $s_1^a = s_{01}$, and $\pi(s|s_{01})$ if $s_1^a = (s_{01}, s)$. Then,

$$D_{x^h,\lambda^h,m^h,e^h}(f_1^h, W^h) = \begin{bmatrix} H_1 & -\Psi^T & 0 & 0 \\ -\Psi & 0 & R & \alpha \\ 0 & R^T\Pi & 0 & 0 \\ 0 & 0 & y_s & 0 \end{bmatrix}$$

where $H_1$ is the block-diagonal Hessian matrix (symmetric and negative definite), with $s^a$ block given by $D^2u^{ha}(x^{ha}(s_1^a)) = H(s_1^a)$, and $\alpha$ is a column vector of dimension $(S+1)\times 1$, with $\alpha(s^a) = p(s_1^a)$ if $s_1^a = (s_{01}, s^*)$, and zero otherwise.

It suffices to show that for some $(\Delta x^h, \Delta\lambda^h, \Delta m^h, \Delta e^h)$ the following system of equations has a solution:

$$\begin{aligned} H_1\Delta x^h - \Psi^T\Delta\lambda^h &= 0, \\ -\Psi\Delta x^h + R\Delta m^h + \alpha\Delta e^h &= 0, \\ R^T\Pi\Delta\lambda^h &= 0, \\ y_{s^*}\Delta m^h &= 1. \end{aligned}$$

First get rid of $\Psi\Delta x^h$, by setting

$$\Psi\Delta x^h = Q\Delta\lambda^h,$$

where $Q = \Psi H_1^{-1}\Psi^T$ is a diagonal matrix of dimension $(S+1)\times(S+1)$ with elements equal to $p(s_1^a)H^{-1}(s_1^a)p(s_1^a)^T$. Since $H(s_1^a)$ is negative definite, $Q(s_1^a) \equiv p(s_1^a)H^{-1}(s_1^a)p(s_1^a)^T < 0$.

Then, substituting the last equation into the second of the system we get:

$$\Delta\lambda^h = Q^{-1}[R\Delta m^h + \alpha\Delta e^h].$$

Hence, by exploiting the fact that $R^T\Pi\Delta\lambda^h = 0$, we get

$$0 = R^T\Pi\Delta\lambda^h = R^T\Pi Q^{-1}[R\Delta m^h - a\Delta e^h]$$

Since $R$ has full column rank the matrix $R^T\Pi Q^{-1}R$ is negative definite. For suppose $\varsigma \in \mathbb{R}^J$ and consider $\varsigma^T(R^T\Pi Q^{-1}R)\varsigma = (R\varsigma)^T\Pi Q^{-1}(R\varsigma)$. If $\varsigma \neq 0$, then $R\varsigma \neq 0$, and since $\Pi Q^{-1}$ is negative definite, $\varsigma^T(R^T\Pi Q^{-1}R)\varsigma < 0$, as we wanted to show. Therefore, $R^T\Pi Q^{-1}R$ is invertible. Hence

$$\Delta m^h = (R^T\Pi Q^{-1}R)^{-1}R^T\Pi Q^{-1}\alpha\Delta e^h.$$

However, by the definition of $\alpha$, $\Pi Q^{-1}\alpha = (0, ..., (\frac{\pi(s^*|s_{01})}{Q(s_{01},s^*)})p(s_{01}, s^*), 0, ...0)$ and as a consequence

$$R^T\Pi Q^{-1}\alpha = (0, ..., y_{s^*}^T\frac{\pi(s^*|s_{01})}{Q(s_{01}, s^*)}p(s_{01}, s^*), 0, ...0)$$

Thus, the equation $y_{s^*}\Delta m^h = 1$ becomes

$$1 = y_{s^*}\Delta m^h = y_{s^*}(R^T \Pi Q^{-1} R)^{-1} y_{s^*}^T (\frac{\pi(s^*|s_{01})}{Q(s_{01}, s^*)}) p(s_{01}, s^*) \Delta e^{h1}(s^*)$$

Since $(R^T \Pi Q^{-1} R)$ is negative definite, $y_{s^*}(R^T \Pi Q^{-1} R)^{-1}(y_{s^*})^T < 0$. However, $(\frac{\pi(s^*|s_{01})}{Q(s_{01},s^*)}) < 0$, and hence $y_{s^*}\Delta m = 1$ if

$$p(s_{01}, s^*)\Delta e^{h1}(s^*) = \frac{1}{y_{s^*}(R^T \Pi Q^{-1} R)^{-1} y_{s^*}^T (\frac{\pi(s^*|s_{01})}{Q(s_{01},s^*)})}.$$

This ends the proof.■

### 3.1.3 Main result

We are now ready to state and prove our main result.

**Theorem 6** *Under A1, recursive equilibria exist on the residual set $\Omega^* \subset \Omega$.*

**Proof.** Let $\omega \in \Omega^*$ be given as above. By construction, there is a $\xi \in E(\omega)$ which also has the property $\xi \in E_M^{NC}(\omega)$. Let $w = (..., w^{ha}, ...)_{h\in H, a>0}$, and $W = \{w \in \mathbb{R}^{HG} \mid w^{ha} = y_{s_{\tau+1}} m^{h(a-1)}(s^\tau) \text{ for some } s^\tau \in \widetilde{S}, s_{\tau+1} \in S\}$. Then, for all $s^t \in \widetilde{S}$ the vectors $\xi(s^t)$ can be expressed as $\xi(s^t) = \xi(s_t, w(s^t))$, i.e., they are the image of a function $\xi : S \times W \to \Xi$. For suppose not. First, it must be that $w = w(s^\tau)$ also for some $s^\tau \neq s^t$, otherwise the existence of such function is trivial. Second, suppose there exist $\xi^1, \xi^2 \in \xi(s, w)$ with $\xi^1 \neq \xi^2$ ($\xi(s, w)$ is not a singleton). Then, $\xi^1 = \xi(s^t)$ and $\xi^2 = \xi(s^\tau)$, with $\xi(s^t) \neq \xi(s^\tau)$. By construction of $\xi \in E_M(\omega)$, for all $s^t$,

$$\xi(s^t) = \Big(\big(x^{ha}(s^t), \lambda^{ha}(s^t), m^{ha}(s^t)\big)_{h\in H, a\leq G}, p(s^t), q(s^t)\Big) = \hat{\xi}(p, \lambda, w, s)$$

for $p = p(s^t)$, $\lambda = \left(\lambda^{ha}\right)(s^t)_{h\in H, 0<a<G}$, $w = w(s^t)$ and $s = s_t$, and where $\hat{\xi}(.)$ is the function from Lemma 1. So if $\xi^1, \xi^2 \in \xi(s, w)$, it must be that

$$(p^1, \lambda^1) = (p(s^t), \left(\lambda^{ha}\right)(s^t)_{h\in H, 0<a<G}) \neq (p(s^\tau), \left(\lambda^{ha}\right)(s^\tau)_{h\in H, 0<a<G}) = (p^2, \lambda^2)$$

while $w^1 = w(s^t) = w(s^\tau) = w^2$, a contradiction to $\xi \in E_M^{NC}(\omega)$.

The final step to time-invariance is accomplished by establishing the following law of motion on $S \times W$: given any $(s, w) \in S \times W$, let $(s', w')$ be given by:

(1) $s' \in \{s\}_+$;

(2) $w^{h(a+1)'} = (w)_{s'}^{h(a+1)} = y_{s'} m^{ha}(s, w)$ all $a < G$.

Clearly, since $\xi(s^t)$ satisfies conditions $(H), (M)$ of a competitive equilibrium, then $\xi(s, w)$ also does, i.e., it satisfies $(H'), (M')$ in the definition of recursive equilibrium. Now, transitions $T_a$, all $a$, can be constructed by applying (1) and (2) repeatedly, and conditions $(R)$ are easily established, so that the state space $S \times W$, the functions $\xi(s, w)$ and the transitions $(T_a)_{a \geq 0}$ form a recursive equilibrium for the economy $\omega$, proving our assertion.■

Note that our construction delivers recursive equilibria which are competitive equilibria, since initial conditions can always be included in $W$. Of course, we can say nothing about the uniqueness of such equilibria. Also, we do not know whether the function $\xi(s, w)$ is continuous or even measurable on $S \times W$.

# 4 Appendix

**Proof of Density when $G > 1$**

We show the computations for the case of complete markets. We can get rid of asset portfolios and look at the economy where individuals face a unique budget constraint. Thus, individual $h$ solves the standard programming problem, for $k = 1, 2$

$$\begin{array}{rl} \max \sum_a \sum_{s_k^a} \pi(s_k^a) u^{ha}(x^{ha}(s_k^a)) & \qquad s.t. \\ \sum_a \sum_{s_k^a} \pi(s_k^a) p(s_k^a)[x^{ha}(s_k^a) - e^{ha}(s_k^a)] = 0, & \end{array}$$

where $e^{ha}(s_k^a) = e^{ha}(s_a)$, for $s_k^a = (s_{0k}, ..., s_a)$, and with some abuse of notation we write $\pi(s_k^a)$ short for $\pi(s_k^a | s_{0k})$. The first order conditions associated to this problem are

$$\begin{array}{rr} Du^{ha}(x^{ha}(s_k^a)) - \lambda_k^h p(s_k^a) = 0, & (1, h; s_k^a) \\ \sum_a \sum_{s_k^a} \pi(s_k^a) p(s_k^a)[x^{ha}(s_k^a) - e^{ha}(s_k^a)] = 0. & (2, h; k) \end{array} \qquad \text{(h,k)}$$

First, equilibria in $E_M(\omega)$ can equivalently be described by dropping any reference to prices. Indeed, the minimum state space can be identified with $s \in S$ and

$$\lambda^{ha}(s^{a-1}, s), w^{ha}(s^a), \text{ all } h \in H \text{ and } 0 < a \leq G$$

The argument is identical to the one for Lemma 1, so it is omitted. The advantage of this formulation is that we treat $a = G$ symmetrically, writing condition $(NC.n)$ then as $\left\|(\lambda^{ha}(s_1^{a*})_{h\in H})] - (\lambda^{ha}(s_2^{a*})_{h\in H})]\right\| \geq 1/n$, all $a$. Second, since when markets are complete from the first order conditions $(h,k)$ we have $\lambda^{ha}(s_k^a) = (p_1(s_k^{a^*})\lambda_k^h$ for all $h$, condition $(NC.n)$ can be written as the set of prices and multipliers such that

$$\left\|(p_1(s_1^{a^*})\lambda_1^h)_{h\in H}) - (p_1(s_2^{a^*})\lambda_2^h)_{h\in H})]\right\| \geq 1/n. \tag{NC.n}$$

Next, equation $(4,h)$ must here be replaced by equality of net commodity expenditures. Let

$$T(\hat{s}^a) = \{s^{a'} : a' > a \text{ and } s^{a'} = (\hat{s}^a, s^{a'-a}) \text{ for some } s^{a'-a} \in S^{a'-a}\}$$

be the set of histories of lenght $a' \geq a$ that go through $\hat{s}^a$. Using the budget constraints and the no arbitrage conditions of the original sequential budget economy, taking into account again that $\lambda^{ha}(s^a) = p_1(s_1^{a^*})\lambda_1^h$ for all $h$, $(4,h)$ becomes

$$\begin{aligned} & \sum_{s_1^a \in T(s_1^{a^*})} \frac{\pi(s_1^a)}{\pi(s_1^{a*})p_1(s_1^{a*})} p(s_1^a)[x^{ha}(s_1^a) - e^{ha}(s_1^a)] \\ = & \sum_{s_2^a \in T(s_2^{a^*})} \frac{\pi(s_2^a)}{\pi(s_2^{a*})p_1(s_2^{a*})} p(s_2^a)[x^{ha}(s_2^a) - e^{ha}(s_2^a)] \end{aligned} \tag{4,h}$$

with the usual convention that $e^{ha}(s_k^a) = e^{ha}(s_a)$, for $s_k^a = (s_{0k}, ..., s_a)$.

Let $F$ now denote the left-hand side of the system of equations $(h,k)$, $k = 1,2$, and $(4,h)$, $h \in H$. Hereafter we use the following convention: $F$ restricted to equations $(1,h;s_k^a)$ and $(2,h;k)$ for one $k$ is $f_k^h$; restricted to $(4,h)$ is $W^h$; and restricted to equations $(h,k)_{k=1,2}$ is $F^h$.

We show that for each given pairs of prices $p(k)$, $k = 1,2$, $F$ is transversal to zero. Since $F(\cdot;\omega) = 0$ separates into $H$ disjoint and independent systems $(F^h, W^h)(\cdot;\omega^h) = 0$, we just need to prove that $(F^h, W^h)(\cdot;\omega^h)$ is transversal to zero, i.e., the Jacobian matrix $D_{(x^h,\lambda^h,\omega^h)}(F^h, W^h)$ has full rank.

We start by covering a case where we use only endowment perturbations. Let $(\pi\otimes p)(k) = (..., \pi(s_k^a)p(s_k^a), ...)$, and $(\hat{\pi}_+\otimes\hat{p})(k) = (..., \frac{\pi(s_k^a)}{\pi(s_k^{a*})p_1(s_k^{a*})}p(s_k^a), ...)$, where $s^a \in T(s_k^{a^*})$. Thus $(\pi \otimes p)(k)$ is the price vector as it appears in the overall budget constraint, while $(\pi_+ \otimes p)(k)$ is the price vector as it appears in $(4,h)$. Again, $H(s^a_k) = D^2u^{ha}(x^{ha}(s^a_k))$ is a negative definite, symmetric $C$-dimensional matrix, and $H_k$ is the block-diagonal matrix with blocks $H(s^a_k)$, therefore itself a negative definite and symmetrix matrix. We let $Q(s^a_k) \equiv p(s^a_k)H^{-1}(s^a_k)p(s^a_k)^T$, and $Q_k \equiv \sum_a \sum_{s^a_k \in S^a} \pi(s^a_k)Q(s^a_k)$, $k = 1, 2$.

**Lemma 7** *If $s_{01} \neq s_{02}$ or $p(s_{01}) \neq p(s_{02})$, $D_{(x^h,\lambda^h,\omega^h)}(F^h, W^h)$ has full rank.*

**Proof.** Since $s_{01} \neq s_{02}$ or $p(s_{01}) \neq p(s_{02})$, we can always choose $\Delta e^{h0} = \left(\Delta e^{h0}(s_{0k})\right)_{k=1,2}$ such that $p(s_{02})\Delta e^{h0} = 0$ and $p(s_{01})\Delta e^{h0} = \Delta w$. Then,

$$D_{(x^h,\lambda^h,e^h)}(F^h, W^h) = \begin{bmatrix} H_1 & -p(1)^T & 0 & 0 & 0 \\ -(\pi \otimes p(1)) & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & H_2 & -p(2)^T \\ 0 & 0 & 0 & -(\pi \otimes p(2)) & 0 \\ (0, (\pi_+ \otimes p)(1)) & 0 & 0 & -(0, (\pi_+ \otimes p)(2)) & 0 \end{bmatrix}$$

Since $D_{(x^h_k,\lambda^h_k)} f^h_k$ is invertible, it suffices to find vectors $((\Delta x^{ha}(s^a_k), \Delta\lambda^h_k)_{k=1,2}, \Delta w)$, such that

$$\begin{aligned} H(s^a_k)\Delta x^{ha}(s^a_k) - p(s^a_k)^T\Delta\lambda^h_k &= 0, \\ (\pi \otimes p)(1)\Delta x^h_1 - \Delta w &= 0, \\ (\pi \otimes p)(2)\Delta x^h_2 &= 0, \\ (\pi_+ \otimes p)(1)\Delta\hat{x}^h_1 - (\pi_+ \otimes p)(2)\Delta\hat{x}^h_2 &= 1, \end{aligned}$$

where $\Delta\hat{x}^h_k = (\Delta x^{ha}(s^a_k))_{s^a_k \in T(s^{a*}_k)})$. Since $D_{(x^h_2,\lambda^h_2)} f^h_2$ is invertible and $(\pi \otimes p)(2)\Delta e^h = p(s_{02})\Delta e^{h0} = 0$, we get $(\Delta x^{ha}(s^a_2), \Delta\lambda^h_2) = 0$. Hence, we just consider the equations with $k = 1$. Transform the first equations into

$$\Delta x^{ha}(s^a_1) = H^{-1}(s^a_1)p(s^a_1)^T\Delta\lambda^h_1$$

Premultiplying these by $\pi(s^a_1)p(s^a_1)$ and summing them up we get

$$(\pi \otimes p)(1)\Delta x^h_1 = \Delta\lambda^h_1 \sum_a \sum_{s^a_1 \in S^a} \pi(s^a_1)[p(s^a_1)H^{-1}(s^a_1)p(s^a_1)^T] = \Delta w$$

The terms $Q(s^a_1)$ are negative since $H^{-1}(s^a_1)$ is a negative definite matrix. Letting $Q^* = \sum_a \sum_{s^a_1 \in S^a} \pi(s^a_1)Q(s^a_1)$, we have $\Delta\lambda^h_1 = \Delta w/Q^*$. Hence,

$$(\pi_+ \otimes p)(1)\Delta\hat{x}^h(1) = \Delta\lambda^h_1 \sum_{s^a_1 \in T(s^{a*}_1)} \frac{\pi(s^a_1)}{p_1(s^{a*}_1)\pi(s^{a*}_1)} Q(s^a_1) = 1$$

if and only if

$$\Delta w = Q^* / \sum_{s_1^a \in T(s_1^{a^*})} \frac{\pi(s_1^a)}{p_1(s_1^{a*})\pi(s_1^{a^*})} Q(s_1^a)$$

proving the result.■

Next, we need to consider cases where previous endowment perturbations are useless, i.e., where $s_{01} = s_{02} = \bar{s}$ and $p(s_{01}) = p(s_{02})$. We first show that under some conditions on prices, multipliers and probabilities —beyond $s_{01} = s_{02} = \bar{s}$ and $p(s_{01}) = p(s_{02})$— $D_{(x^h, \lambda^h, \omega^h)}(F^h, W^h)$ has full rank: it is condition C1 below. We then show that C1 cannot be violated at $(F^h, W^h)(\cdot; \omega^h) = 0$ in a dense set of the parameter space.

To this end, we need to introduce gradient perturbations as follows. For each $a$, let

$$X^{ha} = \{x^{ha} : x^{ha} = x^{ha}(s_k^a), \text{ for some } s_k^a \in S_k^a \text{ and some } k \in \{1, 2\}\}$$

$X^{ha}$ is a finite set. Then, for $x_t^h \in X^{ha}$, pick a pair of open balls $B_{\varepsilon_t}(x_t^h), B_{\hat{\varepsilon}_t}(x_t^h)$ centered around $x_t^h$ and such that: 1) $B_{\varepsilon_t}(x_t^h) \subset B_{\hat{\varepsilon}_t}(x_t^h(s^*))$; 2) $\cap_t cl B_{\hat{\varepsilon}_t}(x_t^h) = \varnothing$. Also, pick smooth bump functions $\Phi_t$ such that $\Phi_t(x) = 1$, for $x \in Cl B_{\varepsilon_t}(x_t^h)$ and $\Phi_k(x) = 0$, for $x \notin B_{\hat{\varepsilon}_t}(x_t^h)$. For given vectors $(\Delta D u_t^{ha})_{t=1}^T \in \mathbb{R}^{TC}$, with norm $||(\Delta D u_t^{ha})_{t=1}^{X_k}||$ arbitrarily close to zero, the perturbed utility function

$$\hat{u}^{ha^*}(x) = u^{ha^*}(x) + \sum_{t=1}^{X^{ha}} \Phi_t(x)(\Delta D u_t^{ha} x)$$

is arbitrarily close to $u^{ha^*}(x)$ in the $C^2$ topology, and therefore it satisfies all the maintained assumptions. This parametrization allows for independent perturbations of the utility function on the disjoint sets $B_\varepsilon(x_t^h)$. We identify the partial derivatives of the map $\left(F^h, W^h\right)$ with respect to $u^{ha}$ with the partial derivatives of $D\widehat{u}^{ha}$ with respect to $(\Delta D u_t^{ha})_{t=1}^T$ taken at $(\Delta D u_t^{ha})_{t=1}^T = 0$ (i.e., at $D\widehat{u}^{ha} = D u^{ha}$). Individual $(h, a)$'s utility gradient is then perturbed independently around each point $x_t^h \in X^{ha}$.

We also introduce some further notation. Identify $x_t^{ha} \in X^{ha}$ with a pair $(a, t)$. For each such pair, by the first order conditions $\lambda_{k'}^h p(\bar{s}_{k'}^a) = \lambda_k^h p(\bar{s}_k^a)$, for $k \neq k'$ if $\bar{s}_k^a, \bar{s}_{k'}^a \in \{s_k^a : x^{ha}(s_k^a) = x_t^{ha}\}$. Hence, let $p(a, t) \equiv p(\bar{s}_k^a)/\alpha_k$, with $\alpha_1 = 1$, and $\alpha_2 = \frac{\lambda_1^h}{\lambda_2^h} > 0$, and $H(a, t) \equiv H(\bar{s}_k^a)$ for all $\bar{s}_k^a \in \{s_k^a : x^{ha}(s_k^a) =$

$x_t^{ha}\}$. Note that $H^{-1}(a,t)$ exists. Further, let

$$M_k^h(a,t) \equiv \sum_{s_k^a \in \{\hat{s}_k^a | x^{ha}(\hat{s}_k^a) = x_t^{ha}\}} \pi(s_k^a)$$

and $c(a,t) \equiv p(a,t)H^{-1}(a,t)\Delta Du_t^{ha}$. Finally, for $a \geq a^*$ and $x_t^{ha} \in X^{ha}$, let $S_{k+}(t) = \{s_k^a \in T(s_k^{a^*}) \mid x^{ha} = x^{ha}(s_k^a)\}$ and let

$$\Pi_k(a,t) \equiv \sum_{s_k^a \in S_{k+}(t)} \frac{\pi(s_k^a)}{\pi(s_k^{a^*})}.$$

The condition we will use to establish transversality is

**C1** There exist scalars $c(a,t)$, $a \leq G$, $t = 1,..,X^{ha}$ such that

$$[\sum_{a,t} M_1^h(a,t)c(a,t)] \sum_{s_1^a \in T(s_1^{a^*})} \frac{\pi(s_1^a)}{p_1(s_1^{a*})\pi(s_1^{a*})} \frac{Q(s_1^a)}{Q_1} - \sum_{a \geq a^*,t} \frac{\Pi_1(a,t)c(a,t)}{p_1(s_1^{a^*})}$$
$$\neq [\sum_{a,t} M_2^h(a,t)c(a,t)] \sum_{s_2^a \in T(s_2^{a^*})} \frac{\pi(s_2^a)}{p_1(s_2^{a*})\pi(s_2^{a*})} \frac{Q(s_2^a)}{Q_2} - \sum_{a \geq a^*,t} \frac{\Pi_2(a,t)c(a,t)}{p_1(s_2^{a^*})}.$$

**Lemma 8** *Suppose that $s_{01} = s_{02} = \bar{s}$ and $p(s_{01}) = p(s_{02})$. Then $D_{(x^h,\lambda^h,\omega^h)}(F^h, W^h)$ has full rank if and only if C1 holds.*

**Proof.** We have $D_{(x^h,\lambda^h,\omega^h)}(F^h, W^h) =$

$$\begin{bmatrix} H_1 & -p(1)^T & D_{\omega^h}F_{h.1} & 0 & 0 \\ -(\pi \otimes p)(1) & 0 & 0 & 0 & 0 \\ 0 & 0 & D_{\omega^h}F_{h,2} & H_2 & -p(2)^T \\ 0 & 0 & 0 & -(\pi \otimes p)(2) & 0 \\ (0,(\pi_+ \otimes p)(1)) & 0 & 0 & -(0,(\pi_+ \otimes p)(2)) & 0 \end{bmatrix}$$

Here $D_{\omega^h}F_{h.k}$ is matrix of dimension $C(\sum_{a=0}^G S^a) \times C(\sum_a X^{ha})$, that is, the number of rows is equal to the number of equations $(1,k,s_k^a)$, while the number of columns is equal to the number of independent perturbations, which coincide with the number of distinct vectors $x^{ha}(s_k^a)$. The entries of column $\omega^h = x^{ha}$ of the matrix $D_{\omega^h}F_{h.k}$ are

$$D_{x^{ha}} F_{h.k}(x^{ha}(s_k^a)) = \begin{cases} 0 & \text{if } (a, s_k^a) : x^{ha}(s_k^a) \neq x^{ha}, \\ I_C & \quad \textit{otherwise.} \end{cases}$$

As is well-known, $D_{(x_k^h, \lambda_k^h)} F_k^h$ is invertible and, as it is obvious, $D_{(x_k^h, \lambda_k^h)} F_{k'}^h = 0$ for $k \neq k'$. Hence, to show that $D_{(x^h, \lambda^h, \omega^h)}(F^h, W^h)$ has full row rank, it is enough to prove that there exist vectors $\Delta x_k^h = (\Delta x_k^{ha}(s_k^a))_{a, s_k^a}, \Delta D u_t^{ha}$ and scalars $\Delta \lambda_k^h$ such that

$$\begin{aligned} H(s_k^a)\Delta x^{ha}(s_k^a) - p(s_k^a)^T \Delta\lambda_k^h - \Delta D u_t^{ha} &= 0, \text{ for } x^{ha}(s_k^a) = x_t^{ha} \\ (\pi \otimes p)(k)\Delta x_k^h &= 0 \\ (\pi_+ \otimes p)(1)\Delta\hat{x}^h(1) - (\pi_+ \otimes p)(2)\Delta\hat{x}^h(2) &= 1. \end{aligned}$$

where $\Delta \hat{x}_k = (x^{ha}(s_k^a))_{s^a \in T(s_k^{a^*})})$. Transform the first equations into

$$\Delta x^{ha}(s_k^a) = H_k^{-1}(s_k^a)[p(s_k^a)^T \Delta\lambda_k^h - \Delta D u_t^{ha}] \tag{1}$$

Premultiplying equations (1) by $\pi(s_k^a)p(s_k^a)$ and summing them up we get

$$0 = (\pi \otimes p)(k)\Delta x_k^h = \alpha_k \Delta\lambda_k^h \sum_{a,t} M_k^h(a,t) p(a,t) H^{-1}(a,t) p_k(a,t)^T -$$
$$-\alpha_k \sum_{a,t} M_k^h(a,t) p(a,t) H^{-1}(a,t) \Delta D u_t^{ha}] = 0$$

By the negative definiteness of $H^{-1}$ we have $Q_k < 0$, for $k = 1, 2$. The last equation implies

$$\Delta\lambda_k^h = \frac{\sum_{a,t} M_k^h(a,t) c(a,t)}{Q_k}. \tag{2}$$

Then we get

$$\begin{aligned} (\pi_+ \otimes p)(1)\Delta\hat{x}^h(1) &= \Delta\lambda_1^h \sum_{s_1^a \in T(s_1^{a^*})} \frac{\pi(s_1^a)}{p_1(s_1^{a*})\pi(s_1^{a^*})} Q(s_1^a) - \sum_{a \geq a^*, t} \frac{\Pi_1(a,t)c(a,t)}{p_1(s_1^{a^*})} \\ (\pi_+ \otimes p)(2)\Delta\hat{x}^h(2) &= \Delta\lambda_2^h \sum_{s_2^a \in T(s_2^{a^*})} \frac{\pi(s_2^a)}{p_1(s_2^{a*})\pi(s_2^{a^*})} Q(s_2^a) - \sum_{a \geq a^*, t} \frac{\Pi_2(a,t)c(a,t)}{p_1(s_2^{a^*})}. \end{aligned}$$

Hence, $D_{(x^h,\lambda^h,\omega^h)}(F^h, W^h)$ has full rank if and only if $(\pi_+ \otimes p)(1)\Delta\hat{x}^h(1) \neq (\pi_+ \otimes p)(2)\Delta\hat{x}^h(2)$ for some choice of scalars $c(a,t)$, $a \leq G$, $t = 1,..,X^{ha}$. Exploiting (2), the rank of $D_{(x^h,\lambda^h,\omega^h)}(F^h, W^h)$ is then full if and only if C1 holds.$\blacksquare$

To show that C1 holds in a dense subset of parameters, we characterize C1 in equivalent but more convenient terms.

**Lemma 9** *Let $s_{0k}$ be $k$-invariant. C1 does not hold if and only if $p_1(s_k^{a^*})$ and $\lambda_k^h$ are $k$-invariant.*

**Proof.** Pick $x_{\hat{t}}^{h\widehat{a}} \in X^{h\widehat{a}}$ such that $x_{\hat{t}}^{h\widehat{a}} \neq x^{ha}(\hat{s}_k^a)$ for all $\hat{s}_k^a \in T(s_k^{a^*})$ — there always exists such an $x_{\hat{t}}^{h\widehat{a}}$, for instance $x^{h0}$. Then, C1 does not hold if and only if

$$M_1^h(\widehat{a},\widehat{t}) \sum_{s_1^a \in T(s_1^{a^*})} \frac{\pi(s_1^a)}{p_1(s_1^{a*})\pi(s_1^{a*})} \frac{Q(s_1^a)}{Q_1} = M_2^h(\widehat{a},\widehat{t}) \sum_{s_2^a \in T(s_2^{a^*})} \frac{\pi(s_2^a)}{p_1(s_2^{a*})\pi(s_2^{a*})} \frac{Q(s_2^a)}{Q_2}$$

In particular, the latter implies: a) since this can be done for $\hat{a} = 0$, $x^{h0}(s_{01}) = x^{h0}(s_{02})$, otherwise, $1 = M_1^h(0,\hat{t}) \neq M_2^h(0,\widehat{t}) = 0$ for $\hat{t}$ with $x_{\hat{t}}^{h0} = x^{h0}(s_{01})$, violating the absence of C1; hence, from the first order condition, $\lambda_1^h = \lambda_2^h$; b) since ($\#X^{h0} = 1$ and) $M_k^h(0,1) = 1$,

$$\sum_{s_1^a \in T(s_1^{a^*})} \frac{\hat{\pi}(s_1^a)}{p_1(s_1^a)} \frac{Q(s_1^a)}{Q_1} = \sum_{s_2^a \in T(s_2^{a^*})} \frac{\hat{\pi}(s_2^a)}{p_1(s_2^a)} \frac{Q(s_2^a)}{Q_2} \equiv \overline{Q};$$

c) finally, $M_1^h(a,t) = M_2^h(a,t)$ for all $(a,t)$ such that $x_t^{ha} \neq x^{ha}(\hat{s}_k^a)$ for all $\hat{s}_k^a \in T(s_k^{a^*})$, $k = 1,2$.

If $x^{ha}(s_1^{a*}) = x^{ha}(s_2^{a^*})$, since $\lambda_1^h = \lambda_2^h$, obviously $p_1(s_1^{a^*}) = p_1(s_2^{a^*})$.

If $x^{ha}(s_1^{a*}) \neq x^{ha}(s_2^{a^*})$, denote by $(a^*, k)$ the elements in $X^{ha^*}$ associated with $x^{ha^*}(s_k^{a^*})$. Then, Condition C1 does not hold if and only if

$$[M_1^h(a^*,1) - M_2^h(a^*,1)]\overline{Q} = \frac{1}{p_1(s_1^{a^*})} \quad (3)$$

$$[M_1^h(a^*,2) - M_2^h(a^*,2)]\overline{Q} = \frac{1}{p_1(s_2^{a^*})} \quad (4)$$

However, since $s_{01} = s_{02}$, for any given $a$, $\sum_{s_k^a \in S^a} \pi(s_k^a)$ is $k$-invariant and, hence, $\sum_t M_k^h(a,t)$ is $k$-invariant. Thus, (c) implies that

$$\sum_t M_1^h(a^*,t) - \sum_t M_2^h(a^*,t) = M_1^h(a^*,1) + M_1^h(a^*,2) - M_2^h(a^*,1) - M_2^h(a^*,2) = 0$$

Together with equations (3) and (4), this implies that $p_1(s_1^{a^*}) = p_1(s_2^{a^*})$.■

Next, let

$$P^a = \{\hat{p} \in R_+^C : p(s_k^a) = \hat{p}, \text{ for some } s_k^a, \text{ some } k\}.$$

For each $\hat{p}_t \in P^a$, let

$$M_k^p(a,t) \equiv \sum_{\hat{s}_k^a \in \{s_k^a : p(s_k^a) = \hat{p}_t\}} \pi(s_k^a).$$

The following lemma collects some useful observations. Let $r^h(k) = (\pi \otimes p)(k)e^h$.

**Lemma 10** *Let $s_{0k}$ be $k$-invariant. i) For each $h$, $M_k^p(a,t) = M_k^h(a,t)$; ii) if $M_k^p(a,t) = M^p(a,t)$ for all $(a,t)$, then $\lambda_k^h$ is strictly decreasing in $r^h(k)$; iii) if $r^h(k)$ is $k$-invariant for all $h$, $F(\cdot) = 0$ cannot have a solution.*

**Proof.** i) This is an immediate consequence of the state invariance and strict concavity of $u^{ha}$.

ii) Let $M_k^p(a,t) = M^p(a,t)$, for all $(a,t)$. From i), $(..., x_k^{ha}(a,t), ...)$ is an optimal solution to the programming problems, for $k = 1, 2$,

$$\begin{array}{c} \max \sum_{a,t} M^p(a,t) u^{ha}(x_t^{ha}) \qquad s.t. \\ \sum_{a,t} M^p(a,t) p(a,t) x_k^{ha}(a,t) = r^h(k). \end{array} \tag{5}$$

To prove the strict monotonicity claims we take the derivatives of the various items with respect to $r^h(k)$. In order to do that we consider the first order conditions associated to the programming problem (5), whose left-hand side we denote by $\hat{f}_k^h$. Then we have

$$D_{x_k^h, \lambda_k^h, r(k)} \hat{f}_k^h = \begin{bmatrix} H_k & -p(k)^T & 0 \\ (M^p \otimes p) & 0 & -1 \\ 0 & 0 & D_{\omega^h} F_{h,2} \end{bmatrix}$$

Computations virtually identical to the one already performed yield, using the same notation,

$$\frac{d\lambda_k^h}{dr^h(k)} = \frac{1}{Q_k} < 0.$$

iii) Just observe that if $r^h(k)$ is $k$-invariant for all $h$, then $\lambda_k^h$ is $k$-invariant for all $h$, and inequalities $(NC.n)$ are violated, a contradiction.■

Thus, for the rank of $D_{(x^h,\lambda^h,\omega^h)}(F^h, W^h)$ to collapse, or C1 not to hold, it must be that

$$\lambda_1^h = \lambda_2^h. \tag{6}$$

We are going to show that (6) cannot be verified in a dense set of parameters. Then, by Lemma 9 C1 must hold in the same dense set, and by Lemma 8 rank of $D_{(x^h,\lambda^h,\omega^h)}(F^h, W^h)$ will be full, as desired. To show that (6) does not hold, we show that the system of equations $(1,h)$, $(2,h)$, for all $h$, and $(4,h)$ for $h \notin \mathcal{H}_1$, but $\lambda_1^h = \lambda_2^h$, for $h \in \mathcal{H}_1$, does not have a solution, where $\mathcal{H}_1$ is the set of individuals such that $\lambda_1^h = \lambda_2^h$. We denote this system by $\hat{F}_{\mathcal{H}_1}$. Note that by Lemma 9 inequality $(NC.n)$ now becomes

$$\left\| \left(\lambda_1^h\right)_{h\notin\mathcal{H}_1} - \left(\lambda_2^h\right)_{h\notin\mathcal{H}_1} \right\| \geq \frac{1}{n}. \tag{NC-H$_1.n$}$$

**Lemma 11** *$D_{(x^h,\lambda^h,\omega^h)}\hat{F}^h_{\mathcal{H}_1}$ has full rank.*

**Proof.** We divide the proof in two possible cases:
1) $M_k^p(a,t)$ is $k$-invariant for all $(a,t)$;
2) $M_1^p(a,t) \neq M_2^p(a,t)$ for some $(a,t)$.

Case 1. For $h \notin \mathcal{H}_1$, since $\lambda_1^h \neq \lambda_2^h$ and $\hat{F}^h_{\mathcal{H}_1} = (F^h, W^h)$, C1 holds and $D\hat{F}^h_{\mathcal{H}_1}$ is surjective. Since $M_k^p$ is $k$-invariant, while $\lambda_k^h$ is not, by Lemma 10.ii $r^h(1) \neq r^h(2)$ for $h \notin \mathcal{H}_1$. Without loss of generality, let $1 \notin \mathcal{H}_1$. By Lemma 10.ii and since $M_k^p$ is $k$-invariant, $r^h(1) = r^h(2)$ for $h \in \mathcal{H}_1$. Pick an arbitrary $h \in \mathcal{H}_1$ and consider perturbation $\Delta e^h = \mu e^1$, for some $\mu > 0$. Then, $\Delta r^h(1) = (\pi \otimes p)(1)\mu e^1 = \mu r^1(1)$. For $h \in \mathcal{H}_1$, the Jacobian of $\hat{F}^h_{\mathcal{H}_1}$ is:

$$D_{(x^h,\lambda^h,\omega^h)}\hat{F}^h_{\mathcal{H}_1} = \begin{bmatrix} H_1 & -p(1)^T & 0 & 0 & 0 \\ -(\pi\otimes p)(1) & 0 & \mu r^1(1) & 0 & 0 \\ 0 & 0 & & H_2 & -p(2)^T \\ 0 & 0 & \mu r^1(2) & -(\pi\otimes p)(2) & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

By the usual argument, we need to show that for some $(\Delta x^h, \Delta\lambda^h, \Delta e^h)$ the following system of equations has a solution:

$$\begin{aligned} H_k(s_k^a)\Delta x^{ha}(s_k^a) - p(s_k^a)^T\Delta\lambda_k^h &= 0, \\ (\pi\otimes p)(k)\Delta x_k^h &= r^1(k)\mu \\ \Delta\lambda_1^h - \Delta\lambda_2^h &= 1. \end{aligned}$$

By Lemma 10.i, since $M_k^p$ is $k$-invariant, $M_k^h$ is $k$-invariant for $h \in \mathcal{H}_1$, and hence $Q_1 = Q_2 = Q$. Then, the usual computations show that $\Delta\lambda_k^h = \frac{r^1(k)}{Q}\mu$ and hence

$$\Delta\lambda_1^h - \Delta\lambda_2^h = \frac{r^1(1) - r^1(2)}{Q}\mu.$$

Then, $r^1(1) \neq r^1(2)$ implies the conclusion.

Case 2. Let $h \in \mathcal{H}_1$. Perturb the utility function $u^{ha}$ around $x_t^{ha}$ and $u^{h0}$ around $x^{h0} = x^{h0}(s_{01}) = x^{h0}(s_{02})$ as already illustrated. The Jacobian of $\hat{F}_{H_1}^h$ is:

$$D_{(x^h,\lambda^h,\omega^h)}\hat{F}_{\mathcal{H}_1}^h = \begin{bmatrix} H_1 & -p(1)^T & -D_{u^{ha}}\hat{F}_{\mathcal{H}_1,1}^h & 0 & 0 \\ (\pi\otimes p)(1) & 0 & 0 & 0 & 0 \\ 0 & 0 & -D_{u^{ha}}\hat{F}_{\mathcal{H}_1,2}^h & H_2 & -p(2)^T \\ 0 & 0 & 0 & (\pi\otimes p)(2) & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

where $D_{u^{ha}}\hat{F}_{\mathcal{H}_1,k}^h$ is a matrix of dimension $C(\sum_{a=0}^G S^a) \times C$, with entries:

$$D_{u^{ha}}\hat{F}_k^h = D_{u^{ha}}F_k^h(x^{ha}(s_k^a)) = \begin{cases} 0 & \text{if } a > 1, \text{ and } s_k^a \text{ is such that } x^{ha}(s_k^a) \neq x_t^{ha} \\ I_C & \text{otherwise.} \end{cases}$$

By the usual argument, we need to show that for some $(\Delta x^h, \Delta\lambda^h, \Delta Du^h)$ the following system of equations has a solution:

$$\begin{aligned} H_k(s_k^{a'})\Delta x^{ha}(s_k^{a'}) - p(s_k^{a'})^T\Delta\lambda_k^h &= \Delta Du^{ha}(a,t), \text{ if } x^{ha}(s_k^{a'}) = x_t^{ha}; \\ H_k(s_k^{a'})\Delta x^{ha}(s_k^{a'}) - p(s_k^{a'})^T\Delta\lambda_k^h &= 0, \text{ if } x^{ha}(s_k^{a'}) \neq x_t^{ha}; \\ H_k(\bar{s})\Delta x^{h0} - p(\bar{s})^T\Delta\lambda_k^h &= \Delta Du^{h0}(0,1), \\ (\pi\otimes p)(k)\Delta x_k^h &= 0, \\ \Delta\lambda_1^h - \Delta\lambda_2^h &= 1. \end{aligned}$$

The by-now usual computations show that

$$\Delta\lambda_k^h = \frac{M_k^h(a,t)p(a,t)H_k^{-1}(a,t)\Delta Du^{ha}(a,t) + p(0,1)H_k^{-1}(0,1)\Delta Du^{h0}(0,1)}{Q_k}.$$

By assumption, $H_k^{-1}(a,t)$ is $k$-invariant, while $M_k^h(a,t)$ is not. Thus, there exists $\Delta Du^h$ such that $\Delta\lambda_2^h = 0$, while $\Delta\lambda_1^h > 0$. ■

# References


[1] Balasko, Y., Shell, K., The Overlapping-Generations model: I. The case of pure exchange without money. Journal of Economic Theory **23**, 281-306 (1980)

[2] Cass, D., Green, R., Spear, S., Stationary equilibria with incomplete markets and overlapping generations. International Economic Review **33**, 495-512 (1992)

[3] Citanna, A., Siconolfi, P., On the nonexistence of recursive equilibrium in stochastic OLG economies. Mimeo, HEC – Paris, 2006.

[4] Constantinides, G., Donaldson, J., Mehra, R., Junior can't borrow: a new perspective on the equity premium puzzle. Quarterly Journal of Economics **117**, 269-276 (2002)

[5] Duffie, D., Geanakoplos, J., Mas-Colell, A., McLennan, A., Stationary Markov equilibria. Econometrica **62**, 745-781 (1994)

[6] Geanakoplos, J., Magill, M., Quinzii, M., Demography and the long-run predictability of the stock market. Brookings Papers on Economic Activity **1**, 241-307 (2004)

[7] Geanakoplos, J., Polemarchakis, H., Existence, Regularity and Constrained Suboptimality of Competitive Allocations When The Asset Market Is Incomplete. In: Starrett, D., Heller, W., Starr, R. (eds.), Uncertainty, Information and Communication: Essays in Honor of K. Arrow, Vol. III, 65-96. Cambridge: Cambridge University Press 1986

[8] Gottardi, P., Stationary monetary equilibria in OLG models with incomplete markets. Journal of Economic Theory **71**, 75-89 (1996)

[9] Kubler, F., Polemarchakis, H., Stationary Markov equilibria for overlapping generations. Economic Theory **24**, 623-643 (2004)

[10] Kubler, F., Schmedders, K., Approximate versus exact equilibria in dynamic economies. Econometrica **73**, 1205-35 (2005).

[11] Mas-Colell, A., Nachbar, J., On the finiteness of the number of critical equilibria, with an application to random selections. Journal of Mathematical Economics **20**, 397-409 (1991)

[12] Radner, R., Existence of equilibrium of plans, prices, and price expectations in a sequence of markets. Econometrica **40**, 289-303 (1972)

[13] Rios Rull, J.V., Life-cycle economies and aggregate fluctuations. Review of Economic Studies **63**, 465-489 (1996)

[14] Samuelson, P., An exact consumption-loan model of interest without the social contrivance of money. Journal of Political Economy **66**, 467-482 (1958)

[15] Spear, S., Rational expectations equilibria in the OLG model. Journal of Economic Theory **35**, 251-275 (1985)

[16] Spear, S., Srivastava, S., Markov rational expectations equilibria in an OLG model. Journal of Economic Theory **38**, 35-62 (1986)

[17] Storesletten, S., Telmer, C., Yaron, A., Consumption and risk sharing over the life cycle. Journal of Monetary Economics **51**, 609-633 (2004a)

[18] Storesletten, S., Telmer, C., Yaron, A., Cyclical dynamics in idiosyncratic labor-market risk. Journal of Political Economy **112**, 695-717 (2004b)