

# Revealed Willpower

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Preliminary Draft

## Abstract

In this paper we are interested in the behavior of individuals who have imperfect control over their immediate urges. Willpower as cognitive resource model has been proposed by experimental psychologists and has also been used by economists to explain various behavioral paradoxes. This paper complements this literature by providing a foundation for the choice behavior that can be represented by the limited willpower model. Working with choices over menus of alternatives, we behaviorally characterize a representation that reflects the behavior of a decision maker who has cognitive limitation in exercising self restraint. In our model, there are only two periods. We observe the agent's ranking of menus of alternatives at time 0. From this ranking, we derive a representation that reveals the agent's choices, and hence the willpower problems that she believes she will be facing, in period 1. This two-period model captures key behavioral traits of willpower constrained decision making, and leads to a simple characterization which is mathematically challenging to derive.

## 1 Introduction

People have limited ability for self control when they face tempting alternatives resulting in procrastination, preference for small immediate rewards despite their important future

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costs, and preference for commitment. One plausible explanation of these behaviors is that the ability to self-regulate, or willpower, is a valuable cognitive resource with limited stock. Choosing an alternative when other more tempting alternatives are available depletes the willpower stock and only those alternatives that do not entirely deplete the willpower stock can feasibly be chosen. For example, the possibility of relaxing in front of the TV might make working infeasible and lead to procrastination. Procrastination would be more severe when the agent's willpower stock is already depleted – say at the end of a difficult day. Commitment eliminates tempting alternatives and makes it easier to make choices and is therefore desirable. In this paper we aim to provide a choice theoretic foundation for the willpower as a cognitive resource model.

Willpower as cognitive resource model has been proposed by experimental psychologists (Baumeister and Vohs, 2003, Vohs and Faber, 2004, Muraven et al., 2006, Vohs and Faber, 2004 and Dewitte et al., 2005) and has also been used by economists to explain various behavioral paradoxes (Ozdenoren, Salant and Silverman, forthcoming, Fudenberg and Levine, 2010.) Our paper complements this literature by fully characterizing the choice behaviour that can be represented by the limited willpower model. As a first step, in this paper we focus on choices among menus of alternatives.<sup>1</sup> In our model, there are only two periods. We observe the agent's ranking of menus of alternatives at time 0. From this ranking, we derive a representation that reveals the agent's choices, and hence the willpower problems that she believes she will be facing, in period 1. As we will see the two-period model captures key behavioral traits of willpower constrained decision making, and leads to a simple characterization which is mathematically challenging to derive.

Formally, we assume that the decision maker (DM) faces at most finitely many distinct alternatives and has a complete and transitive preference relation  $\succeq$  over subsets of these alternatives which are also called menus. We call the preference relation restricted to singleton menus the agent's commitment preference. Since the DM's commitment preference is also complete and transitive, we represent it by a utility function  $u$ . In the standard economic

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<sup>1</sup>Ozdenoren, Salant and Silverman, forthcoming, and Fudenberg and Levine (2010) study decision making by willpower constrained agents in a dynamic setting. In those models agents take into account willpower consequences of their current choices for their future choices.

models DM's have no willpower problem and facing a menu  $A$ , they choose the alternative that maximizes  $u$  from the menu. An agent who has limited willpower also maximizes  $u$  but she faces a constraint. Suppose that each alternative,  $x$ , has a temptation value  $v(x)$  and the DM's willpower stock is  $w \geq 0$ . The DM is able to consider an alternative  $x$  in  $A$  only if  $\max_{y \in A} v(y) - v(x) \leq w$ . Otherwise, the DM does not have enough willpower to choose this alternative. Notice that the amount of willpower needed to choose an alternative depends on the menu from which the alternative is chosen. This is because willpower depletion not only depends on the temptation value of the chosen alternative but also the temptation value of the most tempting alternative in the menu. As a concrete example consider an individual who needs to choose among going to the gym, reading a book or watching TV. In our model, the DM can choose any alternative if she could simply commit to that alternative, i.e., choose that alternative from a singleton menu. Let's assume that the agent's commitment preferences are such that she prefers going to the gym over reading a book, and reading a book over watching TV. It is possible that she, absent the ability to commit, does not have enough willpower to go to the gym when watching TV is an option but she has enough willpower to read a book. Therefore, when all three alternatives are available DM ends up reading a book.

We characterize this model using three axioms: irrelevance, set betweenness and consistency. Irrelevance says that an alternative that is neither chosen nor most tempting can be dropped without affecting the attractiveness of the menu. Set Betweenness says that combining two menus leaves the preference ranking of the combined menu between its two pieces. Intuitively, the combined menu is weakly worse than the better menu because it might contain alternatives that are more tempting and better than the worse menu because it might contain alternatives with a higher commitment ranking. Our final axiom, Consistency, has two parts. The first part says roughly that alternatives that are infeasible when a less tempting alternative is available should also be infeasible when a more tempting alternative is available in its place. The second part of Consistency says that any alternative that makes a more tempting alternative infeasible should also make a less tempting alternative infeasible.

Our main theorem is that a complete and transitive preference relation  $\succeq$  on menus of

alternatives satisfies Irrelevance, Set Betweenness and Consistency if and only if  $\succsim$  can be represented by

$$U(A) = \max_{x \in A} u(x) \text{ subject to } \max_{y \in A} v(y) - v(x) \leq w$$

where  $A$  is a menu of alternatives,  $u$  and  $v$  are functions over outcomes and  $w \geq 0$ . We call preferences with the above representation willpower as cognitive resource preferences, or willpower preferences for short. The two extreme cases of willpower as cognitive resource preferences are the standard utility maximization model and the Strotz model. When  $w$  is very large the willpower constraint is no longer binding and the DM becomes the standard economic agent who simply maximizes  $u$ . On the other hand, as  $w$  becomes small the DM has Strotz preferences where she considers only the alternatives that maximize  $v$  and among those chooses the one that maximizes  $u$ .

A closely related literature is based on the costly self control model of Gul and Pesendorfer (2001). A more directly related paper is Gul and Pesendorfer (2005). We present a comparison of the limited willpower and the costly self control models in Section 3.1.

As mentioned above willpower model was proposed by psychologists (for example Baumeister *et al.*, 1994; Baumeister and Vohs, 2003) who have experimentally demonstrated that individuals depleted by prior acts of self-restraint tend to behave later as if they have less self-control. In Section 3.2, we discuss the implication of our results for the interpretation of this class of experiments. Broadly speaking, our results suggest a more nuanced interpretation and need for further experiments to separate various models of self control and willpower.

## 2 Model

Here we investigate the behavior of a DM who has cognitive limitation in exercising self restraint in a menu choice setup.<sup>2</sup> Specifically, given a menu, DM may end up with an alternative which is different from the alternative that she would commit to if she had the ability to commit, because the best alternative even when physically available may not be

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<sup>2</sup>Cognitive limitation can also capture coming from complexity of a choice problem. In this paper, when we refer to cognitive limitation we mean limitations coming from exercising self restraint.

cognitively available. Let  $X$  be the finite set of potentially available alternatives. Menus are non-empty subsets of  $X$ . Let  $\succsim$  represent the DM's preferences over menus.

To get some intuition for how cognitive limitation affects choices, consider the following example with three alternatives: going to the gym ( $g$ ), reading a book ( $b$ ), and watching TV ( $t$ ). Assume that commitment preferences are such that  $\{g\} \succ \{b\} \succ \{t\}$ , so going to the gym is best for the decision maker and watching TV is worst. Suppose  $\{g, b, t\} \sim \{b\} \succ \{t\} \sim \{g, t\}$ . Our interpretation is that even though going to the gym is the best alternative, DM is not able to choose this action when the tempting alternative watching TV is available. However, reading a book is a compromise that the DM can choose even when watching TV is an available alternative.

Formally, our model is

$$U(A) = \max_{x \in A} u(x) \text{ subject to } \max_{y \in A} v(y) - v(x) \leq w$$

where  $u : X \rightarrow R$  represents  $\succsim$  over singleton menus (commitment preference),  $v : X \rightarrow R$  captures the temptation value and  $w$  is the willpower stock.

Our first axiom is standard.

**Axiom 1** *Preference Relation:*  $\succsim$  is complete and transitive.

To understand our next axiom, Irrelevance, consider the earlier example where DM is choosing between going to the gym, reading a book and watching TV. Committing to going to the gym is strictly better than having the all three options because when watching TV is available going to the gym is not feasible. Clearly, going to the gym is not the most tempting alternative either (since otherwise, it would have been feasible to choose it.) Irrelevance says that given that it is neither chosen nor the most tempting alternative, going to the gym is irrelevant in the sense that dropping it from the menu should not affect the decision maker's ranking of the menu.<sup>3</sup>

**Axiom 2** *Irrelevance:* If  $x \succ A \cup x$ , then  $A \sim A \cup x$ .

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<sup>3</sup>Implicit here is that a menu is as good as the alternative chosen from it. This is indeed implied by Irrelevance and our next axiom, Set Betweenness. In contrast, in costly self control models a menu can be strictly worse than the alternative chosen from it.

To understand the next axiom, Set Betweenness, let  $A = \{\text{going to the gym, reading a book}\}$  and  $B = \{\text{watching TV}\}$ . Notice that having a choice between going to the gym or reading a book is strictly better than watching TV, ie.  $A \succeq B$ . Clearly having all three options is weakly better than watching TV which is the worst alternative so  $A \cup B \succsim B$ . Recall, when all three alternatives are available, DM ends up reading a book. However, when the alternatives are going to the gym and reading a book the worst that can happen is reading a book and it might even be possible to go to the gym. Set Betweenness generalizes the idea behind this example.

**Axiom 3** *Set Betweenness*:  $A \succsim B \Rightarrow A \succsim A \cup B \succsim B$

An implication of the first three axioms is that for every  $A$ , there exists  $x_A \in A$  such that  $A \sim \{x_A\}$ . We interpret this to mean that each menu is indifferent to the the alternative that is chosen from the menu. In our model, unchosen alternatives constrain the set of feasible coices but they do not create a direct utility cost. One implication is that a willpower constrained DM would not pay to commit to an alternative that she chooses. Commitment is useful only because it enables her to choose alternatives that are otherwise infeasible. In contrast, in costly self control models agents may pay to commit to alternatives that they are able to choose. We view this as one way of distinguishing between these models in the labaratory or the field.

To explain our third axiom, suppose that in addition to watching TV, reading a book, and going to the gym, DM has a fourth option, taking a walk. Suppose that when watching TV is available, DM cannot take a walk,  $\{w\} \succ \{w, t\}$ . Moreover, having the option to read a book in addition to watching TV is better than having watching TV as the only option,  $\{b, t\} \succ \{t\}$  (presumably because than the DM exercises the option and reads a book instead of watching TV.) We interpret this to mean that reading a book is a more tempting option than taking a walk. Our third axiom, Consistency, says that if going to the gym is not feasible when taking a walk is available, then between reading a book and going to the gym, the DM must choose reading a book. In other words,  $\{g\} \succ \{w, g\}$  implies  $\{b, g\} \sim \{b\}$ . Now we state the axiom:

**Axiom 4 Consistency:** Suppose  $\{y\} \succ \{y, z\}$  and  $\{t, z\} \succ \{z\}$ . If  $\{x\} \succ \{x, y\}$  then  $\{t\} \sim \{x, t\}$ .

Suppose  $U : 2^X \setminus \emptyset \rightarrow \mathfrak{R}$  represents  $\succ$ . We say that  $U$  is a Willpower (as Cognitive Resource) representation of  $\succ$  if there exist  $u : X \rightarrow \mathfrak{R}$  and  $v : X \rightarrow \mathfrak{R}$  such that

$$U(A) = \max_{x \in A} u(x) \text{ subject to } \max_{y \in A} v(y) - v(x) \leq w$$

where  $u$  is the function representing  $\succsim$  over all singletons.

**Theorem 1**  $\succsim$  satisfies Axiom 1-4 if and only if it has a Willpower representation.

Next, we give a sketch of the proof of sufficiency of the representation. The sufficiency proof proceeds in two steps. In the first step, we show that if Axioms 1, 2 and 3 hold then  $\succsim$  is represented by:

$$U(A) = \max_{x \in A} u(x) \text{ subject to } \max_{y \in A} v(y) - v(x) \leq w(x).$$

In the second step, we show that when Axiom 4 is added to the first three axioms then the willpower stock is independent of the chosen alternative. To prove the first step we define a binary relation  $\triangleright'$ . We say that  $x \triangleright' y$  if  $y \succ x \sim xy$ . In words,  $x$  blocks  $y$  if  $x$  is worse than  $y$  but DM cannot choose  $y$  when  $x$  is available. Next, we define a second binary relation  $\triangleright''$ . We say that  $x \triangleright'' y$  if  $x \succsim y$  and there exist  $a$  and  $b$  such that  $a \triangleright' y$ ,  $x \triangleright' b$ , and  $a \not\triangleright' b$ . We say that  $x \triangleright y$  if  $x \triangleright' y$  or  $x \triangleright'' y$ . Next we show that  $\triangleright$  is an interval order, i.e. it is irreflexive and  $x \triangleright b$  or  $a \triangleright y$  holds whenever  $x \triangleright y$  and  $a \triangleright b$ . The binary relation  $\triangleright$  is an interval order if and only if there exist functions  $v$  and  $w$  such that

$$\Gamma_{\triangleright}(S) = \{x \in S : \max v(y) - v(x) \leq w(x)\}.$$

Finally, to complete the proof of the first step, we show that  $S$  is indifferent to the  $\succsim$ -best element in  $\Gamma_{\triangleright}(S)$ .

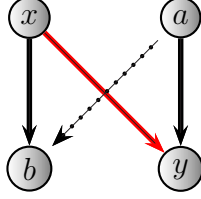


Figure 1: Black and Red arrows represent  $\triangleright'$  and  $\triangleright''$ , respectively. Solid and dashed arrows indicate the existence and non-existence of relations, respectively.

In the second step of the proof we use consistency to show that we construct a semi order  $\bar{\triangleright}$  (i.e.,  $\bar{\triangleright}$  is an interval order and if  $x\bar{\triangleright}y\bar{\triangleright}z$  then  $x\bar{\triangleright}t$  or  $t\bar{\triangleright}z$  for any  $t$ ) by properly modifying  $\triangleright$  such that  $S$  is indifferent to the  $\succsim$ -best element in  $\Gamma_{\bar{\triangleright}}(S)$ . To complete the proof we note that the binary relation  $\bar{\triangleright}$  is a semi order if and only if there exist a function  $v$  and a scalar  $w$  such that

$$\Gamma_{\bar{\triangleright}}(S) = \{x \in S : \max v(y) - v(x) \leq w\}.$$

### 3 Discussion

#### 3.1 Related Representations

We begin this section with a comparison of the willpower model with the closely related costly self-control model of Gul and Pesendorfer (2001). Gul and Pesendorfer (2001) show that preference on sets of lotteries satisfies Continuity, Set Independence and Set Betweenness if and only if it can be represented by

$$W(A) = \max_{x \in A} (u(x) + v(x)) - \max_{y \in A} v(y).$$

Since both models assume set betweenness, the first question is whether our model is a special case of the costly self-control model in a lottery setting. The answer is no. Recall that Irrelevance together with set betweenness implies that each set is indifferent to one of its elements which we call Indifference to Choice. In fact, Gul and Pesendorfer (2001) show that if Indifference to Choice holds for binary sets than in their model No Compromise condition must hold; i.e. either  $A \sim A \cup B$  or  $B \sim A \cup B$ . Moreover, no compromise implies that



preferences must be Strotz, that is,

$$W(A) = \max_{x \in A} u(x) \text{ subject to } v(x) \geq v(y) \text{ for all } y \in A.$$

To summarize, adding Irrelevance to Continuity, Set Independence and Set Betweenness implies the extreme case of Strotz preferences. The benefit of working with sets of lotteries is that the representation can be uniquely identified. However, this discussion reveals that there is also a cost. Set Independence assumption, although intuitive, seems orthogonal to the issue of self-control. However, assuming Set Independence rules out willpower preferences which capture an intuitive aspect of self control<sup>4</sup>.

In a paper that is more directly related to ours, Gul and Pesendorfer (2005) provide an ordinal analogue of the self-control preferences in Gul and Pesendorfer (2001). In Gul and Pesendorfer (2005), as in our paper, the set of all alternatives is finite and preference relation  $\succ$  is over sets of alternatives. They say  $x \succ_t y$  if there exists  $A$  containing  $y$  such that  $A \succ A \cup \{x\}$  and  $x \succ_c y$  if there exists  $A$  containing  $y$  such that  $A \cup \{x\} \succ A$ . They prove that  $x \succ_t y$  and  $x \succ_c y$  are acyclic if and only if  $\succ$  can be represented by

$$U(A) = u(\max_{x \in A} w(x), \max_{y \in A} v(y))$$

where  $u$  is non-decreasing in its first and non-increasing in its second argument. There are several points to note. First, the above temptation-self control representation is more general than the earlier costly self-control representation. Second, acyclicity assumption that characterizes temptation-self control preferences is stronger than Set Betweenness. And third, the willpower model may violate acyclicity and hence distinct from the temptation-self control model. To see this consider the following preferences that have a willpower representation:  $\{x\} \sim \{x, y\} \succ \{y\} \sim \{y, z\} \sim \{x, y, z\} \succ \{z\} \sim \{x, z\}$ . Here, the best alternative  $x$  cannot be chosen when  $z$  is available and DM chooses the best alternative from the set given this constraint. Note that  $\succ_c$  generated by this preference relation is cyclic:  $x \succ_c y$  because  $\{x, y\} \succ \{y\}$  and  $y \succ_c x$  because  $\{x, y, z\} \succ \{x, z\}$ .

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<sup>4</sup>Set Independence also rules out increasing marginal cost of resisting temptation. Noor and Takeoka (2010) axiomatize this class by relaxing Set Independence while maintaining Continuity and Set Betweenness.

### 3.2 Psychology Experiments on Willpower

Psychologists (Baumeister *et al.*, 1994; Baumeister and Vohs, 2003) have experimentally demonstrated that individuals who perform prior acts of self-restraint tend to behave later as if they have less self-control. The typical experiment has two phases. Every subject participates in the second phase but only a randomly chosen subset participates in the first, with the remainder serving as a control group. In the first phase, subjects are asked to perform a task that requires self restraint; in the second phase, their endurance in an unrelated activity also requiring self control is measured.<sup>5</sup> Subjects who participate in the first phase display substantially less endurance in the second phase. Experimental psychologists view these experiments as an apparent demonstration of limited willpower. We show that observing less self control in the second phase is not sufficient to conclude that the agent's willpower stock is depleted – the temptation values of alternatives may be changing as well. To see this consider  $\succ_1$  and  $\succ_2$  such that  $\{x\} \succ_2 \{x, y\} \sim_2 \{y\} \succ_2 \{y, z\} \sim_2 \{z\}$  and  $\{x\} \sim_1 \{x, y\} \sim_1 \{x, z\} \succ_1 \{y\} \succ_1 \{y, z\} \sim_1 \{z\}$ . Notice first that the two preferences have the same commitment ranking over alternatives. Moreover, DM with  $\succ_2$  cannot choose  $x$  when  $y$  is available and  $x$  or  $y$  when  $z$  is available. DM with  $\succ_1$  cannot choose  $y$  when  $z$  is available. Thus, it is apparent that,  $\succ_1$  has more self control than  $\succ_2$ . However, these preferences cannot be represented with a common  $(u, v)$ . To see this suppose there was a common  $(u, v)$  and the willpower levels are such that  $w_1 > w_2$ . Then we would have  $v(z) - v(y) > w_1$  and  $v(z) - v(x) < w_1$  implying  $v(y) < v(x)$ . But then we must have  $v(y) - v(x) < 0 < w_2$  which is a contradiction.

The psychology experiments are interesting to the extent that they show future ability for self-restraint might depend on prior acts of self control. However, our example shows that we need to be careful in interpreting the experimental results as evidence of willpower depletion as opposed to changing tastes.

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<sup>5</sup>For example, in the first phase subjects have been asked not to eat tempting foods, not to drink when thirsty, and to inhibit automated/habitual behaviors such as reading the subtitles of film. Typically, the self-control tasks differ in the two phases of these experiments. However, in Vohs and Heatherton (2000), tempting food is used in both phases.

# A Proof of Theorem 1

(Axioms 1-4  $\Rightarrow$  Representation )

We start by proving the following intermediate result:

**Theorem 2** *If  $\succsim$  satisfies Axiom 1-3 then there exist functions  $v : X \rightarrow R$  and  $\varepsilon : X \rightarrow R_+$  such that  $\succsim$  is represented by:*

$$U(A) = \max_{x \in A} u(x)$$

subject to  $\max_{y \in A} v(y) - v(x) \leq \varepsilon(x)$

where  $u$  is the function representing  $\succsim$  over all singletons.

We first show that Axiom 1-3 imply two important implications of our model. The first one shows that every menu is indifferent to one of its elements.

**Claim 1** *Suppose  $\succsim$  satisfies Axiom 1-3. Then for every  $A$ , there exists  $x_A \in A$  such that  $A \sim \{x_A\}$ .*

**Proof 1** *First, we shall prove that every menu  $A$  has at least one element that is weakly preferred to  $A$ . This statement is trivial if  $|A| = 1$ . Assume that this is true when  $|A| \leq n - 1$  and let  $|A'| = n$ . If every element in  $A'$  is strictly worse than  $A'$ , then by Axiom 3,  $A' \setminus \{x\} \succsim A'$  for every  $x \in A'$ . By the inductive hypothesis,  $A' \setminus \{x\}$  has at least one element that is weakly preferred to  $A' \setminus \{x\}$  and this element is weakly better than  $A'$ .*

*Now we shall prove the claim. Let  $y \in A$  be an arbitrary element in  $A$  such that  $y \succsim A$ . If  $y \sim A$ , then we are done. If  $y \succ A (= (A \setminus \{y\}) \cup \{y\})$ , Axiom 2 implies  $A \sim A \setminus \{y\}$ .*

*By Axiom 3,  $A$  must include an element  $x$  which is weakly better than  $A$ . Applying this step recursively, we must eventually find an element that is indifferent to  $A$ .  $\square$*

**Claim 2** *Suppose  $\succsim$  satisfies Axiom 1-3. Then, if  $x \succ A \cup x$ ,  $B \sim B \cup x$  for all  $B \supset A$ .*

**Proof 2** *Let  $L2(n)$  stand for the statement of claim 2 that is restricted to when  $|B - A| \leq n$ . Notice that Axiom 2 is  $L2(0)$ . First, we shall show  $L2(1)$ . That is,  $x \succ A \cup x$  (so  $A \sim A \cup x$  by Axiom 2) implies  $A \cup y \sim A \cup x \cup y$  for any  $y$ .*

**Case 1:**  $y \succ A \cup x \cup y$ :

*By Axiom 2,  $A \cup x \sim A \cup x \cup y$ . By the assumption, we have  $x \succ A \cup x \sim A \cup x \cup y$ . By applying Axiom 2, we get  $A \cup y \sim A \cup x \cup y$ .*

**Case 2:**  $y \prec A \cup x \cup y$ :

*By Axiom 3,  $y \prec A \cup x \cup y \prec A \cup x \prec x$ . By Axiom 2, we get the desired result,  $A \cup y \sim A \cup x \cup y$ .*

**Case 3:**  $y \sim A \cup x \cup y$ :

- *If  $y \sim A \cup y$ , then they are indifferent to  $A \cup x \cup y$ .*
- *If  $y \succ A \cup y$ , then Axiom 2 implies  $A \cup y \sim A (\sim A \cup x)$ . Applying Axiom 3, we get  $A \cup x \cup y \sim A \cup y$ , which is a contradiction because  $A \cup x \cup y \sim y \succ A \cup y$ .*

- If  $y \prec A \cup y$ , then Axiom 3 implies  $(A \cup x \sim)A \succsim A \cup y$ . Applying Axiom 3 again, it must be  $A \cup x \cup y \succsim A \cup y \succ y$ , which is a contradiction because  $A \cup x \cup y \sim y$  by Axiom 2.

Now suppose that  $L2(k)$  is true up when  $1 \leq k \leq n-1$ . We shall prove  $L2(n)$ . Assume  $x \succ A \cup x$  and let  $B = A \cup \{y_1, y_2, \dots, y_n\}$  where all of  $y_i$ 's are distinct and excluded from  $A$ . Our goal is to show  $B \sim B \cup x$ . Without loss of generality, assume  $y_1 \succsim y_2 \succsim \dots \succsim y_n$ .

**Case 1:**  $y \succ A \cup x \cup y$  for some  $y \in \{y_1, y_2, \dots, y_n\}$ :

By  $L2(n-1)$ , we have  $(B \setminus y) \cup x \sim (B \setminus y) \cup x \cup y = B \cup x$ . (Notice that  $(B \setminus y) \cup x \supset A \cup x$  and the difference of their cardinality is  $n-1$ ). Applying  $L2(1)$  to  $x \succ A \cup x$ , we have  $(y \succ)A \cup x \cup y \sim A \cup y$ . Applying  $L2(n-1)$  to this yields  $B \setminus y \sim (B \setminus y) \cup y = B$ . Notice that  $B \setminus y \sim (B \setminus y) \cup x$  because  $x \succ A \cup x$  and  $L2(n-1)$ . These three indifference relations imply  $B \sim B \cup x$

**Case 2:**  $y \prec A \cup x \cup y$  for some  $y \in \{y_1, y_2, \dots, y_n\}$ :

By  $L2(1)$ , we have  $A \cup y \sim A \cup x \cup y$  so  $y \prec A \cup x \cup y$ . so it must be  $(x \succ)A \cup x \succsim A \cup x \cup y$  by Axiom 3. Thus by  $L2(n-1)$ , we have  $B \sim B \cup x$  (because  $|B \setminus (A \cup y)| = n-1$ ).

**Case 3:**  $y_i \sim A \cup y_i \cup x$  for all  $i = 1, \dots, n$ . Then, we have

$$y_1 \sim A \cup y_1 \cup x \succsim y_2 \sim A \cup y_2 \cup x \succsim \dots \succsim y_n \sim A \cup y_n \cup x$$

Since  $A \cup y_i \cup x \sim A \cup y_i$  by  $L2(1)$ , the above relations still hold when  $x$  is removed:

$$y_1 \sim A \cup y_1 \succsim y_2 \sim A \cup y_2 \succsim \dots \succsim y_n \sim A \cup y_n$$

Recursively applying Axiom 3 implies

$$(A \cup y_1 \cup x \sim)y_1 \succsim A \cup \{y_1, y_2, \dots, y_n\} (= B) \succsim y_n (\sim A \cup y_n \cup x)$$

In other words,

$$A \cup y_1 \cup x \succsim B \succsim A \cup y_n \cup x$$

Since  $(A \cup y_1 \cup x) \cup B = B \cup x$ , Axiom 3 implies  $B \cup x \succsim B$ . Similarly, we can get  $B \succsim B \cup x$ .  $\square$

For any binary relation  $R$ , let  $\Gamma_R(S)$  be the set of  $R$ -undominated elements in  $S$ . Instead of constructing  $v$  and  $\varepsilon$ , we shall construct a binary relation over  $X$ , denoted by  $\triangleright$  such that  $S$  is indifferent to the  $\succsim$ -best element in  $\Gamma_{\triangleright}(S)$ . It is known (Fishburn (1979)) that, if (and only if)  $\triangleright$  is an interval order<sup>6</sup>, there exist functions  $v$  and  $\varepsilon$  such that

$$\Gamma_{\triangleright}(S) = \{x \in S : \max v(y) - v(x) \leq \varepsilon(x)\}$$

so we can get the desired representation.

Now, define  $x \triangleright y$  when either  $x \triangleright' y$  or  $x \triangleright'' y$  where  $\triangleright'$  and  $\triangleright''$  are defined as follow:

1.  $x \triangleright' y$  if  $y \succ x \sim xy$

2.  $x \triangleright'' y$  if  $x \succsim y$  and there exist  $a$  and  $b$  such that  $a \triangleright' y$ ,  $x \triangleright' b$ , and  $a \not\triangleright' b$ .

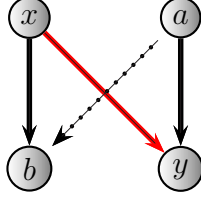


Figure 2: Black and Red arrows represent  $\triangleright'$  and  $\triangleright''$ , respectively. Solid and dashed arrows indicate the existence and non-existence of relations, respectively.

Then all we need to show are (i)  $\triangleright$  is an interval order and (ii) the  $\succsim$ -best element in  $\Gamma_{\triangleright}(A)$  is indifferent to  $A$ .

**Claim 3**  $\triangleright'$  is asymmetric and transitive.

**Proof 3** By construction,  $x \triangleright' y$  and  $y \triangleright' x$  cannot happen at the same time. Suppose  $x \triangleright' y$  and  $y \triangleright' z$  (i.e.  $z \succ yz \sim y \succ xy \sim x$ ). Then by Claim 2,  $xyz \sim xz$  because  $y \succ xy$ . By Axiom 3,  $(z \succ)yz \succsim xyz \succsim xy$ . Hence, we have  $z \succ xyz \sim xz$ . By Claim 1, it must be  $xz \sim x$  so it is  $x \triangleright' z$ .  $\square$

**Claim 4** If  $x \triangleright' y$  and  $a \triangleright' b$  but neither  $x \triangleright' b$  or  $a \triangleright' y$ , then it must be  $x \triangleright'' b$  or  $a \triangleright'' y$  but not both.

**Proof 4** First we shall show that  $x \triangleright'' b$  and  $a \triangleright'' y$  cannot happen at the same time. Suppose it does. Then by definition of  $\triangleright'$  and  $\triangleright''$ , we have  $y \succ x \succsim b \succ a \succsim y$ , which is a contradiction.

Now, we shall show that either  $x \triangleright'' b$  or  $a \triangleright'' y$  must be defined. Suppose not. Then, along with the definition of  $\triangleright'$ , we have  $b \succ x \sim xy$  and  $y \succ a \sim ab$ . Therefore,  $xyab$  must be weakly worse than  $x$  or  $a$  because it must be weakly worse than  $xy$  or  $ab$  by Axiom 3

Since neither  $(x, b)$  or  $(a, y)$  belongs to  $\triangleright'$  or  $\triangleright''$ , we have  $xb \sim b \succ x$  and  $ay \sim y \succ a$ . By Axiom 3,  $xyab$  must be weakly better than  $xb$  or  $ay$  so it must be weakly better than  $y$  or  $b$ .

Hence, either  $x$  or  $a$  must be weakly better than either  $y$  or  $b$ . Since we have already seen  $b \succ x$  and  $y \succ a$ , the only possibilities are  $a \succsim b$  or  $x \succsim y$ , neither of which is possible because  $a \triangleright' b$  and  $x \triangleright' y$ .  $\square$

**Claim 5**  $\triangleright$  is an interval order.

**Proof 5** We need to show that  $\triangleright$  is irreflexive. It is easy to see that we cannot have  $x \triangleright' y$  and  $y \triangleright' x$  or  $x \triangleright' y$  and  $y \triangleright'' x$ . Suppose  $x \triangleright'' y$  and  $y \triangleright'' x$ . Then there exists  $a, b$  such that  $a \triangleright' y$ ,  $x \triangleright' b$  and  $a \not\triangleright' b$  and there exists  $c, d$  such that  $c \triangleright' x$ ,  $y \triangleright' d$  and  $c \not\triangleright' d$ . By transitivity of  $\triangleright'$  we know  $a \triangleright' d$  and  $c \triangleright' b$ . Since  $a \not\triangleright' b$  and  $c \not\triangleright' d$  by Claim 4 it must be that either  $c \triangleright'' d$  or  $a \triangleright'' b$ . However, since  $a \triangleright' y$ ,  $x \triangleright' b$  and  $x \triangleright'' y$ , by Claim 4, we know that  $a \not\triangleright'' b$ . Similarly,  $c \not\triangleright'' d$ . A contradiction.

Next we show that  $x \triangleright b$  or  $a \triangleright y$  holds whenever  $x \triangleright y$  and  $a \triangleright b$ . We shall prove this case by case:

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<sup>6</sup> $\triangleright$  is called an interval order if it is irreflexive and  $x \triangleright b$  or  $a \triangleright y$  holds whenever  $x \triangleright y$  and  $a \triangleright b$ .

**Case 1:**  $x \triangleright' y$  and  $a \triangleright' b$ : It is immediate from the definition of  $\triangleright''$ .

**Case 2:**  $x \triangleright' y$  and  $a \triangleright'' b$ : In this case, by definition of  $\triangleright''$  and claim 4, there exist  $s$  and  $t$  such that  $a \triangleright' t$  and  $s \triangleright' b$  but not  $s \triangleright t$ . Focus on  $x \triangleright' y$  and  $a \triangleright' t$ , we must have either  $a \triangleright y$  (it is done in this case) or  $x \triangleright t$  (so either  $x \triangleright' t$  or  $x \triangleright'' t$ ). If  $x \triangleright' t$ , then by looking at  $x \triangleright' t$  and  $s \triangleright' b$  claim requires  $x \triangleright b$  because it is not  $s \triangleright t$ . Thus, we consider the final subcase:  $x \triangleright'' t$ . If so, we have  $x \triangleright' y$  and  $s \triangleright' b$  so it must be either  $x \triangleright b$  (then done) or  $s \triangleright y$ . If  $s \triangleright y$ , then it must be  $s \triangleright' y$  (i.e. not  $s \triangleright'' y$ ) because  $y \succ x \succsim t \succ a \succsim b \succ s$ . Therefore, we have  $s \triangleright' y$  and  $a \triangleright' t$  with not  $s \triangleright t$ . Hence it must be  $a \triangleright y$ .

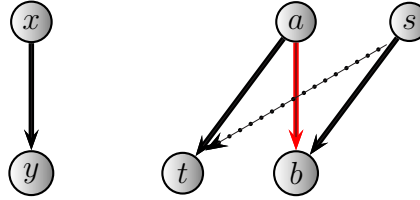


Figure 3: Case 2.

**Case 3:**  $x \triangleright'' y$  and  $a \triangleright'' b$ : By definition of  $\triangleright''$ , there exist  $s$  and  $t$  such that  $x \triangleright' t$  and  $s \triangleright' y$  with not  $s \triangleright t$ . Then by focusing on  $x \triangleright' t$  and  $a \triangleright'' b$ , we must have either  $x \triangleright b$  (done) or  $a \triangleright t$ . Suppose the latter. Then we have  $s \triangleright' y$  and “ $a \triangleright' t$  or  $a \triangleright'' t$ ”, so the previous two cases are applicable so we conclude  $a \triangleright y$  because it is not  $s \triangleright t$ .

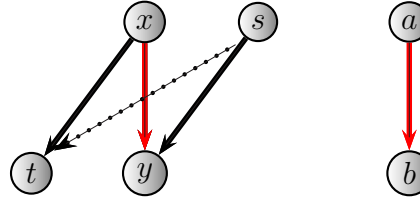


Figure 4: Case 3.

□

**Claim 6**  $S$  is indifferent to the  $\succsim$ -best element in  $\Gamma_{\triangleright'}(S)$ .

**Proof 6** First, we prove that  $\Gamma_{\triangleright'}(S)$  does not include any element that is strictly better than  $S$ . Suppose  $x \in \Gamma_{\triangleright'}(S)$ . Let  $S'$  and  $S''$  be the subsets of  $S \setminus x$  consisting of elements that are weakly better than  $x$  and strictly worse than  $x$ , respectively. Then, we have  $S' \cup x \succsim x$  by Claim 1 and  $x \sim xy \succ y$  for all  $y \in S''$  by the definition of  $\triangleright'$ . Then applying the set betweenness we get  $x \cup S'' \sim x$ . Thus,  $S = (x \cup S'') \cup (S' \cup x) \succsim x$  again by the set betweenness.

Next, we shall show that  $\Gamma_{\triangleright'}(S)$  includes at least one element that is weakly better than  $S$ . Suppose not and let  $\widehat{S}$  be the set of all  $x$  weakly better than  $S$ . Thus if  $x \in \widehat{S}$  then

$x \notin \Gamma_{\triangleright'}(S)$ . Then, for every  $x \in \widehat{S}$ , there exists  $y_x \in \Gamma_{\triangleright'}(S)$  such that  $y_x \triangleright' x$  (we can find such  $y_x$  in  $\Gamma_{\triangleright'}(S)$  because of the transitivity of  $\triangleright'$ ), that is  $x \succsim S \succ \{x, y_x\} \sim y_x$ . Thus, recursively applying the set betweenness, we have  $S \succ \cup_{x \in \widehat{S}} \{x, y_x\}$ . Since  $S \setminus \widehat{S}$  includes only those strictly worse than  $S$ , we have  $S \succ S \setminus \widehat{S}$ . Therefore by set betweenness, it must be  $S \succ (S \setminus \widehat{S}) \cup (\cup_{x \in \widehat{S}} \{x, y_x\}) = S$ , a contradiction.  $\square$

Combining the first and second results, the  $\succsim$ -best element in  $\Gamma_{\triangleright'}(S)$  is indifferent to  $S$ .

**Claim 7**  $S$  is indifferent to the  $\succsim$ -best element in  $\Gamma_{\triangleright}(S)$ .

**Proof 7** Since  $\triangleright \supset \triangleright'$  by construction, we have  $\Gamma_{\triangleright}(S) \subset \Gamma_{\triangleright'}(S)$ . Therefore, by claim 6, it is enough to show is that at least one of  $\succsim$ -best elements in  $\Gamma_{\triangleright'}(S)$  is included in  $\Gamma_{\triangleright}(S)$ . Let  $C$  be the set of  $\succsim$ -best elements in  $\Gamma_{\triangleright'}(S)$ . Suppose  $x \in C$  but  $x \notin \Gamma_{\triangleright}(S)$ . Since  $\triangleright$  is an interval order, it is automatically transitive. Therefore, there exists  $y \in \Gamma_{\triangleright}(S)$  such that  $y \triangleright x$  but not  $y \triangleright' x$ . Therefore, it must be  $y \triangleright'' x$  so  $y \succsim x$ . Since  $y \in \Gamma_{\triangleright'}(S)$   $y$  cannot be strictly better than  $x$  as  $x$  is one of  $\succsim$ -best elements in  $\Gamma_{\triangleright'}(S)$ . Hence, we have  $y \sim x$  and  $y \in \Gamma_{\triangleright}(S)$ .  $\square$

We are now done proving the sufficiency of the axioms for the representation in Theorem 2. Next, we show the sufficiency of Axioms 1-4 for the representation in Theorem 1.

We will use the following claim in the proof.

**Claim 8** If  $\{x\} \succ \{x, y\} \succ \{y, z\}$  then, for all  $t$ ,  $\{x, y, z, t\}$  is indifferent to either  $\{x, t\}$  or  $\{y, z\}$ .

**Proof 8** (a) Suppose  $\{y\} \succ \{y, z\}$  and  $\{t, z\} \succ \{z\}$ . If  $\{x\} \succ \{x, y\}$  then  $\{t\} \sim \{x, t\}$ .  
(b) Suppose  $\{x\} \succ \{x, y\}$  and  $\{x, t\} \succ \{t\}$ . If  $\{y\} \succ \{y, z\}$  then  $\{z\} \sim \{t, z\}$ .

We shall prove Claim 8. Suppose  $\{x\} \succ \{x, y\} \succ \{y, z\}$ , then it must be  $x \succ y \succ z$ ,  $v(y) - v(x) > w(x)$  and  $v(z) - v(y) > w(y)$  (so  $v(z) - v(x) > w(x)$ ).

Suppose  $v(z) > v(t)$ . If  $v(z) - v(t) > w(t)$ , then  $xyzt \sim z \sim yz$ . If  $v(z) - v(t) \leq w(t)$ , then either  $t \succ z$  and  $\{t, z\} \succ z$  or  $z \succeq t$  and  $\{t, z\} \sim z$ . In the first case by Consistency part a  $xyzt \sim \{t\} \sim \{x, t\}$ . In the second case  $xyzt \sim z \sim yz$ .

Suppose  $v(t) \geq v(z)$ . If  $v(t) - v(z) > w(z)$ , then  $xyzt \sim \{t\} \sim \{x, t\}$ . If  $v(t) - v(z) \leq w(z)$ , then either  $z \succeq t$  and  $\{t, z\} \sim z$  or  $t \succ z$  and  $\{t, z\} \succ z$ .  $\square$

Instead of defining  $v$  and  $w$ , we shall construct a binary relation over  $X$ , denoted by  $\overline{\triangleright}$  such that  $S$  is indifferent to the  $\succsim$ -best element in  $\Gamma_{\overline{\triangleright}}(S)$  (i.e. the set of  $\overline{\triangleright}$ -undominated elements in  $S$ ). It is known (Fishburn (1979)) that if (and only if)  $\overline{\triangleright}$  is a semi order<sup>7</sup>, which is a special type of an interval order, there exist function  $v$  and positive number  $w$  such that

$$\Gamma_{\overline{\triangleright}}(S) = \{x \in S : \max v(y) - v(x) \leq w\}$$

so we get the desired representation.

Next we define  $(i, j)$ -representation of an arbitrary binary relation  $P$ .

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<sup>7</sup> $\overline{\triangleright}$  is a semi order if it is an interval order and if  $x \overline{\triangleright} y \overline{\triangleright} z$  then  $x \overline{\triangleright} t$  or  $t \overline{\triangleright} z$  for any  $t$ .

**Definition 1** Two functions  $i : X \rightarrow N$  and  $j : X \rightarrow N$  where  $i(x) \geq j(x)$  for all  $x \in X$  represents a binary relation  $P$  if  $xPy$  if and only if  $i(x) < j(y)$ .

Let  $\triangleright$  be the interval order that is defined in the proof of Theorem 2. First, we argue that  $\triangleright$  has an  $(i, j)$ -representation without any gaps as described in the following claim:

**Claim 9** Any interval order,  $\triangleright$ , has an  $(i, j)$ -representation such that the ranges of  $i$  and  $j$  have no gap: That is if there exist  $x$  and  $y$  such that  $i(x) > i(y)$  then for any integer  $n$  between  $i(x)$  and  $i(y)$  there is  $z$  with  $i(z) = n$ . Similarly for  $j(\cdot)$ .

**Proof 9** The following proof is based on Mirkin (1979). Strong intervality condition ( $x \triangleright y$  and  $z \triangleright w$  imply  $x \triangleright w$  or  $z \triangleright y$ ) implies that, for all  $x$  and  $y$  in  $X$ ,  $L(x) \subseteq L(y)$  or  $L(y) \subseteq L(x)$ , and,  $U(x) \subseteq U(y)$  or  $U(y) \subseteq U(x)$ , where  $L(x)$  and  $U(x)$  are lower and upper contour sets of  $x$  with respect to  $\triangleright$ , respectively. That is,  $L(x) = \{y \in X \mid x \triangleright y\}$  and  $U(x) = \{y \in X \mid y \triangleright x\}$ . Irreflexivity indicates that there are a chain with respect to both lower contour sets, i.e., relabel elements of  $X$ ,  $|X| = n$  such that  $L(x_j) \subseteq L(x_i)$  for all  $1 \leq i \leq j \leq n$ . Moreover, we can strict inclusions such as there exists  $s \leq n$  such that  $\emptyset = L(x_s) \subset L(x_{s-1}) \dots \subset L(x_2) \subset L(x_1)$  where  $\{x_1, x_2, \dots, x_s\} \subseteq X$ . For all  $k \leq s$ , Define

$$I_k = \{x \in X \mid L(x_k) = L(x)\}$$

$I_k$  is not empty for any  $k$  since  $x_k \in I_k$  by construction. Clearly, the system  $\{I_k\}_1^s$  is a partition of the set  $X$ , i.e.  $\bigcup_{k=1}^s I_k = X$ ,  $I_k \cap I_l = \emptyset$  when  $k \neq l$ . Define

$$i(x) := k \text{ if } L(x) = L(x_k) \text{ for some } x_k \text{ in } X.$$

Now construct another family of non-empty sets  $\{J_m\}_1^s$ , as follows

$$J_s = L(x_{s-1}) \setminus L(x_s), \quad \dots, \quad J_2 = L(x_1) \setminus L(x_2), \quad J_1 = X \setminus L(x_1)$$

Clearly, the system  $\{J_m\}_1^s$  is another partition of the set  $X$ . Most importantly, we have  $\emptyset = U(y_1) \subset U(y_2) \dots \subset U(y_{s-1}) \subset U(y_s)$  where  $y_i \in J_i$  for all  $i \leq s$ . Define

$$j(x) := k \text{ if } x \in J_k.$$

Condition i): Let  $i(x) = i$ . That means  $x \in I_i$ . If there exists no element  $z$  such that  $z \triangleright x$ , i.e.  $U(x) = \emptyset$ , then  $j(x) = 1 \leq i(x)$ . Otherwise find the largest integer  $j$  such that  $x \in L(x_j)$ . Note that  $j$  must be strictly less than  $i$ . Then by definition,  $j(x) = j + 1$ , which is less than  $i = i(x)$ .

Condition ii) follows from by the construction since both  $\{I_k\}_1^s$  and  $\{J_k\}_1^s$  are partitions of  $X$ . Finally, we prove the condition iii).  $x \triangleright y \Leftrightarrow y \in L(x) \Leftrightarrow j(x) \geq i(x) + 1 > i(x)$ .  $\square$

Now, let  $i(\cdot)$  and  $j(\cdot)$  represent  $\triangleright$ . By modifying  $i$  and  $j$ , we shall define a semiorder  $\overline{\triangleright}$  that represents  $\succsim$ .

**Claim 10** If  $i(x) = j(y) - 1$ , it must be  $y \succ x$ .



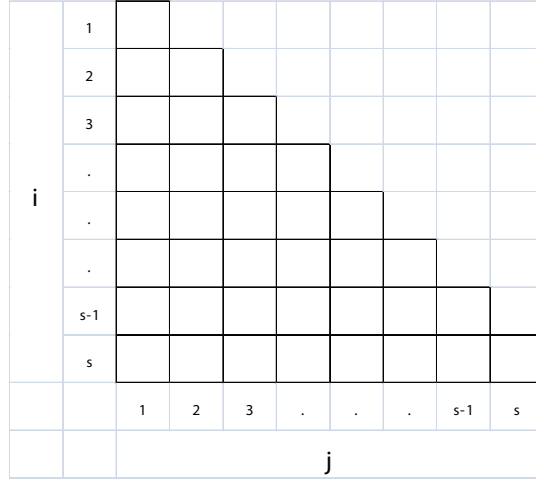


Figure 5: Semiorder structure

**Proof 10** Since  $i(x) < j(y)$  we know that  $x \triangleright y$ . If  $x \triangleright' y$  then we are done since in that case  $y \succ x \sim \{x, y\}$ . So suppose that  $x \triangleright'' y$ . Then by definition of  $\triangleright''$ , there exist  $\alpha$  and  $\beta$  such that  $\alpha \triangleright' y$  and  $x \triangleright' \beta$  and  $\alpha \not\triangleright' \beta$ . Moreover, by Claim 4  $\alpha \not\triangleright'' \beta$ . So  $\alpha \not\triangleright \beta$ . Since  $\alpha \triangleright y$  and  $x \triangleright \beta$ ,  $i(\alpha) < j(y)$  and  $i(x) < j(\beta)$ . Since  $\alpha \not\triangleright \beta$ ,  $i(\alpha) \geq j(\beta)$ . Therefore it must be  $i(x) \leq j(y) - 2$ , a contradiction.  $\square$

**Definition 2**  $(i, j)$  is called a prohibited cell if there exists  $x$  such that  $i(x) < i$  and  $j(x) > j$ . Otherwise, it is called a safe cell.

**Definition 3**  $x$  can be moved to  $(i, j)$  where  $i \geq j$  if:

- $i \leq i(x)$  and  $j \geq j(x)$
- $x \succsim y$  for all  $y$  with  $i < j(y) \leq i(x)$
- $z \succsim x$  for all  $z$  with  $j(x) \leq i(z) < j$ .

To understand this definition consider Figure 4. As we will see later, to obtain a semiorder representation, we need to move outcomes that are in prohibited cells to safe cells. That is we need to move them up and right (which is the first bullet point in the definition) and still represent  $\succeq$ . As an outcome  $x$  is moved up we might have  $i(x) \geq j(y)$  to begin with but  $i < j(y)$  once  $x$  is moved up. Bullet point 2 requires that in this case  $x \succsim y$ . Suppose to the contrary that  $y \succ x$ . Since  $i < j(y)$ , in the new representation  $x \triangleright' y$ . But in the original representation  $x \not\triangleright' y$ . So the two representations must represent different preferences. The third bullet point can be understood similarly.

**Claim 11** Suppose  $\beta \succ y \succ \alpha$  and  $\alpha \triangleright y \triangleright \beta$ . For any  $x$  such that  $x \not\triangleright \beta$  and  $\alpha \not\triangleright x$ ,  $x \succsim \beta$  or  $\alpha \succsim x$ .

**Proof 11** Suppose  $\beta \succ x \succ \alpha$  and we shall get a contradiction. Then we have  $\beta \sim \beta x$  and  $x\alpha \sim x$  because  $x \not\triangleright \beta$  and  $\alpha \not\triangleright x$ . Therefore, by Claim 8,  $\alpha\beta xy$  must be indifferent to either  $\beta$  or  $\alpha$ . Consider two sets  $\beta y$  and  $x\alpha$ , both of which are strictly worse than  $\beta$  and strictly better than  $\alpha$ . Axiom 3 dictates that  $\beta \succ \beta y \succsim \alpha\beta xy \succsim x\alpha \succ \alpha$ . Hence,  $\alpha\beta xy$  cannot be indifferent to  $\beta$  or  $\alpha$ , which is a contradiction.  $\square$

**Claim 12** If  $i(x) > i(y)$  and  $j(x) < j(y)$ , then  $x$  can be moved to either  $(i(y), j(x))$  or  $(i(x), j(y))$  but not to both.

**Proof 12** There exist two alternatives  $\alpha$  and  $\beta$  such that  $i(\alpha) = j(y) - 1$ . and  $j(\beta) = i(y) + 1$ .<sup>8</sup> By Claim 9 and 10, we have  $\beta \succ y \succ \alpha$  and  $\alpha \triangleright y \triangleright \beta$ . Since  $j(\beta) \leq i(x)$  and  $j(x) \leq i(\alpha)$ ,  $x \not\triangleright \beta$  and  $\alpha \not\triangleright x$  by Claim 9. Thus, by Claim 11, we have  $x \succsim \beta \succ \alpha$  (so  $x$  cannot be moved to  $(i(x), j(y))$  because of  $\alpha$ ) or  $\beta \succ \alpha \succsim x$  (so  $x$  cannot be moved to  $(i(y), j(x))$  because of  $\beta$ ). Therefore, all we need to show is that  $x$  can be moved to either of them.

**Case I:**  $x \succsim \beta$ . We show that  $x$  can be moved to  $(i(y), j(x))$ . First, the first condition holds trivially:  $i(y) \leq i(x)$  and  $j(x) \geq j(x)$ . For the second condition, take an element  $z$  such that  $i(y) < j(z) \leq i(x)$  (so  $y \triangleright z$  but  $x \not\triangleright z$ ). Then, it must be either  $y \succsim z$  (which implies  $x \succ z$ ) or  $z \succ y \sim y, z$  in which case we have  $z \succ y \succ \alpha$  and  $\alpha \triangleright y \triangleright z$  (with  $x \not\triangleright z$  and  $\alpha \not\triangleright x$ ). By Claim 11, we should have  $x \succsim z$  or  $\alpha \succsim x$ . Since we are considering the case  $x \succsim \beta(\succ \alpha)$ , it must be  $x \succsim z$ . The third condition is trivial because  $j = j(x)$ .

**Case II:**  $\alpha \succsim x$ : The first and the second conditions will be now trivial while the third condition can be proven in the same way how we prove the second condition in case II.  $\square$

**Claim 13** Let

$$U_x = \{y : i(x) > i(y), j(x) < j(y) \text{ and } x \text{ can be moved to } (i(y), j(x))\} \cup \{x\}$$

$$R_x = \{y : i(x) > i(y), j(x) < j(y) \text{ and } x \text{ can be moved to } (i(x), j(y))\} \cup \{x\}$$

and let

$$i_x = \min_{y \in U_x} i(y) \quad \text{and} \quad j_x = \max_{y \in R_x} j(y)$$

Then

- (i)  $x$  can be moved to  $(i_x, j_x)$
- (ii)  $(i_x, j_x)$  is a safe cell. That is, there is no  $z$  with  $i(z) < i_x$  and  $j(z) > j_x$ .

**Proof 13** Notice that by the definitions of movability,  $i_x \leq i(x)$  and  $j_x \geq j(x)$ .

(i) Clearly,  $i_x \leq i(x)$  and  $j_x \geq j(x)$  as  $x \in U_x, R_x$ . First, we show that  $i_x \geq j_x$ . Take an alternative  $y \in U_x$  such that  $i(y) = i_x$ . Since  $y \in U_x$ ,  $x$  cannot be moved to  $(i(x), j(y))$  by Claim 12. By the definition of movability,  $x$  cannot be moved to  $(i(x), j)$  if  $j \geq j(y)$ . Hence for all  $z \in R_x \setminus \{x\}$ ,  $j(z) < j(y)$ , which means  $j_x = \max_{z \in R_x} j(z) \leq j(y)$ . Since  $i(y) \geq j(y)$ , we have  $j_x \leq j(y) \leq i(y) = i_x$ .

Since  $x$  can be moved to  $(i_x, j(x))$ , then the second condition of the movability is satisfied. Similarly, we can prove the third requirement as well. Therefore,  $x$  can be moved to  $(i_x, j_x)$

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<sup>8</sup>This is because since neither  $i(y)$  is the smallest nor  $i(y)$  is the largest integer within the range of  $i$ .

(ii) If  $z \notin U_x, R_x$ , then by Claim 12, it must be  $(i_x \leq) i(x) \leq i(z)$  or  $j(z) \leq j(x) (\leq j_x)$ . If  $z \in U_x$ , then  $i(z) \geq i_x$ . If  $z \in R_x$  then  $j(z) \leq j_x$ .  $\square$

Now, define  $x \bar{\triangleright} y$  if and only if  $j_y > i_x$ .

**Claim 14**  $\bar{\triangleright}$  is a semi-order.

**Proof 14** Since  $i_x \geq j_x$  by Claim 13 for all  $x$ ,  $\bar{\triangleright}$  is an interval order.

Next, we shall show that if  $(i, j)$  is a safe cell, there is no element  $x$  such that  $i_x < i$  and  $j_x > j$ . Suppose there is such  $x$ . Notice that it must be  $i \leq i(x)$  or  $j \geq j(x)$  because  $(i, j)$  is a safe cell so it must be  $i_x < i(x)$  or  $j > j(x)$ . Suppose  $i_x < i(x)$ . Then there exists  $y$  such that  $i(y) = i_x$  and  $j(x) < j(y)$  such that  $x$  can be moved to  $(i(y), j(x))$ . By Claim 12,  $x$  cannot be moved to  $(i(x), j(y))$ , so it cannot be moved to  $(i(x), j')$  for any  $j' \geq j(y)$ . Since  $(i, j)$  is a safe cell and  $i(y) = i_x < i$ , it must be  $j(y) \leq j (< j_x)$ . Hence,  $x$  cannot be moved to  $(i(x), j_x)$ , which contradicts the definition of  $j_x$  unless  $j_x = j(x)$ . But if so,  $j(y) > j_x > j$  but this contradicts that  $(i, j)$  is a safe cell. Analogously, we can show a contradiction if  $j > j(x)$ .

By Claim 13, all elements have been moved to safe cells, so there is no pair of elements  $x$  and  $y$  such that  $i_x < i_y$  and  $j_x > j_y$ . Therefore, if  $i_x < j_y \leq i_y < j_z$  (i.e.  $x \bar{\triangleright} y \bar{\triangleright} z$ ) then for any  $w$ , it must be either  $j_w > j_y$  or  $i_w \leq i_y$ , which implies  $j_w > i_x$  or  $i_w < j_z$  (i.e.  $x \bar{\triangleright} w$  or  $w \bar{\triangleright} z$ ). Therefore,  $\bar{\triangleright}$  is a semi order.  $\square$

**Claim 15**  $x \bar{\triangleright} y$  whenever  $x \triangleright y$ .

**Proof 15** By definitions of  $i'$  and  $j'$ ,  $i_x \leq i(x)$  and  $j_x \geq j(x)$  for all  $x$ . Therefore, if  $x \triangleright y$ , then  $i_x \leq i(x) < j(y) \leq j_y$  so we have  $x \bar{\triangleright} y$ .  $\square$

**Claim 16** If  $x \bar{\triangleright} y$  but not  $x \triangleright y$ , then  $x \succsim y$ .

**Proof 16** First, we shall note that both  $x$  and  $y$  must be in prohibited cells. If neither of them is in,  $i_x = i(x)$  and  $j_y = j(y)$  so  $x \bar{\triangleright} y$  and not  $x \triangleright y$  cannot happen at the same time. If only  $x$  is in a prohibited cell, then  $i_x < j(y) \leq i(x)$  so  $x$  cannot be moved to  $(i_x, j_x)$ . Similarly we can prove that it is not possible that only  $y$  is in a prohibited cell.

Next we shall show that  $i_x < i(x)$  and  $j_x > j(x)$ . Since  $x$  can be moved to  $(i_x, j_x)$  while  $y \succ x$ , it must be  $i_x \geq j(y)$  because  $j(y) \leq i(x)$  (i.e. not  $x \triangleright y$ ). Combined with  $x \bar{\triangleright} y$ , we get  $j_y > j(y)$ . Flipping  $x$  and  $y$ , one can prove  $i_x < i(x)$ .

Therefore, there must exist  $z$  and  $z'$  with  $i(z) \in [j(y), j_y - 1]$  and  $j(z') \in [i_x - 1, i(x)]$  (notice that these intervals are non-empty). Furthermore, we can take such  $z$  and  $z'$  so that  $i(z) = j(z') - 1$  because  $i_x - 1 < j_y - 1$  and  $i(x) > j(y)$ . Thus,  $z' \succ z$  by Claim 10. Since  $x$  is movable to  $(i_x, j_x)$ , we have  $x \succsim z'$ . Similarly, we have  $z \succsim y$ . Therefore, we conclude  $x \succ y$ .  $\square$

**Claim 17**  $S$  is indifferent to the  $\succsim$ -best element in  $\Gamma_{\bar{\triangleright}}(S)$ .

**Proof 17** We know  $\bar{\triangleright}$  is transitive,  $\bar{\triangleright} \supset \triangleright$  and  $x \bar{\succ} y$  whenever  $x \bar{\triangleright} y$  but not  $x \triangleright y$ . It is easy to see that this claim can be proven in the exactly same way as Claim 7.  $\square$

**(Representation  $\Rightarrow$  Axioms 1-4)** Showing that the first axiom is necessary is straightforward. Let

$$\Gamma(A) = \left\{ x \in A : \max_{y \in A} v(y) - v(x) \leq \bar{w} \right\}$$

For Axiom 2, if  $x \succ A \cup x$  then  $A$  must have an element  $y$  with  $v(y) > v(x) + \bar{w}$ , so it is clear that  $\Gamma(A) = \Gamma(A \cup x)$  so  $A \sim A \cup x$ .

Axiom 3: Let  $x^*$  be the  $u$ -best element in  $\Gamma(A \cup B)$ . Then it must be in  $\Gamma(A)$  or  $\Gamma(B)$  so it is not possible that  $A \cup B$  is strictly preferred to both  $A$  and  $B$ . Now we show that the union cannot be strictly worse than both. Let  $x_A$  and  $x_B$  be the  $u$ -best elements in  $\Gamma(A)$  and  $\Gamma(B)$ , respectively, and take  $v_A$  and  $v_B$  be the maximum values of  $v$  in  $A$  and in  $B$ , respectively. Then we have

$$v_A \leq v(x_A) + \bar{w} \text{ and } v_B \leq v(x_B) + \bar{w}$$

Therefore the maximum value of  $v$  in  $A \cup B$  is the higher one between  $v_A$  and  $v_B$ , either  $x_A$  or  $x_B$  must be in  $\Gamma(A \cup B)$  so  $A \cup B$  must be weakly better than either  $A$  or  $B$ .

Finally we show that the representation implies Axiom 4. Suppose  $\{x\} \succ \{x, y\} \succ \{y, z\}$ , then it must be  $x \succ y \succ z$ ,  $v(y) - v(x) > \bar{w}$  and  $v(z) - v(y) > \bar{w}$  (so  $v(z) - v(x) > \bar{w}$ ). If  $v(z) - v(t) > \bar{w}$ , then  $xyzt \sim z \sim yz$ . If  $v(z) - v(t) \leq \bar{w}$ , then  $xyzt \sim z \sim yz$  or  $xyzt \sim t \sim xt$  (as  $v(t) - v(x) \geq v(y) - v(x) > \bar{w}$ ).