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Smallness of Invisible Dictators

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M E X I C O

Smallness of Invisible Dictators

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Abstract

Fishburn (1970) showed that in an infinite society Arrow's axioms for a preference aggregation rule do not necessarily imply the existence of a dictator. In those cases in which there is no dictator, Kirman and Sondermann (1972) suggested two different approaches to justify that something similar to dictatorship occurs: one measure–theoretic, the other topological. Both approaches have their shortcomings. We develop here a third, set–theoretic, approach, and show its domain of applicability. We consider a model in which there are arbitrarily many agents and alternatives, and admissible coalitions may be restricted to lie in an algebra. In this framework (which includes the standard one), we characterize, in terms of Strict Neutrality, the Ultrafilter Property of preference aggregation rules. Based on this property, we define and characterize the different classes of dictatorial–like rules.
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1 Introduction

In many models of social choice, it is convenient to model the society as composed by an infinite set of individuals (see, e.g., Armstrong [2], Banks, Duggan and Le Breton [5], Gomberg, Martinelli and Torres [9], and Mihara [13], [14]). However, Fishburn [7] showed[1] that in this case there may exist Arrovian preference aggregation rules that are not dictatorial. From then on, several authors have attempted to show that nondictatorial Arrovian rules are in some sense close to a dictatorship. This work was first undertaken by Kirman and Sondermann [12], who suggested two different lines to approach the problem. The first line was measure-theoretic: the authors showed that if the agent set is an atomless probability measure space, then any nondictatorial Arrovian rule has decisive sets (that is, coalitions that can impose their strict preferences on society) of arbitrarily small measure. This approach was contested by Schmitz [15], who showed that, if the agent set has a σ-finite but infinite measure, then one can always construct nondictatorial Arrovian rules for which all decisive sets have infinite measure. A second line suggested by Kirman and Sondermann was topological: restricting themselves to the case in which there are finitely many alternatives, they showed that the agent set can be enlarged in such a way (the Stone-Čech compactification) that the original profile of preferences determines the preferences of all agents in the larger set, and each Arrovian rule has a dictator in the enlarged space. They referred to dictators in the larger space that do not belong to the original one as *invisible dictators*. The topological approach initiated by Kirman and Sondermann was extended by Armstrong [2] to measurable structures and arbitrary sets of alternatives. In the topological approach followed by Armstrong, one constructs a (discrete–like) topology[2] on the agent space and shows that the invisible dictators lie in its Stone-Čech compactification. The problem with this approach is that, if the original agent space is endowed with a topology that for some reason is relevant, this topology is essentially ignored in the construction of the enlarged agent space. Thus, one can always associate "invisible dictators" with nondictatorial Arrovian rules, but in a manner that in some cases may seem somewhat arbitrary.

In this paper, we pursue a third, set-theoretic, approach to show that to nondictatorial Arrovian rules there correspond arbitrarily small decisive coalitions. Our approach works also in the case of agent spaces where coalitons are restricted to

[1] Fishburn attributes prior knowledge of this result to Julian Blau. See Hansson [11].

[2] In this topology, the original agent set (after identifying points that are not separated by the algebra) is viewed as a subspace of the Stone space that corresponds to the algebra of coalitions. The Stone space is a totally disconnected compact topological space. See Sikorski [16].

lie in a measurable structure (an algebra or σ-algebra). In the latter case we cannot always apply this approach (as a matter of fact we construct counterexamples where it does not work), but we show that it is applicable in practically all the cases that have been considered in the literature.

We first consider the (Fishburn's, Kirman and Sondermann's) case in which all coalitions are admissible. These authors were the first[3] to show the *Ultrafilter Property of Arrovian rules*: the collection of all decisive coalitions that correspond to any Arrovian rule is an ultrafilter. With infinitely many agents, there are non-dictatorial Arrovian rules if the corresponding ultrafilter has an empty intersection (it is a *free ultrafilter*). In the latter case, we show that, out of any free ultrafilter, one can select a collection of *nested* decisive coalitions that shrink to the empty set. This is our set-theoretic concept of smallness.

Whenever the sets of alternatives and/or individuals are infinite, to develop a theory of social choice one eventually needs to resort to measurable structures (see, e.g., Armstrong [2], Banks, Duggan and Le Breton [5], Gomberg, Martinelli and Torres [9]). We consider next Arrovian rules in infinite societies with a measurable structure, i.e. an algebra or σ-algebra of admissible coalitions. We begin by showing that in this case there may be two different kinds of "invisible dictators": *non-measurable invisible dictators* (when the intersection of all decisive coalitions is a non-measurable set) and *empty invisible dictators* (when the intersection of all decisive coalitions is empty). We provide a necessary and sufficient condition for non-measurable invisible dictators to exist, and precisely characterize the decisive coalitions, as well as the preference aggregation rules, in this case. It turns out that, though essentially quite different, non-measurable invisible dictators are closest in spirit to the invisible dictators one obtains under the topological approach. Next, we focus on the study of empty invisible dictators, in which we try to investigate under what conditions our "selection problem" (selecting a nested subcollection that has an empty intersection out of the free ultrafilter of all decisive coalitions) has a solution. We show that, for any algebra of admissible coalitions, the selection problem has always a solution when the set of agents is countable[4]. Actually, we show that this result is true for arbitrary agent spaces,

[3]See below for a formal definition of ultrafilters. This result was first shown by Kirman and Sondermann [12], and was implicit in the proof given by Fishburn [7]. Essentially, it was earlier discovered by Guilbaud [10] as a consistency condition for the aggregation of preferences. (I thank Alan Kirman for pointing me in the direction that allowed me to find this earlier reference.) Hansson [11] also found it independently in a working paper circulated before the appearance of Kirman and Sondermann's article.

[4]When the set of agents is countable, Mihara [14] provides a constructive procedure to charac-

provided there is a countable decisive coalition. We display a counterexample to show that, when the admissible coalitions form an algebra and the set of agents is uncountable, the selection may not be possible if no decisive set is countable. For an arbitrary agent set, we show that the selection is always possible whenever the admissible coalitions are restricted to a σ-algebra, provided this σ-algebra is *countably generated*. This covers the usual set-up considered in probability theory: the agent set is a Borel space with its corresponding induced σ–algebra.

Summing up, when there are no measurability constraints on coalitions, one can always select a decreasing collection with an empty intersection out of any free ultrafilter. When one restricts admissible coalitions to lie in an algebra or a σ-algebra, then this selection may not be possible if the collection of admissible coalitions is too narrow with respect to the cardinality of the agent space. We describe broad classes of cases in which this does not happen, but we also give examples when it does happen. Note that the latter case can be attributed to the lack of adequacy of the collection of admissible coalitions, rather than to a fault in the selection as a solution concept.

As we mentioned before, Schmitz [15] showed that, if the agent space has an infinite σ-finite measure, then, given a free ultrafilter of decisive coalitions, there do not necessarily exist decisive coalitions of arbitrarily small measure. As examples of spaces satisfying his requirements, Schmitz cites the naturals with a counting measure, and $\mathbb{R}^n$ with its Borel subsets and Lebesgue measure. In both cases our selection result applies, and it implies that it is still possible to refer to "arbitrarily small" decisive coalitions, though in set-theoretic terms rather than in terms of measures.

The model we use is presented in section 2. In section 3, we characterize the Ultrafilter Property in the general framework (arbitrary sets of individuals and alternatives, measurable structures) we are considering. We resort to the Strict Neutrality property which has been recently used by Geanakoplos [8] and Úbeda [17] to provide very efficient proofs of Arrow's theorem in the finite case. We show that, in general, the Ultrafilter Property holds if, and only if, the preference aggregation rule satisfies Unanimity and Strict Neutrality. In particular, this implies the known fact that dictatorial rules are not necessarily Arrovian when one moves out of the domain of linear orders.

Our main contributions about invisible dictators and the selection problem are presented in section 4. In section 5 we conclude. We relegate technical results to an appendix.

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terize the invisible dictators. His focus, though, is quite different from ours.

2 The model

The (non-empty) set of individuals is denoted by N, which can be either finite or infinite. Admissible coalitions of individuals are members of an algebra $\mathscr{L}$ of subsets of N.

The (non-empty) set of alternatives is denoted by X. This set can be either finite or infinite, but it must have at least three different elements. Each individual has a (weak) preference relation on X, i.e. a reflexive, transitive and complete binary relation (a complete preorder). Let $\mathscr{R}$ denote the set of all preference relations on X. Given a preference relation R on X, we define its *indicator function*, $I(R) : X \times X \to \{-1,0,1\}$, by

$$I(R)(x,y) = \begin{cases} -1 & \text{if } yRx \text{ and not } xRy; \\ 0 & \text{if } xRy \text{ and } yRx; \\ 1 & \text{if } xRy \text{ and not } yRx. \end{cases}$$

A *preference profile* is a mapping $\rho : N \to \mathscr{R}$. Given a profile of preferences ρ, we define its *indicator function*, $\mathbf{I}(\rho) : X \times X \to \{-1,0,1\}^N$, by

$$\mathbf{I}(\rho)(x,y) = \Big(I[\rho(i)](x,y) \Big)_{i \in N}$$

We consider the following assumptions:

(FC) For any $i \in N$, the singleton $\{i\}$ belongs to $\mathscr{L}$.

(UD) For any $x,y \in X$, the mapping $i \mapsto I[\rho(i)](x,y)$, from N to $\{-1,0,1\}$, is $\mathscr{L}$–measurable.

Assumption (FC) means that all finite sets, and hence their complements, the cofinite sets, belong to $\mathscr{L}$ (the letters FC stand for "finite and cofinite"). Assumption (UD) means that all rankings among any given finite subset of alternatives are admissible (the letters UD stand for "universal domain"). Technically, assumption (UD) imposes a measurability requirement on the profiles of preferences that are allowed.

3 The Ultrafilter Property

Denote by $\mathscr{R}^N_{\mathscr{L}}$ the set of all preference profiles that satisfy assumption (UD). A preference aggregation rule is a map

$$f : \mathscr{R}^N_{\mathscr{L}} \to \mathscr{R}$$

That is, with each preference profile ρ the rule f associates a social preference relation on X.

Definition 1 *Given a preference aggregation rule f, we say that a coalition $A \in \mathscr{L}$ is f–*decisive *(for short,* decisive*) if*

$$\forall i \in A,\ I[\rho(i)](x,y) = 1 \implies I[f(\rho)](x,y) = 1.$$

Definition 2 *Given a preference aggregation rule f, we say that an individual $i \in N$ is a* dictator *if the coalition $\{i\}$ is f–decisive.*

We consider the following properties a preference aggregation rule f may satisfy. We define later the concept of an ultrafilter.

- *Unanimity* or *weak Pareto*: The coalition N of all individuals is f–decisive.
- *Independence of irrelevant alternatives* or *Pairwise Independence* (PI):

$$\mathbf{I}(\rho)(x,y) = \mathbf{I}(\rho')(x,y) \implies I[f(\rho)](x,y) = I[f(\rho')](x,y).$$

- *Strict Neutrality* (SN):

$$\begin{aligned}\mathbf{I}(\rho)(x,y) = \mathbf{I}(\rho')(x',y') \in \{-1,1\}^N \\ \implies I[f(\rho)](x,y) = I[f(\rho')](x',y') \in \{-1,1\}.\end{aligned}$$

- *Ultrafilter Property* (UP): The set of all f–decisive coalitions is an ultrafilter.

Geanakoplos [8] and Úbeda [17] have recently developed very efficient proofs of Arrow's theorem in the finite case, by showing first that Arrovian rules satisfy Strict Neutrality. We actually show something stronger: Strict Neutrality together with Unanimity is equivalent to the Ultrafilter Property. The fact that Arrovian rules satisfy the Ultrafilter Property was proved for the infinite, but not measurable, case by Kirman and Sondermann [12]. Armstrong [2] was the first to show that the same result is valid when admissible coalitons are restricted to an algebra. See also Austen-Smith and Banks [4], section 2.4, for generalizations in the finite case.

Lemma 1 (Geanakoplos, Úbeda) *Let* (UD) *hold, and assume the preference aggregation rule f satisfies Unanimity and Pairwise Independence* (PI). *Then:*

(i) Strict individual preferences result in a strict social preference:

$$\mathbf{I}(\rho)(x,y) \in \{-1,1\}^N \implies I[f(\rho)](x,y) \in \{-1,1\}.$$

(ii) f satisfies Strict Neutrality (SN).

PROOF: Suppose that

$$\mathbf{I}(\rho)(x,y) \in \{-1,1\}^N.$$

Since X has at least 3 different elements, there is $z \in X$ such that $z \notin \{x,y\}$. Assume, without loss of generality, that $I[f(\rho)](x,y) \in \{0,1\}$ (otherwise, reverse the roles of x and y).

Let ρ' be a profile that satisfies $\mathbf{I}(\rho')(x,y) = \mathbf{I}(\rho')(z,y) = \mathbf{I}(\rho)(x,y)$.

Let ρ'' be a profile that satisfies $\mathbf{I}(\rho'')(x,y) = \mathbf{I}(\rho)(x,y)$, $\mathbf{I}(\rho'')(z,y) = \mathbf{I}(\rho')(z,y)$, and $\mathbf{I}(\rho'')(z,x) \in \{1\}^N$.

By unanimity, $I[f(\rho'')](z,x) = 1$. By (PI), $I[f(\rho'')](x,y) \in \{0,1\}$, so by transitivity of $f(\rho'')$, we must have $I[f(\rho'')](z,y) = 1$. Apply again (PI) to get $I[f(\rho')](z,y) = 1$.

Let now ρ''' be a profile that satisfies $\mathbf{I}(\rho''')(x,y) = \mathbf{I}(\rho)(x,y)$, $\mathbf{I}(\rho''')(z,y) = \mathbf{I}(\rho')(z,y)$, and $\mathbf{I}(\rho''')(z,x) \in \{-1\}^N$.

By unanimity, $I[f(\rho''')](z,x) = -1$. By (PI), $I[f(\rho''')](z,y) = 1$, so by transitivity of $f(\rho''')$, we must have $I[f(\rho''')](x,y) = 1$.

(PI) then implies that $I[f(\rho)](x,y) = 1$. This concludes the proof of the first statement.

Let us now consider Strict Neutrality. Assume

$$\mathbf{I}(\rho)(x,y) = \mathbf{I}(\rho')(x',y') \in \{-1,1\}^N.$$

We have just shown that $I[f(\rho)](x,y) \in \{-1,1\}$. Assume, without loss of generality, that $I[f(\rho)](x,y) = 1$ (otherwise, reverse the roles of x and y, and x' and y').

If $(x,y) = (x',y')$, apply (PI) to obtain $I[f(\rho')](x',y') = I[f(\rho)](x,y)$. Therefore, suppose in what follows that $(x,y) \neq (x',y')$.

Let ρ'' be a profile that satisfies $\mathbf{I}(\rho'')(x,y) = \mathbf{I}(\rho)(x,y)$, $\mathbf{I}(\rho'')(x',y') = \mathbf{I}(\rho')(x',y')$, and, additionally,

$$\begin{cases} x \neq x' & \Rightarrow \mathbf{I}(\rho'')(x,x') \in \{-1\}^N & \Rightarrow I[f(\rho'')](x,x') = -1; \\ y \neq y' & \Rightarrow \mathbf{I}(\rho'')(y,y') \in \{1\}^N & \Rightarrow I[f(\rho'')](y,y') = 1. \end{cases}$$

Where the second implications follow because of unanimity. Also, (PI) implies $I[f(\rho'')](x,y) = I[f(\rho)](x,y) = 1$.

Now $(x,y) \neq (x',y')$ means that either $x \neq x'$ or $y \neq y'$, and in either case transitivity of $f(\rho'')$ implies $I[f(\rho'')](x',y') = 1$. A final application of (PI) results in $I[f(\rho')](x',y') = 1$. $\square$

The next lemma is similar in spirit to the pivotal argument used by Geanakoplos [8] and Úbeda [17].

Lemma 2 *Let* (UD) *hold, and assume the preference aggregation rule f satisfies Unanimity and Strict Neutrality* (SN). *Let $x,y \in X$, $x \neq y$, and let $S,T \in \mathscr{L}$ satisfy $S \subset T \subset N$. Let ρ_0 and ρ_1 be profiles in which:*

$$\begin{array}{rlll} \forall i \in S, & I[\rho_0(i)](x,y) = 1, & I[\rho_1(i)](x,y) = 1; \\ \forall i \in T \setminus S, & I[\rho_0(i)](x,y) = -1, & I[\rho_1(i)](x,y) = 1; \\ \forall i \in N \setminus T, & I[\rho_0(i)](x,y) = -1, & I[\rho_1(i)](x,y) = -1. \end{array}$$

Then $I[f(\rho_0)](x,y) = -1$ and $I[f(\rho_1)](x,y) = 1$ imply that $T \setminus S$ is a decisive coalition.

PROOF: Let $a,b \in X$, with $a \neq b$, and assume that ρ is an arbitrary profile in which, for all $i \in T \setminus S$, $I[\rho(i)](a,b) = 1$.

Let $c \in X$ be such that $c \notin \{a,b\}$ (this is possible because X has at least three different elements). Let ρ' be a profile in which

$$\mathbf{I}(\rho')(a,b) = \mathbf{I}(\rho)(a,b),$$

and, additionally,

$$\begin{array}{rlll} \forall i \in S, & I[\rho(i)](a,c) = 1, & \text{and} & I[\rho(i)](b,c) = 1; \\ \forall i \in T \setminus S, & I[\rho(i)](a,c) = 1, & \text{and} & I[\rho(i)](b,c) = -1; \\ \forall i \in N \setminus T, & I[\rho(i)](a,c) = -1, & \text{and} & I[\rho(i)](b,c) = -1. \end{array}$$

Now $\mathbf{I}(\rho')(b,c) = \mathbf{I}(\rho_0)(x,y)$ and $I[f(\rho_0)](x,y) = -1$ imply, by (SN), that $I[f(\rho')](b,c) = -1$.

Analogously, $\mathbf{I}(\rho')(a,c) = \mathbf{I}(\rho_1)(x,y)$ and $I[f(\rho_1)](x,y) = 1$ imply, by (SN), that $I[f(\rho')](a,c) = 1$.

By transitivity, $I[f(\rho')](a,b) = 1$, and by (SN), $I[f(\rho)](a,b) = 1$. $\square$

Lemma 3 *Let* (UD) *hold, and assume the preference aggregation rule f satisfies Unanimity and Strict Neutrality* (SN). *Let $x,y \in X$, $x \neq y$, let $T \in \mathscr{L}$, and let ρ be a profile in which:*

$$\begin{aligned} \forall i \in T, \qquad & I[\rho(i)](x,y) = 1; \\ \forall i \in N \setminus T, \qquad & I[\rho(i)](x,y) = -1. \end{aligned}$$

Then $I[f(\rho)](x,y) = 1$ implies that T is a decisive coalition.

PROOF: Let $S = \emptyset$, let ρ_0 be the profile in which all individuals strictly prefer y to x, and let $\rho_1 = \rho$. Then use Unanimity and apply lemma 2. □

Lemma 4 *Let* (UD) *hold, and assume the preference aggregation rule satisfies Unanimity and Strict Neutrality* (SN). *Suppose $T \in \mathscr{L}$ is decisive, and $S \subset T$, $S \in \mathscr{L}$. Then either S or $T \setminus S$ (but not both) is decisive.*

PROOF: Let ρ_0 and ρ_1 be as in the statement of lemma 2.

By (SN), we must have $I[f(\rho_0)](x,y) \in \{-1,1\}$.

If $I[f(\rho_0)](x,y) = 1$, lemma 3 implies that S is decisive.

Since by assumption T is decisive, we know that $I[f(\rho_1)](x,y) = 1$, so if $I[f(\rho_0)](x,y) = -1$, lemma 2 implies that $T \setminus S$ is decisive. □

Lemma 5 *Let* (UD) *hold, and assume the preference aggregation rule satisfies Unanimity and Strict Neutrality* (SN). *Then the intersection of any two decisive coalitions is a decisive coalition.*

PROOF: Let S and T be decisive coalitions in $\mathscr{L}$. Then T is the union of the disjoint sets $S \cap T$ and $T \setminus S$. Now $T \setminus S$ is disjoint from the decisive coalition S, so by definition it cannot be decisive. Therefore, lemma 4 implies that $S \cap T$ is decisive. □

A (non-empty) collection $\mathscr{C}$ of non-empty subsets of N is (downward) *filtering* if for each $A,B \in \mathscr{C}$ there exists $C \in \mathscr{C}$ such that $A \supset C$ and $B \supset C$, that is, $A \cap B \supset C$. A filter is the collection of all supersets of a filtering collection. In other words, a collection $\mathscr{D}$ of subsets of N is a *filter* (Bourbaki [6], I.6.1) if: (i) $\emptyset \notin \mathscr{D}$; (ii) $A,B \in \mathscr{D}$ implies $A \cap B \in \mathscr{D}$; and (iii) $A \in \mathscr{D}$ and $A \subset B$ implies $B \in \mathscr{D}$. A filter is an *ultrafilter* if there is no other filter that strictly contains it. Using this maximality property, it can be shown (Bourbaki [6], I.6.4), that a filter $\mathscr{D}$ is an

ultrafilter iff $\forall A$, precisely one of A and $N \setminus A$ is in $\mathscr{D}$. An ultrafilter $\mathscr{D}$ is called *free* if $\cap\{D : D \in \mathscr{D}\} = \emptyset$.

Given an algebra $\mathscr{L}$, a collection $\mathscr{D}$ of sets of $\mathscr{L}$ is an $\mathscr{L}$*-filter* if: (i) $\emptyset \notin \mathscr{D}$; (ii) $A, B \in \mathscr{D}$ implies $A \cap B \in \mathscr{D}$; and (iii) $A \in \mathscr{D}$, $B \in \mathscr{L}$, and $A \subset B$ implies $B \in \mathscr{D}$. An $\mathscr{L}$-filter is an $\mathscr{L}$*-ultrafilter* if there is no other $\mathscr{L}$-filter that strictly contains it. The (ultra)filters not restricted to lie in an algebra can be identified with 2^N-(ultra)filters. Given any 2^N-(ultra)filter $\mathscr{F}$, the intersection $\mathscr{F} \cap \mathscr{L}$ is an $\mathscr{L}$-(ultra)filter. Any collection of nonempty sets closed under finite intersections is always contained in a 2^N-filter; hence, given an $\mathscr{L}$-filter $\mathscr{D}$ there is a 2^N-filter $\mathscr{F}$ that contains it, and we have that $\mathscr{F} \cap \mathscr{L} \supset \mathscr{D}$. In particular, if $\mathscr{D}$ is an $\mathscr{L}$-ultrafilter and $\mathscr{F}$ is any 2^N-filter that contains it, we must have $\mathscr{F} \cap \mathscr{L} = \mathscr{D}$. Concluding, $\mathscr{D}$ is an $\mathscr{L}$-ultrafilter if, and only if, there is a 2^N-ultrafilter $\mathscr{U}$ such that $\mathscr{D} = \mathscr{U} \cap \mathscr{L}$. That is, an $\mathscr{L}$-filter $\mathscr{D}$ is an $\mathscr{L}$-ultrafilter iff $\forall A \in \mathscr{L}$, precisely one of A and $N \setminus A$ is in $\mathscr{D}$. The theory of measurable filters and ultrafilters has been developed in the context of the study of Boolean Algebras (see Sikorski [16]).

Theorem 1 (Ultrafilter Property) *Let* (UD) *hold. Then a preference aggregation rule f satisfies Unanimity and Strict Neutrality* (SN) *if, and only if, the collection of all f–decisive sets forms an $\mathscr{L}$–ultrafilter.*

PROOF: Assume f satisfies Unanimity and (SN). Let $\mathscr{D}$ be the collection of all f–decisive coalitions. Unanimity implies that $N \in \mathscr{D}$ and $\emptyset \notin \mathscr{D}$. We have seen (lemma 5) that $\mathscr{D}$ is closed under (finite) intersections, and by definition supersets of elements of $\mathscr{D}$ are in $\mathscr{D}$. This implies that $\mathscr{D}$ is a filter. Now, taking $T = N$ in lemma 4, we obtain that $\mathscr{D}$ is an ultrafilter.

Assume now that f satisfies the Ultrafilter Property, that is, the collection $\mathscr{D}$ of all f–decisive sets forms an $\mathscr{L}$–ultrafilter. Since $N \in \mathscr{D}$, Unanimity is satisfied. Let us show that (SN) holds as well. Assume that

$$\mathbf{I}(\rho)(x,y) = \mathbf{I}(\rho')(x',y') \in \{-1,1\}^N.$$

Let $S = \{i \in N : I[\rho(i)](x,y) = 1\}$. If $S \in \mathscr{D}$, then $I[f(\rho)](x,y) = I[f(\rho')](x',y') = 1$. Otherwise, $N \setminus S \in \mathscr{D}$, which implies $I[f(\rho)](x,y) = I[f(\rho')](x',y') = -1$. $\square$

Theorem 1 gives a complete characterization of the Ultrafilter Property. It is important to notice that, in our domain of preferences, if f satisfies the Ultrafilter Property, then f is not necessarily an Arrovian rule. This is only true over the domain of linear orders, where indifference between distinct alternatives is not

allowed. Actually, it is easy to construct simple (finite) examples in which a dictatorial rule does not satisfy Pairwise Independence. For example, it is possible for a dictatorial rule to have x strictly preferred to y and the opposite preference, for two different profiles in which individual preferences between x and y do not change, as long as the dictator is indifferent between both alternatives. What is true, however, is that, corresponding to any such f, there exists an Arrovian preference aggregation rule that has exactly the same decisive sets: the *simple rule* generated by those decisive sets (see section 6.1 in the appendix).

Theorem 1 together with lemma 1 imply:

Corollary 1 (Arrovian Rules and Ultrafilters) *Let* (UD) *hold, and assume the preference aggregation rule f satisfies Unanimity and Pairwise Independence. Then f satisfies the Ultrafilter Property.*

When the set of individuals N is finite, and assumption (FC) holds, then all singletons are admissible coalitions, and any ultrafilter $\mathscr{D}$ is of the form: $\mathscr{D} = \{S \subset N : S \ni i\}$, for some $i \in N$. In particular, $\{i\}$ is a decisive coalition—that is, a dictator.

Corollary 2 (Arrow's Theorem) *Let* (UD) *and* (FC) *hold. Assume the preference aggregation rule f satisfies Unanimity and Pairwise Independence. Assume additionally that N is a finite set. Then f is dictatorial.*

4 Invisible Dictators

Whenever there are no measurability constraints on the admissible coalitions, an ultrafilter of decisive coalitions is either *free* (it has an empty intersection) or *fixed* (its intersection is a single point). Let us consider the non measurable case; that is, assume for the moment that $\mathscr{L} = 2^N$. We have seen that an Arrovian rule f has associated a collection of decisive sets $\mathscr{D}$ that is an ultrafilter. Define $\mathscr{N} = \left\{A \subset N : N \setminus A \in \mathscr{D}\right\}$; we say that the coalitions in $\mathscr{N}$ are *negligible*: the preference aggregation rule f acts independently[5] of the preferences of the members of any such coalition. If $\mathscr{D}$ is a free ultrafilter, one can say that the rule f is *invisibly*

[5] Almost independently, to be precise. Consider a lexicographic chain of dictators: the first dictator's strict preferences decide; when the first dictator is indiferent, the second dictator's strict preferences decide; and so on. In this case, a set is decisive if, and only if, it contains the first dictator. Overlooking this possibility led to a mistake in Armstrong's [2] initial paper, which was corrected in [3].

dictatorial, because then the union of all negligible coalitions is the entire agent set N.[6] Therefore, when $\mathscr{L} = 2^N$ Arrovian rules are either dictatorial or invisibly dictatorial.

Unfortunately, this characterization is not necessarily valid when $\mathscr{L}$ is different from 2^N. In the measurable case there is a third possibility. We need to define some concepts to illustrate it.

We say that a nonempty set $A \in \mathscr{L}$ is an *atom* of $\mathscr{L}$ if, for any $B \in \mathscr{L}$, $B \subset A$ implies that $B \in \{A, \emptyset\}$. Whenever assumption (FC) holds, the atoms of $\mathscr{L}$ are precisely the singletons, but in general this need not be the case. An algebra need not have any atoms, as example 1 below shows.

Let $i, j \in N$, $i \neq j$; we say that an algebra $\mathscr{L}$ *separates* i *and* j if there exists $A \in \mathscr{L}$ such that $i \in A$ and $j \in N \setminus A$. We say that $\mathscr{L}$ *separates the points* of N if it separates any two distinct points. Given an algebra $\mathscr{L}$, any two points that are not separated by it are in a sense indistinguishable; this is formalized by the following definition. We say that $i \sim j$ if $\mathscr{L}$ does not separate i and j, i.e. if $[\forall A \in \mathscr{L},\ i \in A \Leftrightarrow j \in A]$. It is immediate to see that $\sim$ is an equivalence relation (a reflexive, symmetric and transitive binary relation). Given $i \in N$, let $a(i)$ denote the equivalence class of i. It follows from the definition of the equivalence relation that, for any $i \in N$ and $A \in \mathscr{L}$, either $a(i) \subset A$ or $a(i) \cap A = \emptyset$. We denote by $N/_\sim$ the quotient space, the set of all equivalence classes.

In general, an equivalence class need not be measurable, as example 1 below shows. If all the equivalence classes are measurable, we say that the algebra $\mathscr{L}$ is *fully atomic*. In subsection 6.2 in the appendix we show that an atom is always an equivalence class, and that equivalence classes are atoms whenever they are measurable. In the following example the algebra separates the points, so that the equivalence classes are the singletons, but they are not measurable, and therefore there are not any atoms.

Example 1. Let $N = [0,1)$, and let $\mathscr{L}$ be the algebra generated by the intervals of the form $[a,b)$, with $a < b$ (this algebra is formed by finite unions of such intervals, and *all* intervals that are in sets belonging to the algebra have positive length). Note that: (1) the algebra separates the points of $[0,1)$; (2) there are no atoms, since singletons are not measurable. Given any $i \in [0,1)$, let $\mathscr{D}$ consist of all the elements of $\mathscr{L}$ that contain i. Then the ultrafilter is not free, since its intersection is $\{i\}$, but $\{i\}$ itself is not an admissible coalition. □

Since Kirman and Sondermann [12] considered the case in which all coali-

[6] I thank an anonymous referee for suggesting this argument.

tions are admissible[7], nondictatorial Arrovian rules (invisible dictators) always correspond in their case to free ultrafilters. When there is an algebra of admissible coalitions, the above example shows that we have another kind of "invisible dictator," for which actually the terminology is even more appropriate. We therefore refine that terminology to include the new case.

Definition 3 *Given a preference aggregation rule, let $\mathscr{D}$ be the collection of admissible (belonging to $\mathscr{L}$) decisive coalitions. Assume $\mathscr{D} \neq \emptyset$, and let $M = \cap\{D : D \in \mathscr{D}\}$. We say that a preference aggregation rule has:*

*(i) A **measurable dictator** if M is an atom of the algebra $\mathscr{L}$ of admissible coalitions.*

*(ii) An **empty invisible dictator** if $M = \emptyset$.*

*(iii) A **non-measurable invisible dictator** if $M \neq \emptyset$ and $M \notin \mathscr{L}$.*

If the agent set N is finite, then a trivial extension of Arrow's theorem implies that, whenever the preference aggregation rule satisfies unanimity and (PI), there is always a measurable dictator. Note that a measurable dictator need not be a single individual unless assumption (FC) is satisfied. In the general case, we have the following result.

Proposition 1 (Visible or Invisible Dictators) *Let* (UD) *hold. Assume the preference aggregation rule satisfies Unanimity and Pairwise Independence* (PI). *Then there is either a measurable dictator, an empty invisible dictator, or a non-measurable invisible dictator.*

PROOF: By corollary 1, we know that the collection $\mathscr{D}$ of all decisive coalitions forms an ultrafilter.

Consider the intersection $M = \cap\{D : D \in \mathscr{D}\}$. If $M = \emptyset$, then it is an empty invisible dictator by definition. Assume now that $M \neq \emptyset$. In the appendix (proposition 3) we show that in this case M is an equivalence class of the equivalence relation that identifies points that are not separated by $\mathscr{L}$. If $M \in \mathscr{L}$, then it is an atom of $\mathscr{L}$ (see lemma 8 in the appendix): in this case we have a measurable dictator. Otherwise, if $M \notin \mathscr{L}$ we have a non-measurable invisible dictator. □

[7] Except in the case in which they assumed the agent space is an atomless probability space, in which they implicitly assume that only coalitions belonging to the underlying σ-algebra are admissible.

4.1 Non-measurable invisible dictators

Example 1 above suggests that the existence of non-measurable dictators is related to the measurability of the equivalence classes (equivalently, to the existence of atoms in $\mathscr{L}$). The next proposition formalizes this intuition.

Proposition 2 (Existence of non-measurable dictators) *Let* (UD) *hold. There exists a preference aggregation rule that satisfies Unanimity and Pairwise Independence* (PI) *and has a non-measurable dictator if, and only if, there exists a non-measurable equivalence class of the relation that identifies points that are not separated by* $\mathscr{L}$*. In particular, if assumption* (FC) *is satisfied there do not exist any non-measurable invisible dictators.*

PROOF: Assume first there exists $i \in N$ such that the equivalence class of i is not a member of $\mathscr{L}$. Let $\mathscr{D}$ be the ultrafilter of all sets in $\mathscr{L}$ that contain i. Consider the simple rule defined by $\mathscr{D}$ (see subsection 6.1 in the appendix). In lemma 6 in the appendix we show that this preference aggregation rule satisfies unanimity and (PI), and the collection of its decisive sets is $\mathscr{D}$. By construction, this rule has as a non-measurable dictator the equivalence class of i (see proposition 3 in the appendix).

On the other hand, if a preference aggregation rule satisfies unanimity and (PI), we know from corollary 1 that the collection of decisive coalitions forms an ultrafilter, and we know from proposition 3 in the appendix that the intersection of all decisive coalitions is an equivalence class. □

Corollary 3 *Let* (UD) *hold. If there exists a set* $A \in \mathscr{L}$ *that does not contain any atoms, then there is a preference aggregation rule that satisfies Unanimity and Pairwise Independence* (PI) *and has a non-measurable dictator which is a subset of* A*.*

PROOF: Given any $i \in A$, the equivalence class of i is entirely contained in A and is not measurable because then it would be an atom (see lemma 8 in the appendix), so we can construct an Arrovian rule as in the proof of the previous result. □

The proof of proposition 2 actually shows us that non-measurable dictators corresponding to Arrovian rules are "small" in a precise sense.

Theorem 2 (Smallness of non-measurable dictators) *Let* (UD) *hold. Assume that a preference aggregation rule satisfies Unanimity and Pairwise Independence* (PI) *and has a non-measurable dictator. Then the dictator is an equivalence class of the relation that identifies points that are not separated by $\mathscr{L}$, and the collection of decisive sets consists precisely of all admissible coalitions that contain this equivalence class.*

PROOF: We know from corollary 1 that, given an Arrovian rule, the collection of all decisive coalitions forms an ultrafilter. Proposition 3 in the appendix implies therefore that if an Arrovian rule has a non-measurable dictator, this latter is necessarily an equivalence class. By definition, all decisive sets contain this equivalence class. Since *all* admissible coalitions that contain the equivalence class form an ultrafilter, both ultrafilters must necessarily be the same. □

Example 2. In the space $\mathbb{N}$ of the natural numbers, consider the algebra generated by the collection of sets $E_k = \{2kn : n \in \mathbb{N}\}$, for $k \in \mathbb{N}$. Note first of all that this algebra contains no finite sets. The odd numbers are an atom of the algebra, therefore all of them belong to the same equivalence class. On the other hand, the algebra separates any two even numbers, so that the equivalence class of any even number is a singleton. Since the algebra does not contain any finite set, the equivalence class of any even number is not measurable. For any $k \in \mathbb{N}$, theorem 2 shows how we can construct an Arrovian rule that has $2k$ as a non-measurable invisible dictator. □

4.2 Empty invisible dictators

From now on we will concentrate on *empty* invisible dictators. We formulate the smallness of empty invisible dictators as a selection problem: out of any ultrafilter of decisive coalitions that has a empty intersection, one can select a *nested* collection that shrinks to the empty set. If there are no measurability constraints, this selection is always valid. In what follows, by an Arrovian (preference aggregation) rule we mean a preference aggregation rule that satisfies Unanimity and Pairwise Independence.

Theorem 3 (Smallness of Empty Invisible Dictators) *Assume that all coalitions are admissible: $\mathscr{L} = 2^N$. Then for any Arrovian rule that has an empty invisible dictator there exists a nested collection of decisive coalitions that has an empty intersection.*

PROOF: Let $\mathscr{D}$ be the ultrafilter of all decisive coalitions, which by hypothesis has an empty intersection.

If we partially order $\mathscr{D}$ by (reverse) inclusion there can be no maximal element, since if one existed it would coincide with the intersection. (If $\hat{D} \in \mathscr{D}$ is maximal, then given any $D \in \mathscr{D}$, we must have $\hat{D} \subset D$, since the facts that $\hat{D} \cap D \in \mathscr{D}$ and $\hat{D}$ is maximal imply that $\hat{D} \cap D = \hat{D}$, i.e. $\hat{D} \subset D$.)

By (the contrapositive of) Zorn's lemma, there must exist a chain $\mathscr{C}$ in $\mathscr{D}$ that has no upper bound. Let $B = \cap\{C : C \in \mathscr{C}\}$. Note first that $B \notin \mathscr{D}$, since if B were a member of $\mathscr{D}$ then it would be an upper bound of the chain. Since $\mathscr{D}$ is an ultrafilter, we have that $N \setminus B \in \mathscr{D}$. Therefore, for each $C \in \mathscr{C}$, the set $C \setminus B = C \cap (N \setminus B)$ belongs to $\mathscr{D}$. But then the chain $\mathscr{C}' = \{C \setminus B : C \in \mathscr{C}\}$ has an empty intersection. □

Kirman and Sondermann [12] showed that, whenever the space of agents is an atomless finite measure space, then an empty invisible dictator is characterized by the fact that there are coalitions of arbitrarily small measure that are decisive. Whenever there are no measurability requirements on coalitions, the previous theorem shows that one does not need to define a measure to see that, in a precise sense, invisible dictators correspond to arbitrarily small decisive coalitions. We next show that, even if admissible coalitions are restricted to an algebra, the result is still true whenever N is a countable set. A constructive proof works in this case.

Theorem 4 (The countable case) *Assume that N is a countable set, and $\mathscr{L}$ any algebra of its subsets. Then for any Arrovian rule that has an empty invisible dictator there exists a nested collection of decisive coalitions that has an empty intersection.*

PROOF: Let $D_1 \in \mathscr{D}$ be any decisive coalition. Let $\{i_k : k \in \mathbb{N}\}$ be an enumeration of the elements of D_1. Since the intersection of all decisive coalitons is empty, given i_1 there exists $D_2' \in \mathscr{D}$ that does not contain i_1. Let now $D_2 = D_1 \cap D_2'$. Next, if $i_2 \notin D_2$ then let $D_3 = D_2$, and otherwise proceed as above to construct a set $D_3 \subset D_2$ that does not contain i_2. In this manner, we construct inductively a nested collection of decisive sets D_k, for $k \in \mathbb{N}$, with an empty intersection. □

The steps followed in the proof of the previous theorem can be used to prove a much more general result.

Corollary 4 (Countable decisive sets) *Let N be an arbitrary agent set, and $\mathscr{L}$ any algebra of its subsets. Suppose that an Arrovian rule has among its decisive*

sets one that is countable. Then, if the rule has an empty invisible dictator there exists a nested collection of decisive coalitions that has an empty intersection.

PROOF: Let D_1 be the countable decisive set which exists by hypothesis. Enumerate its elements: $D_1 = \{i_k : k \in \mathbb{N}\}$. By proceeding as in the proof of theorem 4, we can construct a sequence of nested decisive coalitions $(D_k)_{k\in\mathbb{N}}$ that has an empty intersection. □

However, when the condition of the previous corollary is not met, then the selection problem need not have a solution, as the following example shows.

Example 3. Let N be an uncountable agent space, and let $\mathscr{L}$ be the algebra of finite and cofinite (complements of finite) sets. Then the only free ultrafilter on $\mathscr{L}$ consists of the collection of all cofinite sets. Notice that no countable coalition is admissible. By taking complements, our problem can be stated in terms of looking for an increasing collection of finite sets whose union is N. Assume such a collection exists, and define a mapping from it to the natural numbers associating with each set its cardinality. The mapping must be injective because the collection is increasing (two different sets have different cardinality), and this then implies that the collection can be at most countable[8], and hence its union must be at most countable, and therefore cannot coincide with N. □

Assume from now on that $\mathscr{L}$ is a σ-algebra, that is, an algebra closed under countable unions and intersections. We will show next that, for practically all uncountable cases that have been considered in the literature, the selection problem has a solution.

We say that a σ-algebra $\mathscr{L}$ is *countably generated* if there is a countable collection $\mathscr{S}$ of subsets of N, for which $\mathscr{L}$ is the smallest σ-algebra containing $\mathscr{S}$. For instance, the Borel subsets of the real line form a countably generated σ-algebra, since it is the smallest σ-algebra that contains all open intervals with rational endpoints. In general, the Borel subsets of any separable metric space form a countably generated σ-algebra.

Theorem 5 (Countably generated σ-algebras) *Assume that $\mathscr{L}$ is a countably generated σ-algebra. Then for any Arrovian rule that has an empty invisible dictator there exists a nested collection of decisive coalitions that has an empty intersection.*

[8] I thank Luis Úbeda for suggesting this argument.

PROOF: Let $\mathscr{S} = (S_n)_{n\in\mathbb{N}}$ be a countable collection of sets such that $\mathscr{L} = \sigma(\mathscr{S})$. Let $\mathscr{D}$ be the free ultrafilter of all decisive coalitions. We can assume without loss of generality that, for all n, $S_n \in \mathscr{D}$, because otherwise we replace that set by its complement. Hence, assume from now on that $\mathscr{S} \subset \mathscr{D}$.

For each $m \in \mathbb{N}$, let $T_m = \cap_{n\leq m} S_n$. Since $\mathscr{D}$ is closed under finite intersections, we have that $\emptyset \neq T_m \in \mathscr{D}$ for all m. We also have by construction that $m' \geq m \Rightarrow T_{m'} \subset T_m$, and

$$B := \bigcap_{m\in\mathbb{N}} T_m = \bigcap_{n\in\mathbb{N}} S_n$$

If $B = \emptyset$ then we are done. Assume therefore that $B \neq \emptyset$.

Since $\mathscr{L}$ is a σ-algebra, $B \in \mathscr{L}$. We claim that B is an atom of $\mathscr{L}$. If this were not true, then the σ-algebra $\mathscr{L}/_{\sim_B} \cup \{N, \emptyset\}$ (see subsection 6.3 in the appendix, and especially proposition 4) would be strictly smaller than $\mathscr{L}$ but would also contain $\mathscr{S}$, which contradicts the fact that $\mathscr{L} = \sigma(\mathscr{S})$.

Now, if the atom B were in $\mathscr{D}$, this would contradict the fact that $\mathscr{D}$ is a free ultrafilter. Therefore, $N \setminus B \in \mathscr{D}$, and hence for all m, $T_m \setminus B = T_m \cap (N \setminus B) \in \mathscr{D}$. Thus, $(T_m \setminus B)_{m\in\mathbb{N}}$ is a nested collection of decisive coalitions that has an empty intersection. □

Two measurable spaces are isomorphic if there is a bijection between them that is measurable, and whose inverse is also measurable. A Borel space is a measurable space which is isomorphic to a Borel subset of the real line with its induced Borel σ-algebra. Since the σ-algebra in Borel spaces is countably generated, we have the following corollary.

Corollary 5 (Borel spaces) *Let $(N, \mathscr{L})$ be any Borel space. Then for any Arrovian rule that has an empty invisible dictator there exists a nested collection of decisive coalitions that has an empty intersection.*

Actually, if one is willing to believe the Continuum Hypothesis (which is an unproven conjecture), a much stronger result appears: if the agent space has the cardinality of the continuum, and the admissible coalitions form a σ–algebra that contains the singletons, then the selection problem always has a positive answer. The details are in section 6.4 in the appendix.

We should finally point out that the failure of the selection problem to have a positive answer is rather attributable to a lack of balance between the set of admissible coalitions and the cardinality of the agent space. If the algebra or σ–algebra of admissible coalitions is too narrow with respect to the cardinality of N, then we may run into problems.

5 Conclusions

Since Fishburn [7] first showed the possibility of the existence of non-dictatorial Arrovian preference aggregation rules in societies with infinitely many agents, there has been an interest in precisely characterizing those rules and knowing in what sense the passage to infinity allows the transition from dictatorial to non-dictatorial rules. One may wonder why economists (social scientists) should worry about this fact. The answer is that, in general, infinite societies are a modeling device that is used to study the properties of large, but finite, societies, as is exemplified by the competitive paradigm. In this case, a desirable property of the model is to have "continuity at infinity," that is, that the properties of the infinite society correspond to the limit of the corresponding properties of finite societies when the number of individuals in them tends to infinity. Fishburn's paper shows that, with regard to Arrow's theorem, there is a discontinuity at infinity: in any finite society, an Arrovian rule gives rise to a dictator, while there are infinite societies with no dictator associated to the corresponding limiting Arrovian rule. In this paper, we have tried to show that one can define things in such a way that there is no discontinuity: a dictator is a very small decisive coalition, so if one looks at the right concept of smallness, we may still say that the limiting infinite society has arbitrarily small decisive coalitions.

The case which we have termed "Non–Measurable Invisible Dictators" is different. Here what happens is related to the interpretation of the restrictions one imposes on the admissibility of coalitions. We may think of admissible coalitions as those that can be "seen" by the individuals in our society. Consider now the case in which the "Non–Measurable Invisible Dictator" is a single individual, as in example 1 above. The Arrovian preference aggregation rule the society has, gives all the power to this individual, that is, any coalition that has her as a member is decisive, and any decisive coalition necessarily has her as a member. However, the individual is not "visible" to society, because when seen as a one–member coalition she is not admissible, so she is not publicly identifiable because she always appears surrounded by other individuals. She truly is an "invisible dictator." Of course, in this case there is no question about her smallness: she is small, because she is the smallest unit our algebra of coalitions allows.

An interesting line of work would be to explicitly model the process of coalition formation, and see if the derived coalitional structure is such that the results of this paper are applicable.

6 Appendix

6.1 Simple rules

Given a preference profile ρ and a nonempty collection $\mathscr{D} \subset \mathscr{L}$, define a binary relation P on X by $xPy \Leftrightarrow \exists D \in \mathscr{D}$ such that $\forall i \in D$, $I[\rho(i)](x,y) = 1$. Define also a binary relation R in X by $xRy \Leftrightarrow$ not yPx. The next lemma is basically the proof of part *ii* of theorem 1 in Kirman and Sondermann [12], and proposition 3.1 in Armstrong [2].

Lemma 6 *Let assumption* (UD) *hold. If $\mathscr{D}$ is an ultrafilter, then R is reflexive, transitive and complete, and P is asymetric and negatively transitive. That is, R is a weak preference relation and P the corresponding strict preference. The preference aggregation rule thus defined satisfies unanimity and* (PI)*, and $\mathscr{D}$ coincides with the collection of its decisive sets.*

PROOF: We claim that xRy if, and only if, $R_{xy} \doteq \{i \in N : I[\rho(i)](x,y) \in \{0,1\}\} \in \mathscr{D}$. The proof is immediate from the definitions and the fact that $\mathscr{D}$ is an ultrafilter.

Now the fact that R is reflexive follows from $R_{xx} = N \in \mathscr{D}$. Transitivity follows because the fact that R_{xy} and R_{yz} are in $\mathscr{D}$ implies that so is $R_{xz} \supset R_{xy} \cap R_{yz} \in \mathscr{D}$. Finally, completeness follows because $R_{xy} \notin \mathscr{D}$ implies $N \setminus R_{xy} = \{i \in N : I[\rho(i)](x,y) = -1\} \in \mathscr{D}$.

Since $N \in \mathscr{D}$, the preference aggregation rule satisfies Unanimity. Pairwise Independence is satisfied because the social preference between each pair of alternatives depends only on the individual preferences between those alternatives. By construction, each $D \in \mathscr{D}$ is decisive with respect to the new rule. Since the aggregation rule is Arrovian, its collection of decisive sets is an ultrafilter. That is, the decisive sets form an ultrafilter that contains the ultrafilter $\mathscr{D}$: by maximality of ultrafilters, both must coincide. □

If $\mathscr{D}$ is an ultrafilter, the (Arrovian) preference aggregation rule defined above is called the *simple rule defined by* $\mathscr{D}$. Note that, among all rules that have $\mathscr{D}$ as their decisive sets, the simple rule is the one that gives rise to the smallest (as a subset of $X \times X$) strict social preference.

6.2 Atoms of algebras

In section 4 we have defined the equivalence relation $\sim$ that identifies points that are not separated by the algebra $\mathscr{L}$. We have also defined the concepts of *atom* of an algebra, and of a *fully atomic* algebra, which we use in this section. We prove next a few auxiliary results.

Lemma 7 *If A is an atom of $\mathscr{L}$, then A is an equivalence class of $\sim$.*

PROOF: Let A be an atom of $\mathscr{L}$. Since $A \neq \emptyset$, there is $i \in A$. By definition of $\sim$, A must contain the entire equivalence class of i. Suppose now there is $j \in A$ such that $j \not\sim i$. By definition of $\sim$, there is $B \in \mathscr{L}$ such that $i \in B$ and $j \notin B$. But then $j \in A \setminus B$ and $i \notin A \setminus B$, which implies that $A \setminus B \subset A$, $A \setminus B \neq A$ and $A \setminus B \neq \emptyset$, a contradiction with the fact that A is an atom. □

Lemma 8 *Let $i \in N$. If the equivalence class of i, $a(i)$, belongs to $\mathscr{L}$, then it is an atom.*

PROOF: Suppose not. Then $\exists B \in \mathscr{L}$, $B \subset a(i)$, $B \neq a(i)$ and $B \neq \emptyset$. If we let $j \in a(i) \setminus B$ and $k \in B$, then by assumption $j \sim k$ since both are in $a(i)$, but $k \in B$ and $j \notin B$, a contradiction. □

Corollary 6 *The algebra $\mathscr{L}$ can be identified with an algebra on the quotient space $N/\sim$. This new algebra satisfies assumption* (FC) *if, and only if, $\mathscr{L}$ is fully atomic.*

Proposition 3 *Let $\mathscr{D}$ be an ultrafilter of sets of $\mathscr{L}$, and let $M = \cap\{D : D \in \mathscr{D}\}$. Assume that $M \neq \emptyset$. Then M is an equivalence class of $\sim$.*

PROOF: Let $i \in M$. It is clear that the equivalence class of i, $a(i)$, must also be contained in M, because $\mathscr{D}$ is a subset of $\mathscr{L}$. Suppose now that $M \ni j$ such that $j \not\sim i$. Then there is $B \in \mathscr{L}$ such that $i \in B$ and $j \in B^c$. Since $\mathscr{D}$ is an ultrafilter, either B or B^c must be in it. In either case, this implies a contradiction (e.g., if $B \in \mathscr{D}$, then $j \in B^c$ implies $j \notin M$). □

6.3 Reduction of algebras

Our next objective is to define, given a certain algebra $\mathscr{L}$, a smaller algebra where all subsets of a given set have been taken away. This is not a trivial exercise, since unions of those subsets with other sets in the algebra have to be removed also. We perform the task by resorting again to an equivalence relation.

Given $B \subset N$, define on $\mathscr{L}$ the binary relation $A \sim_B A'$ if $A \triangle A' \subset B$. (Recall that $A \triangle A' = (A \setminus A') \cup (A' \setminus A)$.) Then we have:

Lemma 9 *For any $B \subset N$, the relation $\sim_B$ is an equivalence relation.*

PROOF: Reflexivity and symmetry of $\sim_B$ are immediate from its definition. Transitivity follows from:

$$(A \triangle A'') \subset (A \triangle A') \cup (A' \triangle A'')$$

Which in turn follows from:

$$\begin{array}{rcl} A \setminus A'' & \subset & (A \setminus A') \cup (A' \setminus A'') \\ A'' \setminus A & \subset & (A'' \setminus A') \cup (A' \setminus A) \end{array}$$

□

Proposition 4 *Given any $B \in \mathscr{L}$, the quotient space $\mathscr{L}/_{\sim_B} \cup \{N, \emptyset\}$ can be identified with an algebra of subsets of N. Furthermore, if $\mathscr{L}$ is a σ-algebra, the quotient space can be identified with a σ-algebra. The new (σ-)algebra is strictly contained in $\mathscr{L}$ if, and only if, B is not an atom of $\mathscr{L}$.*

PROOF: Let us assume $\mathscr{L}$ is a σ-algebra (if it is only an algebra, replace countable unions with finite ones in the proof).

The general idea of the proof is to take as representative of any equivalence class its smallest set. We have that $A' \sim_B A$ implies that $A' \sim_B (A \setminus B)$, because $A' \setminus (A \setminus B) \subset (A' \setminus A) \cup B$, and $(A \setminus B) \setminus A' \subset (A \setminus A')$. Next, note that, in this case, we have $(A \setminus B) \subset A'$, because if this were not true there would exist i such that $i \in A$, $i \notin A'$, and $i \notin B$, which would contradict the fact that $A \triangle A' \subset B$. Therefore, we can unambiguously take as representative of the equivalence class of any $A \neq B$ the set $A \setminus B$. We take B as the representative of its equivalence class.

Next, note that

$$A \triangle A' = A^c \triangle (A')^c$$

implies that $A \sim_B A'$ if, and only if, $A^c \sim_B (A')^c$. Taking into account the identity $(A \setminus B)^c = (A^c \setminus B) \cup B$, we can see that $\mathscr{L}/_{\sim_B} \cup \{N, \emptyset\}$ is closed under complementation.

Finally, if $(A_n)_{n \in \mathbb{N}}$ and $(A'_n)_{n \in \mathbb{N}}$ are sequences contained in the σ-algebra $\mathscr{L}$, we have:

$$(\cup_n A_n) \triangle \left(\cup_n A'_n\right) \subset \bigcup_n \left[\left(A_n \setminus A'_n\right) \cup \left(A'_n \setminus A_n\right) \right] = \bigcup_n \left(A_n \triangle A'_n\right)$$

Therefore, if for all n we have $A_n \sim_B A'_n$, this implies that $\cup_n A_n \sim_B \cup_n A'_n$. So closedness of $\mathscr{L}/_{\sim_B} \cup \{N, \emptyset\}$ under countable unions follows because

$$\cup_n (A_n \setminus B) = (\cup_n A_n) \setminus B$$

By construction, the new σ-algebra is a subset of $\mathscr{L}$. If B is an atom of $\mathscr{L}$, it is easy to see that $\mathscr{L}/_{\sim_B} \cup \{N, \emptyset\}$ coincides with $\mathscr{L}$. Otherwise, there is a set $B' \subset B$, $B' \neq \emptyset$, and $B' \neq B$, that is in $\mathscr{L}$ but not in $\mathscr{L}/_{\sim_B} \cup \{N, \emptyset\}$. □

6.4 In the realm of conjecture

Mathematicians have not been able to prove or disprove the following conjecture, called the Continuum Hypothesis: "there is no set with a cardinality strictly larger than that of the natural numbers and strictly smaller than that of the real numbers." It has, nevertheless, been shown that the stronger Generalized Continuum Hypothesis ("there is no set with a cardinality strictly larger than that of a given infinite set and strictly smaller than that of its power set") is independent of the Zermello–Fraenkel set theory plus the axiom of choice (which, by the way, we use freely in this paper).

If one accepts the Continuum Hypothesis as valid, then our selection problem has a positive answer for a wide class of situations in which the agent set has the cardinality of the continuum. The results we present next should be taken with caution, given that they rely on an unproven conjecture.

Lemma 10 *Assume the Continuum Hypothesis is valid. Then there exists an increasing collection of countable subsets of* $[0,1]$ *whose union is* $[0,1]$.

PROOF: The first part of the proof follows closely the proof of theorem 1.11 of Aliprantis and Border [1]. Start with $[0,1]$ and well-order it. Without danger of confusion, denote by $\leq$ the order. For any α, let $I(\alpha) = \{\alpha' \in [0,1] : \alpha' \leq \alpha\}$ be the *initial segment* corresponding to α. Following the steps (and the notation) of the proof of the above mentioned theorem, we end up with an uncountable set Ω, well-ordered by $\leq$, that has a greatest element ω_1 (the first uncountable ordinal), and such that the initial segment of any $\alpha \neq \omega_1$ is countable. Let $\Omega_0 = \Omega \setminus \{\omega_1\}$. By the Continuum Hypothesis, since Ω_0 is uncountable it has the cardinality of $[0,1]$, so there exists a bijection ϕ between the two sets. For any $\alpha \in \Omega_0$, let $C_\alpha \subset [0,1]$ be defined by

$$C_\alpha = \phi[I(\alpha)] = \left\{\phi(\alpha') : \alpha' \leq \alpha\right\}$$

By construction, for any α the set C_α is countable. Also, since Ω_0 is linearly ordered by $\leq$, given any two distinct α and α', either $\alpha \leq \alpha'$ or $\alpha' \leq \alpha$, which implies that either $C_\alpha \subset C_{\alpha'}$ or $C_{\alpha'} \subset C_\alpha$. In other words, the collection $(C_\alpha)_{\alpha \in \Omega_0}$ is an increasing collection of countable sets. Since ϕ is a bijection between Ω_0 and $[0,1]$, it follows that the union of this collection is $[0,1]$.[9] □

Proposition 5 *Assume the validity of the Continuum Hypothesis. Let* $N = [0,1]$ *and let* $\mathscr{L}$ *be a* σ*-algebra of its subsets. Suppose that assumption (FC) is satisfied, i.e. all singletons are admissible coalitions. Then for any Arrovian rule that has an empty invisible dictator there exists a nested collection of decisive coalitions that has an empty intersection.*

[9] I thank Hervé Moulin for suggesting the line of proof used here.

PROOF: If the ultrafilter of decisive coalitions contains a countable set, then we can apply corollary 4.

Otherwise, the fact that $\mathscr{L}$ is a σ-algebra and contains all singletons implies that it contains all finite, countable, cofinite, and cocountable (complements of countable) sets. Since we assume that no countable set is decisive, we have that *all* cocountable sets are decisive. To obtain a decreasing collection of cocountable sets whose intersection is empty, apply lemma 10 and take complements. □

Corollary 7 *Assume the validity of the Continuum Hypothesis. Let $N = [0,1]$, and let $\mathscr{L}$ be a σ-algebra of its subsets. Assume that the algebra $\mathscr{L}$ is fully atomic. Then for any Arrovian rule that has an empty invisible dictator there exists a nested collection of decisive coalitions that has an empty intersection.*

PROOF: By corollary 6 in the appendix, $\mathscr{L}$ can be identified with an algebra on the quotient space, in which all singletons are measurable. If the quotient space is countable, we apply theorem 4, otherwise by the continuum hypothesis its cardinality is that of $[0,1]$, so there is a bijection between both sets. This bijection induces a measurable isomorphism, so we can then apply proposition 5. □

Corollary 8 *Assume the validity of the Continuum Hypothesis. Let $(N,\mathscr{L})$ be such that the agent space N has the cardinality of the continuum, and $\mathscr{L}$ is a σ-algebra. Then for any Arrovian rule that has an empty invisible dictator there exists a nested collection of decisive coalitions that has an empty intersection, provided $\mathscr{L}$ is fully atomic.*

PROOF: There is a bijection between N and $[0,1]$. The image of $\mathscr{L}$ under this bijection is a σ-algebra on $[0,1]$, so we apply corollary 7. □

References

[1] Aliprantis, C. and Border, K., *Infinite Dimensional Analysis: A Hitchhiker's Guide* (1st edition), Springer-Verlag, Berlin, 1994.

[2] Armstrong, T.E. (1980), Arrow's Theorem with Restricted Coalition Algebras, *Journal of Mathematical Economics* (1980), 7: 55-75.

[3] Armstrong, T.E., Precisely Dictatorial Social Welfare Functions: Erratum and Addendum to 'Arrow's Theorem with restricted coalition algebras', *Journal of Mathematical Economics* (1985), 14: 57 - 59.

[4] Austen-Smith, D. and Banks, J.S., *Positive Political Theory I*, University of Michigan Press, Ann Arbor, 1999.

[5] Banks, J.S., Duggan, J., and Le Breton, M., Social Choice and Electoral Competition in the General Spatial Model, mimeo, 2003.

[6] Bourbaki, N., *Élements de Mathématique. Topologie générale. Chapitres 1 à 4*, Diffusion C.C.L.S., Paris, 1971.

[7] Fishburn, P.C., Arrow's Impossibility Theorem: Concise Proof and Infinite Voters, *Journal of Economic Theory* (1970), 2: 103-106.

[8] Geanakoplos, J., Three Brief Proofs of Arrow's Impossibility Theorem, *Economic Theory*, forthcoming.

[9] Gomberg, A., C. Martinelli and R. Torres, Anonymity in Large Societies, working paper, ITAM, 2002.

[10] Guilbaud, Georges, Les théories de l'intérêt général et le problème logique de l'aggrégation, *Économie Appliquée* (1952), 5: 501-584.

[11] Hansson, B., The Existence of Group Preference Functions, *Public Choice* (1976), 28: 89-98.

[12] Kirman, A.P. and Sondermann, D., Arrow's Theorem, Many Agents, and Invisible Dictators, *Journal of Economic Theory* (1972), 5: 267-277.

[13] Mihara, R.H., Anonymity and Neutrality in Arrow's Theorem with Restricted Coalition Algebras, *Social Choice and Welfare* (1997), 14: 503-512.

[14] Mihara, R.H., Arrow's Theorem, Countably Many Agents and More Visible Invisible Dictators, *Journal of Mathematical Economics* (1999), 32: 267 - 287.

[15] Schmitz, N., A Further Note on Arrow's Impossibility Theorem, *Journal of Mathematical Economics* (1977): 189-196.

[16] Sikorski, R., *Boolean Algebras* (3rd edition), Springer-Verlag, Berlin, 1969.

[17] Úbeda, L., Neutrality in Arrow and Other Impossibility Theorems, *Economic Theory* (2003), 23: 195-204.