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Walrasian analysis via two-player games[☆]

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Abstract

We associate to any pure exchange economy a game with only two players, regardless of the number of consumers. In this two-player game, each player represents a different role of the society, formed by all the individuals in the economy. Player 1 selects feasible allocations trying to make Pareto improvements. Player 2 chooses an alternative from the wider range of allocations that are feasible in the sense of Aubin. The set of Nash equilibria of our game is non-empty and our main result provides a characterization of Walrasian equilibria allocations as strong Nash equilibria of the associated society game.

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1. Introduction

Game theoretic approaches to economic equilibrium and particularly to Walrasian or competitive equilibrium, provide insights into the market mechanism through which trade is conducted.

The Walrasian equilibrium reflects the spirit of “the invisible hand” and of decentralization. However, the power and appeal of this equilibrium concept appears to be far greater than that of mere decentralization. This is reflected in the finding that under the appropriate conditions the Walrasian equilibrium may be regarded either as the solution or as the limit solution for several cooperative and non-cooperative game-theoretic notions of equilibria.

Regarding cooperative notions, a great deal of work has been done by showing the connection of Walrasian equilibria with the core of the economy. Edgeworth (1881) proved, in the case of two agents and two commodities, that the core shrinks to the set of Walrasian equilibria and claimed that his result applies for an arbitrary number of commodities and an arbitrary number of agents. Almost eighty years later, Shubik (1959) recognized the importance of Edgeworth’s contribution and pointed out the relationship between the core and Edgeworth’s idea of the contract curve. Debreu and Scarf (1963) provided a rigorous formulation of the Edgeworth’s conjecture and showed that the

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intersection of the cores of the sequence of the replications coincides with the set of Walrasian equilibrium allocations. Further, Aumann (1964) introduced a model of an economy with a continuum of agents and showed the core–Walras equivalence; that is, in the Aumann’s model the core exactly coincides with the set of competitive equilibrium allocations. Following these pioneering contributions, the relations between the core and the set of equilibrium allocations have been a major focus of the literature in mathematical economics during the 1970s and 1980s. Notable contributions on this direction include Arrow and Hahn (1971), Bewley (1973), Hildenbrand (1974), Dierker (1975), Khan (1976), Trockel (1976), Anderson (1978, 1981, 1985), Shubik and Wooders (1982), Hammond et al. (1989), and Kaneko and Wooders (1989). These works establish the existence of an equilibrium price system as a result of a theory whose prime concern is the power of coalitions and makes no mention of prices.

The search for non-cooperative game theoretic foundations of Walrasian equilibrium rests on the Nash equilibrium solution. The seminal paper of Nash (1950), on the existence of equilibrium points in non-cooperative games, was historically critical for Walrasian analysis and founded the genesis for a rapidly growing series of papers on strategic approaches to economic equilibrium. In order to prove existence of Walrasian equilibrium, Debreu (1952), Arrow and Debreu (1954) and Debreu (1962) extended Nash’s model to “generalized games” by adding a fictitious price player whose payoff was the value of the excess demand. Walras equilibrium was then obtained as Nash equilibrium of a “pseudo-game” that included this additional price player. Hence, the game theoretical interpretation of the Walrasian equilibrium in the proof of this existence result is not based on a game but merely in a pseudo-game. One would, therefore, hope for an economic or game theoretical model that formulates an exchange process and/or a price-setting process in addition to the consumer behavior in the market. Actually, this is the aim of all the papers on strategic market games, which was initiated by Shubik (1973), Shapley (1976), and Shapley and Shubik (1977) and at the present moment constitutes a well known alternative to the Walrasian model. Other attempts to provide strategic foundations of competitive equilibria make use of cooperative game theory. In this direction, Shapley and Shubik (1969) showed that the class of market games and the class of totally balanced games are the same and Wooders (1994) proved an equivalence between large games with effective small groups of players and games generated by markets.

Most of the literature on market games deals with the basic problem of redistribution of goods in the framework of an exchange economy and models this procedure by describing explicitly the behavior of the agents and the corresponding process of formation of prices and exchange of goods. There is a first mechanism (game form), where the agents are the players and their strategies are signals (in terms of money and/or commodities to buy or sell on each trading post), which specify as an outcome a new allocation of the quantities announced. Prices appear at this stage as an interim technical device. Once endowments and utilities are added to the model, one can describe the set of feasible strategies for each player and evaluate the outcome in terms of utilities. As a consequence, one faces a strategic game.

Strategic market games may be classified into different categories depending on the underlying strategy sets for players and on the way in which every agent’s signal is used to determine market prices. In any case, these market games can be viewed as extensions of the single market analysis of Cournot (1838) and Bertrand (1883) to multiple markets within a general equilibrium framework.

The extension of the Cournot tradition to general equilibrium was pioneered by the already cited works by Shubik (1973), Shapley (1976), and Shapley and Shubik (1977). In order to overcome the difficulty that an agent might want to sell in one market and buy in another, Shapley and Shubik explicitly introduced money as the stipulated medium of exchange. Their model was carried forward by several other authors, who showed that the Cournot–Nash equilibria converge to Walrasian equilibria (see, for instance, Dubey and Shapley, 1994 and Dubey and Geanakoplos, 2003).

Hurwicz (1979), Schmeidler (1980) and Dubey (1982) followed the Bertrand tradition, which naturally led to discontinuous payoff functions, and established the exact coincidence of Nash and Walrasian equilibria, relying on the existence of Walrasian equilibrium of the economy to show the existence of the Nash equilibrium of the game.

This paper adds to the broad range of literature on strategic approaches to Walrasian equilibrium. Our aim is to show a characterization of Walrasian equilibria within a strategic game approach which differs from those contemplated in the previous papers. Actually, the game we consider is neither a generalized game nor a game in the tradition of Cournot or Bertrand, but a two-player game played by the society, representing all the agents in the economy. Furthermore, money is not considered in our game and prices are not involved either in the strategy sets or the payoff functions defining the game. Another important difference to point out is that we do not need to define outcome functions from the strategy profiles. In fact, in our society game the outcomes are given by the strategies.

Given any pure exchange economy with a finite number of agents, we define an associated game with only two players. We refer to this game as the society game because it can be interpreted as a game in which the society plays two different roles. The first one consists in acting as a Paretian player in pursuit of efficiency. The second role involves an impartial and fair behavior against the Paretian player.

The Paretian player, player 1, selects feasible allocations and tries to make Pareto improvements. On the other hand, the society, acting as player 2, chooses strategies among the wider range of the feasible allocations in the sense of Aubin, that is, allocations that are feasible by considering strictly positive participation or weights of each member of the society.

A strategy profile in this society game is given by a feasible allocation chosen by player 1, and by the weights and the Aubin allocation chosen by player 2. The payoff function for the Paretian player depends on her strategy and on the Aubin allocation proposed by player 2, but it is not directly affected by the selected weights, which only appear explicitly in the second player's payoff function. By definition, the payoffs for both players cannot be strictly positive simultaneously. Moreover, the player 2 can always get a null payoff by choosing the same allocation as player 1 and her payoff can be strictly positive only in the case in which the Paretian player's strategy is not a Walrasian allocation.

The assumption stated for our exchange economy leads to the existence of Walrasian equilibria. It is not difficult to show that the strategy in which both players play the same Walrasian allocation is a Nash equilibrium for the society game. As the Paretian player can increase her payoff when her strategy is not an efficient allocation, at any Nash equilibrium the strategy selected by player 1 is required to be Pareto optimal. Hence, the first player represents efficiency whereas the second one is a weighting player who tries to give balance. Actually, we will show that at any Nash outcome both payoffs are zero, Paretian player achieves efficiency and the Aubin player restores Walrasian equilibrium allocations.

We recall that all the previous works on market games model the economy as a game where each consumer is a player that announces quantities or both quantities and prices and, therefore, the economy is recasted as a game with at least as many players as consumers. However, in this paper the game we associate to a pure exchange economy is just a two-player game, regardless of the number of consumers in the economy.

Our main result provides an equivalence between the Walrasian allocations and the set of Nash equilibria of the associated society game. We obtain a characterization of Walrasian equilibria of the n -agent economy in terms of Nash equilibria of a game with only two players. In other words, we show that the Walrasian mechanism is implementable as a Nash equilibrium of a two-player game.

In order to obtain our results, the key idea is to exploit the veto power of the society. In Hervés-Beloso et al. (2005a, 2005b), we provide characterizations of Walrasian allocations in terms of the blocking power of the "society" called "grand coalition." Precisely, in Hervés-Beloso et al. (2005b), it is shown that the set of Walrasian allocations coincides with the set of allocations which are not blocked, in the sense of Aubin, by the society. Therefore, in order to obtain the equilibria it suffices to consider the Aubin blocking power of just one coalition, namely, the society formed by all the individuals in the economy. This equivalence between the set of Walrasian allocations and the set of Aubin non-dominated allocations by the society, which is already stated for the more general case of differential information economies, will be used in order to show our main result. The proof of that characterization rests on a core–Walras equivalence theorem and on the blocking power of large coalitions in continuum economies (Vind, 1972). That is, we apply results which connect Walrasian equilibria to subtle cooperative solutions. Then, although in this article we follow a non-cooperative approach to Walrasian analysis, the underlying arguments are somehow related to a cooperative approach.

To sum up, this paper is related to the literature on non-cooperative market games and the main result shows a coincidence between Walrasian equilibria and Nash equilibria of this society game. On the other hand, it is also related to a cooperative approach to equilibria due to the implicit use of core equivalence results. Furthermore, it can be easily shown that Nash equilibria of the society game exist, and that they are strong Nash equilibria. Therefore, we can conclude that the Walrasian mechanism is implementable as a strong Nash equilibrium of a game with two players which represent the society. Finally, we remark that, for simplicity, the model we consider in this paper addresses a complete information pure exchange economy with a finite number of consumers and a finite number of commodities. However, the same results hold for more general settings. To be more precise, the equivalence between Walrasian equilibria and Nash equilibria of this society game does still hold for economies with infinitely many commodities and for differential information economies. This is due to the fact that the required tool is core–Walras equivalence which has been established for more general commodity spaces.

The remainder of the paper is organized as follows. Section 2 states the model of a finite exchange economy and recalls the solution concepts and the already mentioned results that will be used in the proof of our theorem. In Section 3, we define the society game associated to a pure exchange economy and discuss the definitions of strategy sets and payoff functions that describe the game. Section 4 includes our main result and some previous lemmas. In Section 5 we modify the game in order to include extreme situations not contemplated in the previous section. Finally, Section 6 is the conclusion.

2. The economy

Consider a pure exchange economy \mathcal{E} with n consumers and ℓ commodities. The commodity space is \mathbb{R}^ℓ . Each consumer i is characterized by the consumption set \mathbb{R}_+^ℓ , a preference relation \succeq_i on \mathbb{R}_+^ℓ and an initial endowment $\omega_i = (\omega_{ij})_{j=1}^\ell \in \mathbb{R}_+^\ell$.

We state the following assumptions on endowments and preference relations for every consumer i :

(H.1) The initial endowment, ω_i , is strictly positive, i.e., $\omega_i \gg 0$.

(H.2) The preference relation, \succeq_i , is continuous, strictly monotone and convex.

Note that the assumption (H.2) means that each preference relation, \succeq_i , is represented by a continuous, quasi-concave and strictly increasing utility function $U_i: \mathbb{R}_+^\ell \rightarrow \mathbb{R}_+$, with $U_i(0) = 0$. So, the economy is defined by $\mathcal{E} \equiv (\mathbb{R}_+^\ell, (U_i, \omega_i)_{i=1}^n)$.

An allocation x is a consumption bundle $x_i \in \mathbb{R}_+^\ell$ for every agent $i = 1, \dots, n$. The allocation x is feasible in the economy \mathcal{E} if $\sum_{i=1}^n x_i \leq \sum_{i=1}^n \omega_i$. A price system is an element of Δ , where Δ denotes the $(\ell - 1)$ -dimensional simplex of \mathbb{R}_+^ℓ , that is, $\Delta = \{p \in \mathbb{R}_+^\ell \text{ such that } \sum_{h=1}^\ell p_h = 1\}$. A Walrasian equilibrium for the economy \mathcal{E} is a pair $(p, x) \in \Delta \times \mathbb{R}_+^{\ell n}$, where p is a price system and x is a feasible allocation such that, for every agent i , the bundle x_i maximizes the preference relation \succeq_i (or equivalently, the utility function U_i) in the budget set $B_i(p) = \{y \in \mathbb{R}_+^\ell \text{ such that } p \cdot y \leq p \cdot \omega_i\}$.

A feasible allocation x is blocked by a coalition S if there exists $y_i, i \in S$, such that $\sum_{i \in S} y_i \leq \sum_{i \in S} \omega_i$ and $U_i(y_i) > U_i(x_i)$ for every member i in the coalition S . The core of the economy is the set of feasible allocations which are not blocked by any coalition of agents. It is well known that under the hypotheses (H.1) and (H.2) the economy \mathcal{E} has Walrasian equilibrium, and that if x is a Walrasian allocation for the economy \mathcal{E} , then x belongs to the core of \mathcal{E} .

Aubin (1979), addressing pure exchange economies with a finite number of agents and commodities, introduced the pondered veto concept. The veto system proposed by Aubin extends the notion of ordinary veto in the sense that allows the agents to participate with a fraction of their endowments when forming a coalition. This veto mechanism is known in the literature as *fuzzy veto*. The term *fuzzy* is usually used in relation to the elements that belong to a set with certain probability. In the veto system introduced by Aubin, agents actually participate in a coalition with a fraction of their endowments (which, under standard assumptions, is equivalent to the classical Debreu–Scarff participation of a coalition in a replicated economy—see Florenzano, 1990). Therefore, we prefer to designate this veto system as *Aubin veto* or *veto in the sense of Aubin*.

Following Aubin (1979), we define the Aubin veto as follows: an allocation x is blocked in the sense of Aubin by the coalition S via the allocation y if there exist $\alpha_i \in (0, 1]$, for each $i \in S$, such that (i) $\sum_{i \in S} \alpha_i y_i \leq \sum_{i \in S} \alpha_i \omega_i$, and (ii) $U_i(y_i) > U_i(x_i)$, for every $i \in S$.

The Aubin core of the economy \mathcal{E} is the set of all feasible allocations which cannot be blocked in the sense of Aubin.

This definition of Aubin core is equivalent to the original one, Aubin (1979). However, it is important to remark that we require the coefficients α_i to be strictly positive for every agent forming the coalition. If we consider (as in the original definition by Aubin) the possibility of null weights or contributions, the coalition formed by all the agents (the society) contains, implicitly, any other coalition (see Hervés-Beloso and Moreno-García, 2001 for more details).

Aubin (1979) showed that, under standard assumptions, any Walrasian allocation is in the Aubin core, and, reciprocally, any non-Walrasian allocation is blocked in the sense of Aubin (see Florenzano, 1990 for economies with an infinite-dimensional commodity space and without ordered preferences).

Hervés-Beloso et al. (2005b), provide a new characterization of Walrasian equilibrium allocations in terms of the blocking power of the “society” called “grand coalition.” Precisely, under assumptions (H.1) and (H.2), a feasible allocation is a Walrasian equilibrium allocation in \mathcal{E} , if and only if x is not blocked by the society in the sense of Aubin. It should be remarked that in the characterization above, the society is able to block, in the sense of Aubin, any non-Walrasian allocation with a contribution of each member close to the total participation.

To be more precise, we can write the above result in the following way:

(*) *Let \mathcal{E} be a pure exchange economy under assumptions (H.1) and (H.2). The next statements hold:*

If x is a feasible allocation which is Aubin blocked by the society, then x is not a Walrasian allocation.

Reciprocally, if x is feasible but not a Walrasian allocation then, for any positive $\alpha < 1$ there exist coefficients $\alpha_i \in [\alpha, 1]$, and consumption bundles $y_i, i = 1, \dots, n$, such that $\sum_{i=1}^n \alpha_i y_i \leq \sum_{i=1}^n \alpha_i \omega_i$, and $U_i(y_i) > U_i(x_i)$, for every agent i .

This equivalence between the set of Walrasian allocations and the set of allocations that the society cannot block in the sense of Aubin, with participations of every member arbitrarily close to the total participation, will be used in the rest of the paper and we will refer to this characterization by the symbol (*).

3. The associated game

Consider the pure exchange economy $\mathcal{E} \equiv (X = \mathbb{R}^\ell, (U_i, \omega_i)_{i=1}^n)$ defined in the previous section.

We define a game \mathcal{G} associated to the economy \mathcal{E} in order to analyze the relation between the non-cooperative solution of Nash equilibrium and the decentralized solution of Walrasian equilibrium.

There are two players. The strategy set for player 1 is denoted by S_1 and is given by

$$S_1 = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}_+^{\ell n} \text{ such that } x_i \neq 0 \text{ and } \sum_{i=1}^n x_i \leq \sum_{i=1}^n \omega_i \right\}.$$

That is, the strategy set for player 1 is the set of feasible allocations that assign a non-zero consumption for each agent. Observe that $\omega = (\omega_1, \dots, \omega_n) \in S_1$.

Let α be a real number such that $0 < \alpha < 1$. The strategy set for player 2 is denoted by S_2 and is defined as follows:

$$S_2 = \left\{ (a, y) \in [\alpha, 1]^n \times \mathbb{R}_+^{\ell n} \text{ such that } \sum_{i=1}^n a_i y_i \leq \sum_{i=1}^n a_i \omega_i \right\}.$$

That is, the strategy set for player 2 is the set of allocations that are feasible in the sense of Aubin with a participation greater or equal to α for every member of the society. Observe that S_2 is a non empty set ($\mathbf{1}, \omega \in S_2$, where $\mathbf{1}$ is the vector in $[\alpha, 1]^n$ whose coordinates are constant and equal to 1).

Let S denote the product set $S_1 \times S_2$. A strategy profile is any $s = (x, a, y) \in S$, that is, a strategy profile is a strategy $x \in S_1$ for player 1 and a strategy $(a, y) \in S_2$ for player 2.

Given a strategy profile $s = (x, a, y) \in S$, the payoff functions Π_1 and Π_2 , for players 1 and 2, respectively, are defined as follows:

$$\begin{aligned} \Pi_1(x, a, y) &= \min_i \{ U_i(x_i) - U_i(y_i) \}, \\ \Pi_2(x, a, y) &= \min_i \{ a_i (U_i(y_i) - U_i(x_i)) \}. \end{aligned}$$

In short, the game \mathcal{G} is defined by the strategy sets and the payoff functions for each player. We write

$$\mathcal{G} \equiv \{ S_1, S_2, \Pi_1, \Pi_2 \}.$$

From the definition of the game \mathcal{G} we can obtain some immediate consequences.

Let s be a strategy profile. If $\Pi_1(s) > 0$, then $\Pi_2(s) < 0$. Reciprocally, if $\Pi_2(s) > 0$, then $\Pi_1(s) < 0$. That is, $\Pi_1(s)$ and $\Pi_2(s)$ cannot be strictly positive for any s , although both $\Pi_1(s)$ and $\Pi_2(s)$ can be strictly negative for some strategy profile s .

Given the strategy profile $s = (x, a, y) \in S$, note that if $x \in S_1$ is not an efficient allocation, then there exists a feasible allocation z such that $U_i(z_i) > U_i(x_i)$ for every $i = 1, \dots, n$; and then $U_i(z_i) - U_i(y_i) > U_i(x_i) - U_i(y_i)$ for every $i = 1, \dots, n$, and for any $(a, y) \in S_2$. In other words, if x is not a Pareto optimum, there exists an allocation $z \in S_1$ such that $\Pi_1(z, a, y) > \Pi_1(x, a, y)$, for any $(a, y) \in S_2$. That is, if x is not a Pareto optimum, player 1 can improve upon her payoff.

On the other hand, if player 2 selects (a, x) , where x is a feasible and efficient allocation, then the best response of player 1 is also the Pareto optimum x . To be precise, if x is a Pareto optimum, we have $\Pi_1(x, a, x) \geq \Pi_1(z, a, x)$, for any $z \in S_1$. To see this, note that $\Pi_1(x, a, x) = 0$ and if there exists $z \in S_1$ such that $\Pi_1(z, a, x) > \Pi_1(x, a, x) = 0$, then $U_i(z_i) > U_i(x_i)$ for every individual in the society, which is in contradiction with the efficiency of x .

Moreover, if $\Pi_2(x, a, y) > 0$, then x is blocked by the society in the sense of Aubin. Reciprocally, if x is an allocation blocked by the society in the sense of Aubin, then there exists $(a, y) \in S_2$ such that $\Pi_2(x, a, y) > 0$. Furthermore, according to the characterization (*) if x is a Walrasian allocation then $\Pi_2(x, a, y) \leq 0$ for any $(a, y) \in S_2$.

Definition 3.1. A Nash equilibrium for the game \mathcal{G} is a strategy profile $s^* = (x^*, a^*, y^*) \in S$ such that

$$\begin{aligned} \Pi_1(s^*) &\geq \Pi_1(x, a^*, y^*), & \text{for every } x \in S_1 & \text{ and} \\ \Pi_2(s^*) &\geq \Pi_2(x^*, a, y), & \text{for every } (a, y) \in S_2. \end{aligned}$$

Proposition 3.1. The set of Nash equilibria in pure strategies for the game \mathcal{G} is not empty.

Proof. This is a consequence of the existence of Walrasian equilibrium of the economy \mathcal{E} . In fact, if x is a Walrasian allocation, then $(x, \mathbf{1}, x)$ is a Nash equilibrium of the society game \mathcal{G} . To see this, note that $\Pi_1(x, \mathbf{1}, x) \geq \Pi_1(z, \mathbf{1}, x)$, for all $z \in S_1$, since x is a Pareto-optimum. On the other hand, if there exists $(a, y) \in S_2$ such that $\Pi_2(x, a, y) \geq \Pi_2(x, \mathbf{1}, x)$, it would imply that x could be blocked in the sense of Aubin, which is a contradiction with the fact that x is Walrasian. \square

Remark. Given any $x \in S_1$, the strategy $(\mathbf{1}, x)$ belongs to S_2 . Therefore the payoff for agent 2 at any Nash equilibrium cannot be negative. That is, if $s^* = (x^*, a^*, y^*)$ is a Nash equilibrium, then $\Pi_2(s^*) \geq 0$.

As previously observed, if $s^* = (x^*, a^*, y^*)$ is a Nash equilibrium of the game \mathcal{G} then the allocation x^* is necessarily Pareto-efficient.

Finally, if $s^* = (x^*, a^*, y^*)$ is a Nash equilibrium of the game \mathcal{G} and $\Pi_2(s^*) = 0$, then the allocation x is a non-dominated allocation in the sense of Aubin and, therefore, applying the characterization (*), x^* is a Walrasian allocation of the economy \mathcal{E} .

4. The main result

In this section, we state our main result which shows the equivalence between the Walrasian equilibria of the economy \mathcal{E} and the set of Nash equilibria of the associated game \mathcal{G} . That is, we obtain a characterization of Walrasian equilibria in terms of Nash equilibria of a game with only two players, independently of the number of consumers in the economy. Thus, we show that the Walrasian mechanism is implementable as a Nash equilibrium of a two-player game.

As we have remarked in the Introduction, the game \mathcal{G} can be interpreted as a society game where the society plays two different roles: in the first role, the society, as player 1, selects feasible allocations and tries to make Pareto improvements, while as player 2, society comes up with alternative allocations that are feasible in the sense of Aubin.

As we have already observed, the Paretian player has an incentive to deviate whenever the strategy she chooses is not an efficient allocation. Hence, at any Nash equilibrium, the strategy for player 1 is required to be Pareto-optimal. Then, we may argue that the society, as player 1, seeks efficiency.

The society, as player 2, acts as an adviser who recommends different assignments trying to dominate the allocation proposed by the Paretian player. Player 2 has incentives to deviate whenever the strategy she selects is a dominated allocation in the sense of Aubin. On the other hand, this ‘‘Aubin’’ player can always get a non-negative payoff (by

choosing the same allocation as player 1) and can reach a strictly positive payoff only in the case that the allocation proposed by player 1 is not Walrasian.

We will show that the values of the payoff functions which come from any Nash equilibrium coincide for both players and are equal to zero. As we will see, this fact avoids the allocation proposed by player 1 to be Aubin dominated at any Nash equilibrium. Thus, at any Nash equilibrium $s^* = (x^*, a^*, y^*)$, the outcome x^* is feasible and efficient while player 2 ensures that it is Walrasian.

In order to show our main result we need some previous lemmas. Given a strategy profile $s = (x, a, y)$ let us define the sets:

$$B(s) = \left\{ k \in \{1, \dots, n\}, \text{ s.t. } U_k(x_k) - U_k(y_k) = \min_i \{U_i(x_i) - U_i(y_i)\} \right\};$$

$$B'(s) = \left\{ k \in \{1, \dots, n\}, \text{ s.t. } a_k(U_k(x_k) - U_k(y_k)) = \min_i \{a_i(U_i(x_i) - U_i(y_i))\} \right\}.$$

Given a real number $a \in [0, 1]$ we denote $\mathbf{a} = (a, \dots, a) \in [0, 1]^n$, i.e., \mathbf{a} is the vector in $[0, 1]^n$ whose coordinates are identical and equal to a .

The next lemmas show that, in a Nash equilibrium, the minima which define the payoff functions Π_1 and Π_2 are attained by every consumer. That is, if s^* is a Nash equilibrium of the game \mathcal{G} , then $B(s^*) = B'(s^*) = \{1, \dots, n\}$.

Lemma 4.1. *If x^* is player 1's best response to the strategy (a^*, y^*) selected by player 2, in particular, if $s^* = (x^*, a^*, y^*)$ is a Nash equilibrium of the game \mathcal{G} , then $U_i(x_i^*) - U_i(y_i^*) = U_j(x_j^*) - U_j(y_j^*)$ for every $i, j \in \{1, \dots, n\}$.*

Proof. Let x^* be player 1's best response to the strategy (a^*, y^*) selected by player 2. Assume that the statement of the lemma does not hold. Then, $B(s^*) \neq \{1, \dots, n\}$. This implies that there exists a consumer j such that $U_j(x_j^*) - U_j(y_j^*) > U_i(x_i^*) - U_i(y_i^*)$ for every $i \in B(s^*)$. By continuity of the utility functions, there exists some $\delta > 0$ such that player 1 can deviate to x' , where $x'_j = (1 - \delta)x_j^*$ and $x'_i = x_i^* + \frac{\delta}{n-1}x_j^*$, $\forall i \neq j$, and still have $U_j(x'_j) - U_j(y_j^*) > U_i(x'_i) - U_i(y_i^*)$ for every $i \in B(s^*)$. By monotonicity of preferences, $U_i(x'_i) > U_i(x_i^*)$ for every $i \neq j$, which implies that $\Pi_1(x', y^*) > \Pi_1(s^*)$. This contradicts the fact that x^* is player 1's best response to (a^*, y^*) . \square

Lemma 4.2. *If (a^*, y^*) is player 2's best response to the strategy x^* selected by player 1, in particular, if $s^* = (x^*, a^*, y^*)$ is a Nash equilibrium of the game \mathcal{G} , then $a_i(U_i(y_i^*) - U_i(x_i^*)) = a_j(U_j(y_j^*) - U_j(x_j^*))$ for every $i, j \in \{1, \dots, n\}$.*

Proof. Let (a^*, y^*) be player 2's best response to the strategy x^* selected by player 1. Since $(\mathbf{1}, x^*) \in S_2$, one has $\Pi_2(s^*) \geq 0$. Assume that the statement of the lemma does not hold. Then, $B'(s^*) \neq \{1, \dots, n\}$. This implies that there exists a consumer j such that $a_j(U_j(y_j^*) - U_j(x_j^*)) > a_i(U_i(y_i^*) - U_i(x_i^*)) \geq 0$ for every $i \in B'(s^*)$. This implies that $U_j(y_j^*) > U_j(x_j^*)$, so, by strict monotonicity of preferences, $y_j^* > 0$. By continuity of the utility functions, we can take a non-null commodity bundle $\varepsilon \in \mathbb{R}_+^{\ell}$ such that $a_j(U_j(y_j^* - \varepsilon) - U_j(x_j^*)) > a_i(U_i(y_i^*) - U_i(x_i^*))$ for every $i \in B'(s^*)$. Let $\delta = \frac{a_j}{\sum_{i \in B'(s^*)} a_i} \varepsilon$. Consider the allocation $y = (y_1, \dots, y_n)$ defined as follows:

$$y_i = \begin{cases} y_i^* - \varepsilon & \text{if } i = j, \\ y_i^* + \delta & \text{if } i \in B'(s^*), \\ y_i^* & \text{otherwise.} \end{cases}$$

By construction, we obtain:

$$\begin{aligned} \sum_{i=1}^n a_i^* y_i &= a_j^* (y_j^* - \varepsilon) + \sum_{i \in B'(s^*)} a_i^* (y_i^* + \delta) + \sum_{\substack{i \in B'(s^*) \\ i \neq j}} a_i^* y_i^* \\ &= a_j^* y_j^* - a_j^* \varepsilon + \sum_{i \in B'(s^*)} a_i^* y_i^* + \sum_{i \in B'(s^*)} a_i^* \delta + \sum_{\substack{i \notin B'(s^*) \\ i \neq j}} a_i^* y_i^* \\ &= \sum_{i=1}^n a_i^* y_i^* \leq \sum_{i=1}^n a_i^* \omega_i. \end{aligned}$$

Then we have that $(a^*, y) \in S_2$. On the other hand, by monotonicity of preferences, $U_i(y_i) > U_i(y_i^*)$ for every $i \in B'(s^*)$. Therefore, we conclude that $\Pi_2(x^*, a^*, y) > \Pi_2(s^*)$, which is a contradiction with the fact that (a^*, y^*) is player 2's best response to x^* . \square

As an immediate consequence of the previous lemmas we obtain the following proposition.

Proposition 4.1. *If $s^* = (x^*, a^*, y^*)$ is a Nash equilibrium for the game \mathcal{G} , then $U_i(y_i^*) = U_i(x_i^*)$ for every $i = 1, \dots, n$, and $\Pi_1(s^*) = \Pi_2(s^*) = 0$.*

Proof. Let $s^* = (x^*, a^*, y^*)$ be a Nash equilibrium for the game \mathcal{G} . Since $(\mathbf{1}, x^*) \in S_2$, we have $\Pi_2(s^*) \geq 0$. Assume that the statement of the proposition does not hold. Then, $\Pi_2(s^*) > 0$, which implies that $\Pi_1(s^*) < 0$. By the two previous lemmas, $a_i^* = a > 0$ for every $i = 1, \dots, n$. This implies that $\sum_{i=1}^n y_i^* \leq \sum_{i=1}^n \omega_i$, so y^* belongs to S_1 . Hence, $\Pi_1(y^*, a^*, y^*) = 0 > \Pi_1(s^*)$, which is a contradiction. \square

Before we state our characterization result, we will show, as an easy consequence of the previous lemmas, that the Nash equilibria of the society game are actually strong Nash equilibria.

Proposition 4.2. *Any Nash equilibrium of the associated game \mathcal{G} is a strong Nash equilibrium.*

Proof. Let $s^* = (x^*, a^*, y^*)$ be a Nash equilibrium for the game \mathcal{G} . Since there are only two players in the game, it is enough to show that the coalition formed by both players has no incentive to deviate. Otherwise, there is a strategy profile $s = (x, a, y) \in S$ such that player 1 and player 2 improve. Then, by Proposition 4.1, we have $\Pi_1(s) > \Pi_1(s^*) = 0$ and $\Pi_2(s) > \Pi_2(s^*) = 0$. But, by the definition of the payoff functions, this is impossible, that is, both inequalities above cannot hold together because if $\Pi_1(s) > 0$, then necessarily $\Pi_2(s) < 0$. \square

The next theorem is our main result in this paper. It shows the relation between the set of Walrasian equilibria of the economy \mathcal{E} and the set of Nash equilibria of the associated game \mathcal{G} . This characterization of Walrasian equilibria allows us to conclude that the Walrasian mechanism is implementable as a Nash equilibrium of the society game.

Theorem 4.1. *Let \mathcal{E} be a pure exchange economy under assumptions (H.1) and (H.2).*

If $s^ = (x^*, a^*, y^*)$ is a Nash equilibrium for the game \mathcal{G} , then x^* is a Walrasian equilibrium allocation for the economy \mathcal{E} .*

Reciprocally, if x^ is a Walrasian equilibrium allocation for the economy \mathcal{E} , then any $s^* = (x^*, a^*, y^*) \in S$, with $U_i(y_i^*) = U_i(x_i^*)$ for every $i = 1, \dots, n$, is a Nash equilibrium for the game \mathcal{G} .*

In particular, x^ is a Walrasian equilibrium allocation for the economy \mathcal{E} , if and only if (x^*, \mathbf{b}, x^*) with $\mathbf{b}_i = b$, for every $i = 1, \dots, n$, (for instance $(x^*, \mathbf{1}, x^*)$) is a Nash equilibrium for the game \mathcal{G} .*

Proof. Let $s^* = (x^*, a^*, y^*)$ be a Nash equilibrium for the game \mathcal{G} . Assume that x^* is not a Walrasian equilibrium allocation. Then, by (*), we can take $a = (a_1, \dots, a_n) \in [\alpha, 1]^n$ and y_i , for each $i \in \{1, \dots, n\}$, such that

- (a) $\sum_{i=1}^n a_i y_i \leq \sum_{i=1}^n a_i \omega_i$, and
- (b) $U_i(y_i) > U_i(x_i^*)$ for every $i = 1, \dots, n$.

Then, there exists $(a, y) \in S_2$ such that $\Pi_2(x^*, a, y) > \Pi_2(s^*)$, which contradicts the fact that s^* is a Nash equilibrium.

Reciprocally, let x^* be a Walrasian equilibrium allocation. Assume that (x^*, a, y) is not a Nash equilibrium and $(a, y) \in S_2$ is such that $U_i(x_i^*) = U_i(y_i)$ for every $i = 1, \dots, n$. Then, we have: (i) there exists $x \in S_1$ such that $\Pi_1(x, a, y) > \Pi_1(x^*, a, y) = 0$; or (ii) there exists $(b, z) \in S_2$ such that $\Pi_2(x^*, b, z) > \Pi_2(x^*, a, y) = 0$.

If (i) is the case, we obtain that $U_i(x_i) > U_i(y_i) = U_i(x_i^*)$ for every consumer i . Since x is a feasible allocation in the economy \mathcal{E} , we conclude that x^* is not an efficient allocation. By the first Welfare Theorem, it is a contradiction with the fact that x^* is a Walrasian allocation.

Assume that (ii) holds. Then, $U_i(z_i) > U_i(x_i^*)$ for every $i = 1, \dots, n$, and $\sum_{i=1}^n b_i z_i \leq \sum_{i=1}^n b_i \omega_i$, with $b_i > 0$ for all i . This implies that x^* is a dominated allocation in the sense of Aubin, which is a contradiction with the fact that x^* is a Walrasian equilibrium allocation. \square

Even though that we may obtain a continuum of Nash equilibria for the game \mathcal{G} with the same strategy for player 1, we can select a canonical representation. If (x^*, a^*, y^*) is a Nash equilibrium then $(x^*, \mathbf{1}, x^*)$ is also a Nash equilibrium for the game \mathcal{G} .

5. A remark

The reader may observe that we have excluded null consumption from the strategy set of player 1. In particular, Pareto-optimal allocations that assign all the initial endowments to one of the agents are excluded as strategies for player 1.

Note that $x_i \neq 0$ was only used in the proof of Lemma 4.1. Without this technical device, there could be an equilibrium of the game that would not correspond to a Walrasian allocation (see the example below).

On the other hand, assuming that the utility functions are concave, we may allow null consumptions and still obtain the same result if we define a new game \mathcal{G}' where the strategy set and the payoff function for player 1 are changed to S'_1 and Π'_1 .

To be precise, S'_1 is the set of feasible allocations:

$$S'_1 = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}_+^n \text{ such that } \sum_{i=1}^n x_i \leq \sum_{i=1}^n \omega_i \right\}.$$

Let $f(x)$ be a positive differentiable function defined in $[\alpha, 1]$, such that $f'(x)x > f(x)$. For example, $f(x) = e^{bx}$, with $b > \frac{1}{\alpha}$. Observe that $\frac{f(x)}{x}$ is a positive and strictly increasing function, therefore: $\max\{f(x)/x\} = f(1)$.

Given a strategy profile $s = (x, a, y) \in S'$, the new payoff function of player 1 is:

$$\Pi'_1(x, a, y) = \min_i \{ f(a_i)(U_i(x_i) - U_i(y_i)) \}.$$

Both strategy sets, S'_1 and S_2 , are compact. Obviously, S_1 is convex, but S_2 is not (we owe this observation to an anonymous referee). Thus, we cannot conclude from the strategies and payoff functions that the game has a Nash equilibrium. Observe that the considerations made for the game \mathcal{G} also apply to \mathcal{G}' . In particular, Proposition 3.1 also establishes existence of Nash equilibrium of the game \mathcal{G}' .

In order to prove Proposition 4.1, assume that $\Pi_2(s^*) = C > 0$.

Begin by seeing that if $\Pi_2(s^*) > 0$, then $a_{\max}^* = \max\{a_i^*\} = 1$. Or else, by deviating to $s = (x^*, a, y^*)$, where $a_i = \frac{a_i^*}{a_{\max}^*}$, player 2 improves its payoff. Since player 2 cannot improve its payoff, $\max\{a_i^*\} = 1$.

If all $a_i^* = 1$, then y^* is feasible and player 1 can obtain a null payoff by selecting $x^* = y^*$. This would give $\Pi_2 = 0$.

Lemma 4.2 is still valid in this modified game. In a Nash equilibrium: $\Pi_1 = \min\{f(a_i)(U_i(x_i) - U_i(y_i))\} = \min\{f(a_i)\frac{C}{a_i}\} = -Cf(1)$. The properties of the function $f(\cdot)$ would guarantee that the payoff of player 1 is determined in the i such that $a_i = 1$.

Now observe that if $a_i^* < 1$, then $x_i^* \neq 0$. Otherwise, player 2 could deviate, in the i th coordinate, to $(a', y') = [(1 + \varepsilon)a_i^*, y_i^*/(1 + \varepsilon)]$, obtaining at least the same payoff:

$$a'_i [U_i(y'_i) - U_i(x_i^*)] = a'_i U_i(y'_i) = (1 + \varepsilon)a_i^* U_i(y_i^*/(1 + \varepsilon)) \geq a_i^* U_i(y_i^*) = a_i^* [U_i(y_i^*) - U_i(x_i^*)].$$

This is an interior Aubin allocation:

$$\sum a'_j y'_j = \sum a_j^* y_j^* \leq \sum a_j^* \omega_j < \sum a_j^* \omega_j + \varepsilon a_i^* \omega_i = \sum a_j \omega_j.$$

Therefore, we can redistribute the remaining resources $(\varepsilon a_i^* \omega_i)$ and obtain a higher payoff. This contradiction implies that if $a_i^* < 1$, then $x_i^* \neq 0$.

The coefficients aren't all equal, so there is some i such that $a_i^* < 1$ (and $x_i^* \neq 0$) for which $f(a_i^*)[U_i(x_i^*) - U_i(y_i^*)] > \Pi_1(s^*)$. Then, by continuity, player 1 can select a strategy x' in which $x'_i = (1 - \delta)x_i^*$ and $x'_j = x_j^* + \frac{\delta}{n-1}x_i^*$, obtaining a higher payoff. This contradiction implies that $\Pi_2 = 0$, proving Proposition 4.1 and, as a consequence, Theorem 4.1.

5.1. An example

We will show that if null consumptions are allowed as strategies for player one in game \mathcal{G} , then the main result is no longer true.

Consider an economy with two agents and one commodity. Both agents have the same preference relation represented by the utility function $U(x) = x$. Let $\omega_1 = \omega_2 = \omega > 0$. We will see that the non-Walrasian allocation that assigns all the resources to one consumer can be a Nash equilibrium strategy for player one. Let $s^* = (s_1^*, s_2^*) \in S'_1 \times S_2$ with $s_1^* = (2\omega, 0)$ and $s_2^* = ((a, b), (x, y))$, such that $\Pi_2(s^*) > 0$. As we have seen, if s^* is a Nash equilibrium, then $a < 1$, $b = 1$ and, by Lemma 4.2, $a(x - 2\omega) = y$. On the other hand, $s_2^* \in S_2$ implies $ax + y = (a + 1)\omega$. This implies $x = 2\omega + \frac{1-a}{2a}\omega$ and $y = \frac{1-a}{2}\omega$. Then $\Pi_2(s^*) = \frac{1-a}{2}\omega$ and $a = \alpha$ guarantees that player 2 has no incentive to deviate. Observe that in the game \mathcal{G} , given s_2^* , when player 1 chooses the strategy (z, t) her payoff $\Pi_1((z, t), s_2^*) = \min\{z - (2\omega + \frac{1-\alpha}{2\alpha}\omega), t - \frac{1-\alpha}{2}\omega\}$ is attained in the first term and then the best response for player 1 is $(2\omega, 0)$. It is now clear that the profile $s^* = ((2\omega, 0), (\alpha, 1), (2\omega + \frac{1-\alpha}{2\alpha}\omega, \frac{1-\alpha}{2}\omega))$ is a Nash equilibrium.

However, in the modified game \mathcal{G}' the profile s^* cannot be a Nash equilibrium because s_1^* is not a Walrasian allocation. Observe that the payoff for player one is $\Pi'_1((z, t), s_2^*) = \min\{f(\alpha)(z - 2\omega - \frac{1-\alpha}{2\alpha}\omega), f(1)(t - \frac{1-\alpha}{2}\omega)\}$ where each term is increasing in z and t respectively. We have $\Pi'_1(s^*) = f(1)(\frac{1-\alpha}{2}\omega)$. Then player 1 increases her payoff by choosing the strategy $(2\omega - \varepsilon, \varepsilon)$ for ε small.

6. Conclusion

In this paper, we have provided a characterization of Walrasian equilibria allocations in terms of Nash equilibria of an associated two-player game that we have referred to as the society game. Moreover we have established that, independently of the number of consumers and commodities, Walrasian equilibrium is implementable as a strong Nash equilibrium of a two-player game.

This equivalence result adds to the great deal of works on strategic approaches to economic equilibrium. However, our society game differs substantially from those games considered in the literature on strategic market games with respect to several points: our society game involves only two players (it makes no difference the number of consumers in the economy); each player represents a role of the society formed by all the consumers in the economy and not an individual; the outcomes are given by the strategies themselves and prices appear neither in the strategy sets nor in the payoff functions.

The parameter α and the utility functions representing preferences are used in the definition of the game. However, as a consequence of our main result, we conclude that the allocations underlying Nash equilibria are the same, independently of the value of α and only depend on preferences.

Finally, as we have pointed out in the Introduction, our results do still hold for economies with infinitely many commodities and also for differential information economies. For this, it suffices to apply the characterization result (*) which is proved in Hervés-Beloso et al. (2005b) for more general settings.

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