INSTITUTO TECNOLÓGICO AUTÓNOMO DE MÉXICO

CENTRO DE INVESTIGACIÓN ECONÓMICA

Discussion Paper Series

Optimal Combinatorial Mechanism Design

Levent Ulku

Instituto Tecnológico Autónomo de México

March 2009

Discussion Paper 09-03

Av. Camino a Santa Teresa # 930
Col. Héroes de Padierna
México, D.F. 10700
M E X I C O

Optimal Combinatorial Mechanism Design*

Levent Ülkü†
CIE and Department of Economics
ITAM

February 2009

Abstract

We consider an optimal mechanism design problem with several heterogenous objects and interdependent values. We characterize ex post incentives using an appropriate monotonicity condition and reformulate the problem in such a way that the choice of an allocation rule can be separated from the choice of the payment rule. Central to our analysis is the formulation of a regularity condition, which gives a recipe for the optimal mechanism. If the problem is regular, then an optimal mechanism can be obtained by solving a combinatorial allocation problem in which objects are allocated in a way to maximize the sum of "virtual" valuations. We identify conditions that imply regularity for two nonnested environments using the techniques of supermodular optimization.

Keywords: Combinatorial mechanism design, Interdependent values, Supermodularity, Regularity

*This paper is a revision of the third chapter of my PhD dissertation submitted to Rutgers University. I would like to thank Richard McLean for his supervision and continued guidance. I am also grateful to Colin Campbell, John Ledyard, Michael Schwarz and Tomas Sjöström for their helpful comments.

†levent.ulku@itam.mx

1 Introduction

In many problems of interest in economics, an uninformed party must allocate several objects among privately informed agents. Such is the nature of the problem faced by an internet search engine in selling advertisement spots displayed after a keyword search, by the FCC in selling radiospectrum licenses, and by the FAA in selling rights to use airport arrival and departure gates. An important common feature in these problems is that the objects offered for sale are heterogeneous and they may be related in a complex way. Different advertisement spots will not generally attract the same number of users. An arrival gate and a departure gate in suitable locations at suitable times may be complements while two arrival gates at the same airport at the same time are substitutes. A wireless communication company may view radiospectrum licenses for two neighboring locations as complements and licenses for two distant locations as substitutes.

The main purpose of this paper is to analyze the combinatorial mechanism design problem in some generality, without making any assumptions on how the objects are related. The literature on mechanism design with independently distributed private information has established many celebrated results. In the seminal paper of this literature, Myerson [1981] considers an environment in which a principal interacts with several privately informed agents in order to allocate a single object and, in return, collect payments. The revelation principle implies that the mechanism can be chosen from among those which collect truthful valuation reports from the agents and then determine an allocation and payments. For agents to report their information to the mechanism truthfully, certain incentive constraints need to be satisfied. Myerson characterizes these incentive constraints via "monotonicity" and "envelope" conditions. For each agent i and each of his types t_i, let $Q_i(t_i)$ be the expected probability of winning the object and let $U_i(t_i)$ be the expected payoff from reporting truthfully. Loosely speaking, monotonicity requires Q_i to be nondecreasing and the envelope condition requires the equality $Q_i(t_i) = U_i'(t_i)$. Myerson then reformulates the principal's revenue maximization problem as one of maximizing the expected sum of "virtual" valuations of the agents subject to the monotonicity constraint, where an agent's virtual valuation for the object is his actual valuation less the reciprocal of the hazard rate of the distribution of his valuation. Next, he asks when the constraints in the reformulated problem will not be binding and shows that under a regularity condition, a solution to the mechanism design problem can be obtained by focusing on the simpler problem of maximizing the expected sum of virtual valuations without the incentive constraints.

In this paper, we are interested in the formulation of regularity conditions in a combinatorial problem with interdependent valuations. Some important features of the model we analyze are as follows. There are several objects but types are unidimensional. Each agent i is equipped with a valuation function v_i which associates a real number with every type vector $t = (t_1, ..., t_n)$ and every set of objects A in a grand set Ω. Thus each t generates a vector of valuations $(v_i(A, t))_{A \subseteq \Omega}$. Each v_i is common knowledge so that the principal need only elicit one dimensional type reports from the agents in order to calculate their valuations for all subsets of Ω. This allows us to focus on the multidimensionality associated with allocating sets, in the absence of the well known problems of incentive characterization in models with multidimensional private information. We identify conditions under which the former kind of multidimensionality is analytically tractable.

We believe that the main contribution of this paper is the identification of the role of supermodularity-related conditions in allocation problems where the decision variables are multidimensional. Such conditions, in conjunction with others, imply that the decision rules satisfy certain monotonicity conditions, which, in return, imply implementability of the decision rules in the face of incomplete information. Analogous supermodularity conditions can be used, for example, in models with public goods, and in more general implementation problems. Most work on mechanism design builds on assumptions that essentially reduce the dimensionality of the decision space to one in which case the supermodularity conditions we identify are trivially satisfied. An important exception is Levin [1997] who analyzes the mechanism design problem with two complementary objects. Levin's analysis depends on supermodularity in a subtle way and our approach allows us to extend his results to an arbitrary number of objects and to interdependent values.

We impose ex post incentive constraints on the mechanisms. This is a line of departure from Myerson [1981] who imposes interim incentive constraints, requiring truthful reporting to constitute a Bayes-Nash equilibrium. Ex post incentive constraints are stronger as they require truthful reporting to be an ex post Nash equilibrium.[1] Consequently, ex post constraints give rise to mechanisms which are robust to changes in agents' beliefs about the distribution of private information. From this perspective, ex post Nash equilibrium stands in relation to dominant strategy equilibrium. In fact the two solution concepts are equivalent in models with private values where agents' valuations depend only on their own information.

Despite this obvious advantage, the imposition of ex post rather than the weaker interim constraints definitely restricts the feasible set in a mechanism design problem. However this restriction typically has no bearing on the solution to the problem. As Mookherjee and Reichelstein [1992] show, under certain conditions usually imposed in the literature, Bayesian mechanism design with private values produces mechanisms that satisfy dominant strategy incentive constraints. Under exactly the same conditions, Bayesian mechanism design when agents have interdependent values gives rise to mechanisms that satisfy ex post incentive constraints. In Section 5, we further discuss the relationship between ex post and interim incentive constraints in models with interdependent values.

This paper is closely linked to a number of studies of the optimal mecha-

[1] A strategy profile is an ex post Nash equilibrium if it involves the play of a Nash equilibrium in the complete information games associated with any realization of collective private information.

nism design problem. Myerson [1981] solves the mechanism design problem for a single object and with a useful and tractable form of interdependence of valuations. Branco [1996] studies a model with interdependent values, multiple identical objects and decreasing marginal utilities. Monteiro [2002] analyzes a private values model with identical objects but without the decreasing marginal utilities assumption of Branco, allowing for synergies or complementarities between objects. Levin [1997] analyzes the mechanism design problem with two complementary objects and private values.[2] Ledyard [2007] analyzes a combinatorial problem with several nonidentical objects and with private values, however with a special valuation structure: each agent has a positive valuation for exactly one specific subset of the grand set of objects. In related work, Maskin and Riley [1989] and Ausubel and Cramton [1999] analyze the mechanism design problem when the principal has a continuum of identical objects with private and interdependent values, respectively. The approach in this paper can be used to analyze all these models as special cases.

The plan of the paper is as follows. Section 2 introduces the model. Section 3 presents a characterization of ex post incentive compatibility using monotonicity and envelope conditions in a way that extends Myerson's analysis. Section 4 identifies regularity as the condition under which the optimal mechanism can be obtained by solving a reformulation of the mechanism design problem. In two subsections, we identify conditions that guarantee regularity for different classes of valuations. First, we develop a supermodularity based analysis of the sufficiency conditions making use of the theory of monotone selection. Next, we analyze problems in which preferences over sets can be represented by valuation functions over real numbers, or more generally over any completely ordered set. In these problems, supermodularity conditions do not impose any restriction. Many examples of combinator-

[2] Levin uses a direct approach tailored for the two object scenario instead of a Myersonesque reformulation.

ial problems studied in the literature fall in this second category, including models of multiple identical units in which supermodularity conditions of the first approach fail to be satisfied. Section 5 concludes by discussing several important extensions. Among these extensions are the related mechanism design problems with different constraint sets and different objectives, as well as models with revision effects (Myerson [1981]) and models with perfectly divisible objects.

2 The Model

Consider a mechanism design problem in which (possibly a strict subset of) a finite set Ω of indivisible objects will be allocated by an uninformed principal among privately informed agents in return for monetary transfers. All actors are risk-neutral. Let $N = \{1, ..., n\}$ be the set of agents. The space of outcomes is $C \times \Re^n$ where

$$C = \{(A_1, ..., A_n) : \bigcup_{i \in N} A_i \subseteq \Omega \text{ and } A_i \bigcap A_j = \emptyset \text{ if } i \neq j\} \tag{1}$$

is the set of lists of n pairwise disjoint subsets of Ω. We will use $\subseteq$ and $\subset$ for weak and strict set inclusion respectively. The set A_i in the list $(A_1, ..., A_n)$ identifies the objects allocated to agent i. Note that a list $(A_1, ..., A_n) \in C$ need not cover Ω, i.e., some members of Ω may remain unallocated to any agent. The requirement that the sets A_i and A_j be disjoint for different agents i and j ensures that no single object is allocated to multiple agents. Note that, except in the special case when $n = 1$, C is not a lattice.

Agents have private information in the form of one dimensional types. We will assume that the private information is independently distributed across agents. The type of agent i is a random variable $\tilde{t}_i$ with a positive density f_i and associated distribution F_i on a support $T_i = [a_i, b_i]$. We denote by t_i a typical element of T_i. We define random vectors $\tilde{t} = (\tilde{t}_1, ..., \tilde{t}_n)$, $\tilde{t}_{-i} = (\tilde{t}_1, ..., \tilde{t}_{i-1}, \tilde{t}_{i+1}, ..., \tilde{t}_n)$, write $\tilde{t} = (\tilde{t}_i, \tilde{t}_{-i})$ and denote the typical realizations of these random vectors by t and t_{-i}. We let f and f_{-i} be the joint densities for $\tilde{t}$ and $\tilde{t}_{-i}$, with associated distributions F and F_{-i}. We denote by $\mathbb{E}_i, \mathbb{E}_{-i}$ and $\mathbb{E}$, the expectation operators with respect to F_i, F_{-i} and F.

We allow for informational externalities but there are no externalities pertaining to the allocation of objects. The payoff of agent i depends on the set of objects he receives, the size of his payment, and the realized collective private information vector. Given an outcome $(A_1, ..., A_n, x_1, ..., x_n) \in C \times \Re^n$, and a type vector t, i's payoff is $v_i(A_i, t) - x_i$ where $v_i : 2^{\Omega} \times T \rightarrow \Re$ is his valuation function. We maintain the following assumptions on valuations throughout the paper.

Assumption 1 For each i, t_{-i} and A, $v_i(A, \cdot, t_{-i})$ is differentiable (right differentiable at a_i and left differentiable at b_i) and nondecreasing.[3]

Assumption 2 For each i, t_i and A, $v_i(A, t_i, \cdot)$ is (Lebesgue) integrable.

We will also normalize $v_i(\varnothing, t)$ to zero for every i and t. These assumptions place minimal restrictions on functions $v_i(A, \cdot, t_{-i})$. In particular, we make no curvature assumption regarding the way in which v_i depends on agent t_i (cf. Maskin and Riley [1984 and 1989], Levin [1997] and Krishna and Maenner [2001]). In addition, we are not making any assumptions on the functions $v_i(\cdot, t)$. For example we may have $v_i(A, t) < v_i(A', t)$ for two sets $A' \subset A$. Moreover, objects may be complements as in Levin [1997], or substitutes as in Branco [1996] and Monteiro [2002]. Agents may be "single-minded," in the sense of having positive valuation for only one specific set of objects as in Ledyard [2007]. In general, of course, an agent may view some objects as complements and others as substitutes. One agent may view some objects as complements, while another agent may view the same objects as substitutes, and at different type vectors an agent's attitude towards objects may change.

[3]We will denote the derivative of v_i with respect to t_i by $\partial v_i(A, \cdot, t_{-i})$.

The principal attaches no value to the objects and his payoff is simply the sum of payments $\sum x_i$. At the cost of additional notation, all our results directly extend to a setting in which the principal has positive valuations for various sets of objects, as long as these valuations do not depend on agents' private information. This is, in fact, a point of departure from Myerson [1981] who assumes that the information held by i affects the valuation of each $j \neq i$ as well as the valuation of the principal through "revision effects." The exact form of Myerson's results depends critically on the symmetry and the linearity of revision effects. Although we allow virtually any form of interdependence between the agents' valuations, in the main body of the paper we rule out the possibility that an agent's information has any effect on the principal's valuation. We discuss the generalization of Myerson's full fledged model with revision effects to multiple objects in Section 5.

3 Mechanisms

The revelation principle tells us that, regardless of his objective, the principal need only consider direct mechanisms which ask agents to report their types, induce truthful reporting, and determine allocation and payments depending on the reported types. We will be interested in deterministic mechanisms that induce truthful reporting as an ex post Nash equilibrium.

A (direct and deterministic) *mechanism* consists of an allocation rule $S : T \to C$ and a payment rule $x : T \to \Re^n$ and is denoted (S, x). We will write $S(t) = (S_1(t), ..., S_n(t))$ and $x(t) = (x_1(t), ..., x_n(t))$. Given a mechanism (S, x), the ex post payoff to i when the type vector is t and all agents report truthfully is $V_i(t|S, x) = v_i(S_i(t), t) - x_i(t)$. Whenever convenient, we will suppress the dependence of the ex post payoff on the underlying mechanism and simply write $V_i(t)$. A mechanism (S, x) satisfies *ex post Nash incentive compatibility* (XIC) if $V_i(t) \geq v_i(S_i(t'_i, t_{-i}), t) - x_i(t'_i, t_{-i})$ for every $i, t = (t_i, t_{-i})$ and $t'_i \neq t_i$, and *ex post individual rationality* (XIR) if $V_i(t) \geq 0$

for every i and t. A mechanism that satisfies both XIC and XIR is said to be *ex post incentive feasible*. We will denote by $\mathcal{F}$ the set of ex post incentive feasible mechanisms. An allocation rule S is *ex post Nash implementable* if there is a payment rule x such that the mechanism $(S, x) \in \mathcal{F}$. Throughout the paper we will restrict attention to allocation rules S for which $\partial v_i(S_i(\cdot, t_{-i}), \cdot, t_{-i}) : T_i \rightarrow \Re$ is Lebesgue integrable for every i and t_{-i} so that the payment rule we will analyze is well defined.

3.1 Characterizing Incentives

We begin by characterizing the class of mechanisms that satisfy XIC using, as is standard in the literature, "monotonicity" and "envelope" conditions that are appropriate for our combinatorial setting. These conditions, in their various versions, are fundamental workhorses of mechanism design theory.

Monotonicity (M): An allocation rule $S : T \rightarrow C$ satisfies *Condition M* if for every $i, t = (t_i, t_{-i})$ and $t_i' \neq t_i$,

$$v_i(S_i(t), t) \geq v_i(S_i(t), t_i', t_{-i}) + \int_{t_i'}^{t_i} \partial v_i(S_i(y, t_{-i}), y, t_{-i}) dy.$$

Envelope Condition (E): A mechanism (S, x) satisfies *Condition E* if for every $i, t = (t_i, t_{-i})$ and $t_i' \neq t_i$,

$$V_i(t) = V_i(t_i', t_{-i}) + \int_{t_i'}^{t_i} \partial v_i(S_i(y, t_{-i}), y, t_{-i}) dy.$$

Note that M is an assumption pertaining to allocation rules, while E is an assumption pertaining to mechanisms. We leave it to the reader to verify that, in the standard single unit environment with private values where $|\Omega| = 1$ and $v_i(\Omega, t) = t_i$, M is equivalent to the following familiar condition: the agent to whom the object is allocated does not change when only that agent's type goes up. This property is easily recognized to be the ex post version of the monotonicity condition in Myerson [1981]. Versions of M appear in Mookherjee and Reichelstein [1992, equation (3)] in the context of general implementation problems with private values and in Branco [1996, equation (8)] in the context of a mechanism design problem with multiple identical units. In the next subsection we will identify two important environments in which M is implied by useful and aesthetically more appealing conditions. Condition E is standard.

Using results in Milgrom and Segal [2002] and Koliha [2006], the following result characterizes XIC. In particular, it relies on the observation that XIC implies E under two standard assumptions: monotonicity and differentiability of valuations in agents' own types.

Lemma 1 *A mechanism (S, x) satisfies XIC if and only if (S, x) satisfies E and S satisfies M.*

Proof. ($\Rightarrow$) Suppose that (S, x) satisfies XIC. Since $v_i(A, \cdot, t_{-i})$ is differentiable and nondecreasing, $\partial v_i(A, \cdot, t_{-i})$ is Lebesgue integrable and Proposition 1 in Koliha [2006] applies, rendering $v_i(A, \cdot, t_{-i})$ absolutely continuous. Now E follows from Theorem 2 in Milgrom and Segal [2002]. To see that S satisfies M, note that for every $i, t = (t_i, t_{-i})$ and $t'_i \neq t_i$

$$\begin{aligned}
\int_{t'_i}^{t_i} \partial v_i(S_i(y, t_{-i}), y, t_{-i}) dy &= V_i(t) - V_i(t'_i, t_{-i}) \\
&= v_i(S_i(t), t) - x_i(t) - [v_i(S_i(t'_i, t_{-i}), t'_i, t_{-i}) - x_i(t'_i, t_{-i})] \\
&\leq v_i(S_i(t), t) - x_i(t) - [v_i(S_i(t), t'_i, t_{-i}) - x_i(t)] \\
&= v_i(S_i(t), t) - v_i(S_i(t), t'_i, t_{-i})
\end{aligned}$$

where the fisr equality is E and the inequality follows from XIC.

($\Leftarrow$) Suppose that S satisfies M and (S, x) satisfies E. For every $i, t = (t_i, t_{-i})$

and $t_i' \neq t_i$

$$\begin{aligned} V_i(t) - V_i(t_i', t_{-i}) &= \int_{t_i'}^{t_i} \partial v_i(S_i(y, t_{-i}), y, t_{-i}) dy \\ &\leq v_i(S_i(t), t) - v_i(S_i(t), t_i', t_{-i}) \end{aligned}$$

from which XIC follows. ∎

We will finish this subsection by recording three corollaries to Lemma 1. Corollary 1 obtains an ex post revenue equivalence result for combinatorial mechanism design problems. Corollary 2 characterizes ex post Nash implementable allocation rules. Corollary 3 characterizes ex post incentive feasible mechanisms.

Corollary 1 *All mechanisms which satisfy XIC, which have the same allocation rule and which leave the lowest type of each agent with the same ex post payoff generate the same ex post revenue.*

Proof. If a mechanism satisfies XIC, then by Lemma 1 it also satisfies E implying that

$$V_i(t) = V_i(a_i, t_{-i}) + \int_{a_i}^{t_i} \partial v_i(S_i(y, t_{-i}), y, t_{-i}) dy$$

for each i and t. Consequently, an agent's payment depends only on the allocation rule and the payoff received by his lowest type. ∎

Corollary 2 *An allocation rule is ex post Nash implementable if and only if it satisfies M.*

Proof. The only if part trivially follows from the definition of ex post Nash implementability and Lemma 1. Suppose that S satisfies M and choose x

such that for every i and t_{-i},

$$\begin{aligned} x_i(a_i, t_{-i}) &\leq v_i(S_i(a_i, t_{-i}), a_i, t_{-i}), \text{ and if } t_i > a_i, \text{ then} \\ x_i(t_i, t_{-i}) &= v_i(S_i(t_i, t_{-i}), t_i, t_{-i}) - \\ & \int_{a_i}^{t_i} \partial v_i(S_i(y, t_{-i}), y, t_{-i}) dy - V_i(a_i, t_{-i}). \end{aligned} \tag{2}$$

This choice of x implies E. Thus (S, x) must satisfy XIC. Since $\partial v_i(A, t_i, t_{-i}) \geq 0$ for every i, A and $t = (t_i, t_{-i})$, XIR follows as well and $(S, x) \in \mathcal{F}$. ■

Corollary 3 *A mechanism (S, x) is ex post incentive feasible if and only if (S, x) satisfies E, S satisfies M, and for every i and t_{-i}, $V_i(a_i, t_{-i}|S, x) \geq 0$.*

Proof. ($\Rightarrow$) If $(S, x) \in \mathcal{F}$, then E and M follow from Lemma 1 and XIR implies that $V_i(a_i, t_{-i}|S, x) \geq 0$.
($\Leftarrow$) Suppose that S satisfies M, (S, x) satisfies E and $V_i(a_i, t_{-i}|S, x) \geq 0$ for every i and t_{-i}. Now XIC follows from Lemma 1. XIR follows because, using E, we can write

$$V_i(t|S, x) = V_i(a_i, t_{-i}|S, x) + \int_{a_i}^{t_i} \partial v_i(S_i(y, t_{-i}), y, t_{-i}) dy$$

which is nonnegative since $V_i(a_i, t_{-i}|S, x) \geq 0$ and v_i is increasing in t_i. ■

3.2 Restricted Environments

We can identify two important and nonnested environments in which M is implied by simpler and more appealing conditions. The concept of "monotone differences" will play a key role in both environments. Let L be a set partially ordered by $\preceq$. A function $\phi : L \times [a, b] \to \Re$ satisfies *nondecreasing differences* if

$$\phi(l, y') - \phi(l', y') \leq \phi(l, y) - \phi(l', y)$$

for every $l' \preceq l$ and $y' \leq y$. Let $\prec$ be the strict part of $\preceq$. The function ϕ satisfies *strictly increasing differences* if

$$\phi(l, y') - \phi(l', y') < \phi(l, y) - \phi(l', y)$$

for every $l' \prec l$ and $y' < y$.

Lemma 2 *Suppose that 2^Ω is partially ordered by set inclusion and that $v_i(\cdot, \cdot, t_{-i}) : 2^\Omega \times T_i \rightarrow \Re_+$ satisfies nondecreasing differences for every i and t_{-i}. Then an allocation rule S satisfies condition M if*

$$\text{for every } i \text{ and } t_{-i},\ t_i' < t_i \Rightarrow S_i(t_i', t_{-i}) \subseteq S_i(t_i, t_{-i}). \quad (3)$$

Proof. The condition that $v_i(\cdot, \cdot, t_{-i})$ satisfies nondecreasing differences and the assumption of differentiability of $v_i(A_i, \cdot, t_{-i})$ imply that $\partial v_i(\cdot, t_i, t_{-i})$ is isotone, i.e., that $A \subseteq A'$ implies that $\partial v_i(A, t_i, t_{-i}) \leq \partial v_i(A', t_i, t_{-i})$. Fix S, i, t_{-i} and $t_i' < t_i$. We have

$$\begin{aligned}\int_{t_i'}^{t_i} \partial v_i(S_i(y, t_{-i}), y, t_{-i}) dy &\leq \int_{t_i'}^{t_i} \partial v_i(S_i(t_i, t_{-i}), y, t_{-i}) dy \\ &= v_i(S_i(t_i, t_{-i}), t_i, t_{-i}) - v_i(S_i(t_i, t_{-i}), t_i', t_{-i})\end{aligned}$$

where the inequality follows from the observation that $\partial v_i(\cdot, t_i, t_{-i})$ is isotone and the hypothesis of the Lemma. The equality follows from the observation that $v_i(A_i, \cdot, t_{-i})$ is absolutely continuous (Koliha [2006]). If $t_i' > t_i$, then, we use similar arguments to conclude that

$$\begin{aligned}\int_{t_i'}^{t_i} \partial v_i(S_i(y, t_{-i}), y, t_{-i}) dy &= -\int_{t_i}^{t_i'} \partial v_i(S_i(y, t_{-i}), y, t_{-i}) dy \\ &\leq -\int_{t_i}^{t_i'} \partial v_i(S_i(t_i, t_{-i}), y, t_{-i}) dy \\ &= -[v_i(S_i(t_i, t_{-i}), t_i', t_{-i}) - v_i(S_i(t_i, t_{-i}), t_i, t_{-i})] \\ &= v_i(S_i(t_i, t_{-i}), t_i, t_{-i}) - v_i(S_i(t_i, t_{-i}), t_i', t_{-i})\end{aligned}$$

and the proof is complete. ■

Thus conditon M is satisfied if the sets allocated to each agent are weakly expanding in that agent's own type. For such allocation rules, payments can be constructed as in (2) to obtain XIC mechanisms. We should note that the nondecreasing differences condition implies a complementarity relationship between an agent's private information and the set of objects he receives. In particular, it can be shown that v_i satisfies nondecreasing differences in (A, t_i) if and only if for every $t_{-i}, t_i' < t_i, \omega \in \Omega$, and $A \subseteq \Omega \backslash \{\omega\}$,

$$v_i(A \cup \{\omega\}, t_i', t_{-i}) - v_i(A, t_i', t_{-i}) \leq v_i(A \cup \{\omega\}, t_i, t_{-i}) - v_i(A, t_i, t_{-i}),$$

that is, the marginal value of attaining another object (when the agent already has the set A and when the collective type vector for the remaining agents is t_{-i}) is higher for higher types. This does not mean that the objects are complements. In fact valuations may satisfy nondecreasing differences even if all agents think that the objects in Ω are perfect substitutes.

A different environment in which conditon M can be deduced from a simpler condition is one in which valuations depend on scalars associated with sets, rather than the sets themselves. This effectively reduces the domain of valuations to a subset of the real line.

Definition 1 (Mookherjee and Reichelstein [1992]) Valuations satisfy *one dimensional condensation* if for each i, there exist maps $\mu_i : 2^\Omega \to \Re$ and $\hat{v}_i : \Re \times T \to \Re_+$ such that for every A and t, $v_i(A, t) = \hat{v}_i(\mu_i(A), t)$.

We will call the maps $\hat{v}_i$ *condensed valuations.* An important example of an allocation problem where one-dimensional condensation is satisfied is the case of perfect substitutes where μ_i is the counting measure given by $\mu_i(S) = |S|$. Note that if μ_i is not monotone, then the condition that $\hat{v}_i(\cdot, \cdot, t_{-i})$ satisfies nondecreasing differences on $\Re \times T_i$ and the condition that $v_i(\cdot, \cdot, t_{-i})$ satisfies nondecreasing differences on $2^\Omega \times T_i$ are nonnested. We record the following lemma whose proof is similar to that of Lemma 2.

Lemma 3 *Suppose that valuations satisfy one-dimensional condensation. Further suppose that the condensed valuations are such that for every i and t_{-i}, $\hat{v}_i(\cdot, \cdot, t_{-i})$ satisfies nondecreasing differences on $\Re \times T_i$. Then an allocation rule S satisfies condition M if*

$$\textit{for every } i \textit{ and } t_{-i},\ t_i' < t_i \Rightarrow \mu_i(S_i(t_i', t_{-i})) \leq \mu_i(S_i(t_i', t_{-i})). \tag{4}$$

In the particular case when objects are perfect substitutes, Lemma 3 implies that an allocation rule is ex post Nash implementable if it does not assign an agent fewer objects when his type increases and other agents' types remain unchanged.

Note that conditions (3) and (4) are usually nonnested. In the environment of Lemma 2, we may have an allocation rule S with nondecreasing $\mu_i(S_i(\cdot, t_{-i}))$, without $S_i(\cdot, t_{-i})$ being weakly expanding. To see this, consider a single agent problem where the type space is $[0, 1]$, the set of objects is $\{\alpha, \beta\}$, valuations are determined by $v(A, t) = \mu(A)t$ where $\mu(\emptyset) = \mu(\{\alpha\}) = 0$ and $\mu(\{\beta\}) = \mu(\{\alpha, \beta\}) = 1$. Consequently, $\hat{v} : \Re \times [0, 1] \to \Re$ is defined as $\hat{v}(z, t) = zt$ and satisfies nondecreasing differences. If $S(t) = \{\alpha\}$ when $0 \leq t < \frac{1}{2}$ and $S(t) = \{\beta\}$ when $\frac{1}{2} \leq t \leq 1$, then $t \mapsto \mu(S(t))$ is nondecreasing but $t \mapsto S(t)$ is not weakly expanding.

4 Optimal Mechanism Design

In this section we will analyze the *optimal mechanism design problem*

$$\max_{(S,x) \in \mathcal{F}} \mathbb{E} \sum_{i \in N} x_i(\tilde{t}). \tag{5}$$

In our framework, agent i's *virtual valuation* is a map $u_i : 2^\Omega \times T \to \Re$ defined by

$$u_i(A, t) = v_i(A, t) - \partial v_i(A, t) \frac{1 - F_i(t_i)}{f_i(t_i)}.$$

Note that $u_i(\varnothing, t) = 0$ since $v_i(\varnothing, t) = 0$ for every i and t. In problems where $\Omega = \{\omega\}$ and valuations are given by $v_i(\{\omega\}, t) = t_i$, agent i's virtual valuation reduces to the familiar expression $t_i - \frac{1-F_i(t_i)}{f_i(t_i)}$ in Myerson [1981]. Other special cases have appeared in, for example, Maskin and Riley [1984] and Branco [1996]. As is typical in the literature, virtual valuations play a crucial role in our analysis. For any mechanism that satisfies condition E, and consequently for every mechanism that is feasible in problem (5), the expected revenue of the principal is equal to the expected sum of all virtual valuations minus the expected sum of surpluses of the lowest type of each agent.

Lemma 4 *If (S, x) satisfies E, then*

$$\mathbb{E}\sum_{i\in N} x_i(\tilde{t}) = \mathbb{E}[\sum_{i\in N} u_i(S_i(\tilde{t}), \tilde{t}) - \sum_{i\in N} V_i(a_i, \tilde{t}_{-i})].$$

Proof. By an integration by parts argument we obtain, for any i and t_{-i}

$$\mathbb{E}_i \int_{a_i}^{\tilde{t}_i} \partial v_i(S_i(y, t_{-i}), y, t_{-i})dy = \mathbb{E}_i \partial v_i(S_i(\tilde{t}_i, t_{-i}), \tilde{t}_i, t_{-i})\lambda_i(\tilde{t}_i) \tag{6}$$

where $\lambda_i(t_i) = (1 - F_i(t_i))/f_i(t_i)$. Using condition E, we can write, for any i and t_i

$$\mathbb{E}_{-i} x_i(t_i, \tilde{t}_{-i}) = \mathbb{E}_{-i}\left[v_i(S_i(t_i, \tilde{t}_{-i}), t_i, \tilde{t}_{-i}) - V_i(a_i, \tilde{t}_{-i}) - \int_{a_i}^{t_i} \partial v_i(S_i(y, \tilde{t}_{-i}), y, \tilde{t}_{-i})dy\right]$$

and computing expectations with respect to t_i we obtain

$$\mathbb{E}x_i(\tilde{t}) = \mathbb{E}v_i(S_i(\tilde{t}), \tilde{t}) - \mathbb{E}_{-i}V_i(a_i, \tilde{t}_{-i}) - \mathbb{E}_{-i}\mathbb{E}_i \int_{a_i}^{\tilde{t}_i} \partial v_i(S_i(y, \tilde{t}_{-i}), y, \tilde{t}_{-i})dy.$$

Using (6) and summing over i, finishes the proof. ■

Following Myerson [1981], the next result reformulates (5) using Lemma 4. The reformulation separates the choice of the allocation rule from the choice of the payment rule and this will play a key role in the ensuing analysis. We will denote by $\mathcal{M}$ the class of allocation rules satisfying condition M.

Proposition 1 *If S^* solves the reformulated problem*

$$\max_{S\in\mathcal{M}} \mathbb{E}\sum_{i\in N} u_i(S_i(\tilde{t}),\tilde{t}) \tag{7}$$

and if

$$x_i^*(t) = v_i(S_i^*(t),t) - \int_{a_i}^{t_i} \partial v_i(S_i^*(y,t_{-i}),y,t_{-i})dy \tag{8}$$

for every i and t, then the mechanism (S^*,x^*) solves the optimal mechanism design problem (5).

Proof. The allocation rule S^* must satisfy M as it is feasible in problem (7). The choice of x^* implies that the mechanism (S^*,x^*) satisfies E. Therefore, by Lemma 1, (S^*,x^*) satisfies XIC. The choice of x^* also indicates that (S^*,x^*) satisfies XIR. Note that for every i and t_{-i}, $V_i(a_i,t_{-i}|S^*,x^*)=0$ since $x_i^*(a_i,t_{-i}) = v_i(S_i^*(a_i,t_{-i}),a_i,t_{-i})$. Thus (S^*,x^*) is feasible in (5). For any other ex post incentive feasible mechanism (S,x) we have,

$$\begin{aligned}
\mathbb{E}\sum_{i\in N} x_i(\tilde{t}) &= \mathbb{E}[\sum_{i\in N} u_i(S_i(\tilde{t}),\tilde{t}) - \sum_{i\in N} V_i(a_i,\tilde{t}_{-i}|S,x)] \\
&\leq \mathbb{E}\sum_{i\in N} u_i(S_i(\tilde{t}),\tilde{t}) \\
&\leq \mathbb{E}\sum_{i\in N} u_i(S_i^*(\tilde{t}),\tilde{t}) \\
&= \mathbb{E}\sum_{i\in N} x_i^*(\tilde{t})
\end{aligned}$$

where the first equality follows from Lemma 4, the first inequality follows because (S,x) must satisfy XIR, the second inequality is by hypotheses and the final equality follows from Lemma 4 and the observation that $V_i(a_i, t_{-i}|S^*, x^*) = 0$ for every i and t_{-i}. ■

Two remarks on Proposition 1 are in order.

Remark 1 Fix a solution S^* to (7) and consider the set $X(S^*) = \{x : (S^*, x) \in \mathcal{F}\}$. This set is nonempty since $(S^*, x) \in \mathcal{F}$ if x is as defined in (8). If $\xi \in X(S^*)$, then for every i and t, we have

$$\begin{aligned} x_i^*(t) &= v_i(S_i^*(t), t) - \int_{a_i}^{t_i} \partial v_i(S_i^*(y, t_{-i}), y, t_{-i}) dy \\ &\geq v_i(S_i^*(t), t) - \int_{a_i}^{t_i} \partial v_i(S_i^*(y, t_{-i}), y, t_{-i}) dy \\ &\quad -[v_i(S_i^*(a_i, t_{-i}), a_i, t_{-i}) - \xi_i(a_i, t_{-i})] \\ &= \xi_i(t) \end{aligned}$$

where the first equality is by definition, the inequality follows from the fact that (S^*, ξ) must satisfy XIR, and the final equality follows because (S^*, ξ) must satisfy E. Thus (S^*, x^*) achieves the highest possible revenue for the mechanism designer within the class of ex post incentive feasible mechanisms with the allocation rule is S^*.

Remark 2 The implication in Proposition 1 can be reversed. Suppose that the mechanism (S^*, x^*) solves (5). Then the following must hold: S^* solves (7) and payments satisfy (2).

Proposition 1 indicates that in order to solve (5), the allocation rule can be chosen to solve (7) and payments can be derived by using this allocation rule and (8). But solving (7) may still be formidable as we don't know much about the structure of the constraint set $\mathcal{M}$. Regularity addresses exactly this issue. Define the *ex post optimal partitioning problem* at the type vector t to be

$$\max_{(A_1,\ldots,A_n)\in C} \sum_{i\in N} u_i(A_i, t) \tag{9}$$

Definition 2 The optimal mechanism design problem (5) is *regular* if, for any allocation rule S, the following condition holds:

$$S(t) \text{ solves (9) at every } t \Rightarrow S \text{ satisfies M.}$$

The following summary result highlights the role of regularity.

Proposition 2 *If the optimal mechanism design problem (5) is regular, then an optimal mechanism is obtained by choosing an allocation rule that solves (9) at every type vector and determining payments as in (8).*

We should note that even under regularity, the principal still needs to solve (9) which is in general a computationally difficult combinatorial opimization problem. In Section 4.2 we will talk about a special case in which a solution to (9) can be obtained more easily.

We move on to establish conditions that imply regularity in the restricted environments considered in Lemmas 2 and 3. Our main goal is to establish conditions under which a monotone comparative static result can be obtained for (9) which, in conjunction with Lemmas 2 or 3, will imply regularity,

4.1 Regularity with Supermodularity

In this section we will analyze regularity conditions under the assumption that $v_i(\cdot, \cdot, t_{-i})$ satisfies nondecreasing differences for every i and t_{-i}. We will need the following definitions. A map $\phi : 2^\Omega \rightarrow \Re$ is *supermodular* if for every $A, B \subseteq \Omega$,

$$\phi(A) + \phi(B) \leq \phi(A \cup B) + \phi(A \cap B).$$

A map $\phi : 2^\Omega \times [a, b] \rightarrow \Re$ satisfies the *strict single crossing property* if

$$\phi(A, y') \geq \phi(A', y') \Rightarrow \phi(A, y) > \phi(A', y)$$

for every $A' \subset A$ and every $y' < y$.

It is instructive to start with the single agent problem in which (9) reduces to

$$\max_{A \in 2^{\Omega}} u(A, t) \tag{9'}$$

where, importantly, the constraint set is a lattice. Using standard techniques, it can be shown that if $u(\cdot, t)$ is supermodular for every t, and if u satisfies the strict single crossing property, then every optimal selection $t \mapsto S(t) \in \arg\max_{A \in 2^{\Omega}} u(A, t)$ is monotonic in the sense of (3). In other words $S(t') \subseteq S(t)$ whenever $t' < t$ so Lemma 2 implies that condition M is satisfied.[4] To sum up, we have the following result.

Lemma 5 *The optimal mechanism design problem with a single agent is regular if*

1. *v satisfies nondecreasing differences, and*
2. *u satisfies strict single crossing property and $u(\cdot, t)$ is supermodular on 2^{Ω} for every t.*

In order to identify sufficient conditions for the general problem with $n > 1$ agents, we need to tackle two issues. First, the constraint set C in (9) is not a lattice which makes it impossible to get monotonicity of its solutions by making supermodularity and single crossing assumptions on the function $(A_1, ..., A_n, t) \mapsto \sum_{i \in N} u_i(A_i, t)$. Second, the complications arising from interdependence of valuations need to be addressed. In the multiagent problem, t_i has an effect on u_j. Loosely speaking, we must make sure that t_i has a larger effect on the set received by i than on the set received by $j \neq i$.

[4]The supermodularity requirement can be weakened. A function $\phi : 2^{\Omega} \to \Re$ is *quasi-supermodular* (Milgrom and Shannon [1994]) if for every $A, B \subseteq \Omega$, $\phi(A \cap B) \leq \phi(A) \Rightarrow \phi(B) \leq \phi(A \cup B)$. Every supermodular function is quasi-supermodular. It can be shown that if u satisfies the strict single crossing property and $u(\cdot, t)$ satisfies quasi-supermodularity for every t, then every solution to (9′) is weakly expanding.

We will need some new notation. Define, for each i, A and t

$$u^*_{-i}(A_i, t) = \max_{(A_i, A'_{-i}) \in C} \sum_{j \neq i} u_j(A'_j, t)$$

where A'_{-i} lists $\{A'_j : j \neq i\}$. In words, $u^*_{-i}(A_i, t)$ is the largest sum of virtual valuations of all other agents conditional on i getting set A_i when the type vector is t. Note that since we can set $A'_j = \varnothing$ for every $j \neq i$, $u^*_{-i}(A_i, t) \geq 0$. We also have $u^*_{-i}(\Omega, t) = 0$ and $u^*_{-i}(A_i, t) \leq u_i(A'_i, t)$ if $A'_i \subset A_i$. Now let

$$U^*_i(A_i, t) = u_i(A_i, t) + u^*_{-i}(A_i, t)$$

denote the largest sum of virtual valuations at t conditional i getting the set A_i. Consider the problem

$$\max_{A_i \in 2^\Omega} U^*_i(A_i, t) \tag{10}$$

As a consequence of the definitions, for every solution $(S_1(t), ..., S_n(t))$ to (9) and for every i, $S_i(t)$ is a solution to (10). Since the constraint set in (10) is a lattice, if the maps U^*_i satisfy the appropriate strict single crossing property and supermodularity conditions, the monotonicity conditions in (3) will be satisfied and regularity will follow.

For expositional purposes we will define these conditions on the maps u_i and u^*_{-i}, rather than on the maps U^*_i.

Definition 3 Virtual valuations satisfy the *extended strict single crossing property* (E-SSCP) if

$$[u_i(A_i, t'_i, t_{-i}) - u_i(A'_i, t'_i, t_{-i}) \geq u^*_{-i}(A'_i, t'_i, t_{-i}) - u^*_{-i}(A_i, t'_i, t_{-i})]$$
$$\Rightarrow [u_i(A_i, t) - u_i(A'_i, t) > u^*_{-i}(A'_i, t) - u^*_{-i}(A_i, t)]$$

for every $i, t = (t_i, t_{-i}), t'_i < t_i$ and $A'_i \subset A_i$.

Definition 4 Virtual valuations satisfy *extended supermodularity* (E-SUPM) if for every i and t, $u_i(\cdot, t) : 2^{\Omega} \to \Re$ and $u^*_{-i}(\cdot, t) : 2^{\Omega} \to \Re$ are supermodular.

Note that E-SSCP is a condition on the set $\{u_i(\cdot, \cdot, \cdot) : i \in N\}$ and E-SUPM is a condition on the set $\{u_i(\cdot, t) : i \in N \text{ and } t \in T\}$. E-SSCP is satisfied if and only if maps $U^*_i(\cdot, \cdot, t_{-i})$ satisfy the strict single crossing property. Moreover E-SUPM implies the supermodularity of the functions $U^*_i(\cdot, t)$.

We are now ready to state our main result.

Proposition 3 *The optimal mechanism design problem is regular if*

1. $v_i(\cdot, \cdot, t_{-i}) : 2^{\Omega} \times T_i \to \Re$ *satisfies nondecreasing differences for every* i *and* t_{-i}, *and*

2. *virtual valuations satisfy Conditions E-SSCP and E-SUPM.*

Proof. Pick an allocation S such that $S(t) = (S_1(t), ..., S_n(t))$ solves (9) at every t. For every i, $S_i(t)$ must also solve (10). Note that the constraint set in (10) is a lattice. Conditions E-SSCP and E-SUPM imply that the objective function in (10) satisfies strict single crossing property and supermodularity. Pick $i, t'_i < t_i$ and t_{-i} and suppose that $S_i(t'_i, t_{-i}) \not\subseteq S_i(t_i, t_{-i})$. Then $S_i(t_i, t_{-i}) \subset S_i(t_i, t_{-i}) \cup S_i(t'_i, t_{-i})$ and

$$\begin{aligned}
0 &\leq U^*_i(S_i(t'_i, t_{-i}), t'_i, t_{-i}) - U^*_i(S_i(t_i, t_{-i}) \cup S_i(t'_i, t_{-i}), t'_i, t_{-i}) \\
&\Rightarrow \\
0 &< U^*_i(S_i(t'_i, t_{-i}), t_i, t_{-i}) - U^*_i(S_i(t_i, t_{-i}) \cup S_i(t'_i, t_{-i}), t_i, t_{-i}) \\
&\Rightarrow \\
0 &< U^*_i(S_i(t_i, t_{-i}) \cap S_i(t'_i, t_{-i}), t_i, t_{-i}) - U^*_i(S_i(t_i, t_{-i}), t_i, t_{-i})
\end{aligned}$$

where the first inequality is by hypothesis, the first implication is by the strict single crossing property and the last implication is by supermodularity. This contradicts the optimality of $S_i(t_i, t_{-i})$ at (t_i, t_{-i}). We therefore must have

$S_i(t_i', t_{-i}) \subseteq S_i(t_i, t_{-i})$ and (3) is satisfied. By Lemma 2, S satisfies M and we conclude that the optimal mechanism design problem (5) is regular. ■

Several remarks on Proposition 3 are in order.

1. To interpret E-SSCP, fix i, t_{-i} and $A_i' \subset A_i$. Now consider two "type dependent plans." In the first plan, A_i is assigned to i and the remaining objects are allocated between the rest of the agents in a way to solve the problem involved in the definition of $u_{-i}^*(A_i, t)$. In the second plan, the A_i' is assigned to i and the remaining objects are allocated between the rest of the agents in a way to solve the problem involved in the definition of $u_{-i}^*(A_i', t)$. If the first plan induces a higher sum of virtual valuations at the type vector (t_i', t_{-i}), then, under E-SSCP, the first plan continues to dominate the second at the type vector (t_i, t_{-i}) where $t_i > t_i'$.

2. Suppose that for every i, A and t, $u_i(A, t) = \mu_i(A)h_i(t)$ where h_i is differentiable. Furthermore, suppose that $\mu_i(A') < \mu_i(A)$ whenever $A' \subset A$ and that $\mu_i(A) + \mu_i(\Omega \backslash A) = \mu_i(\Omega)$ for each $A \subseteq \Omega$.[5] This includes the special case where for all nonempty A,

$$\mu_i(A) = \sum_{a \in A} \omega_i(a)$$

for some collection of positive numbers $\{\omega_i(a)\}_{a \in A}$ and, consequently, the special case in which $\mu_i(A) = |A|$.

In this model, we claim that the E-SSCP will be satisfied if

$$\frac{\partial h_i}{\partial t_i}(t_i, t_{-i}) > \frac{\partial h_j}{\partial t_i}(t_i, t_{-i}) \text{ for every } i \neq j \text{ and } t = (t_i, t_{-i}) \qquad (11)$$

[5]In the theory of simple games, set functions with this additivity property are called *decisive*.

and if

$$h_i(t_i', t_{-i}) \geq 0 \Rightarrow h_i(t_i, t_{-i}) > 0 \text{ for every } i, t_{-i} \text{ and } t_i' < t_i. \qquad (12)$$

Condition (11) says that an increase in i's type has a bigger effect on i's valuation than it does on j's valuation when $j \neq i$. Similar conditions have appeared in other interdependent valuation models. For examples, especially in the context of auctions and auction design, see Cremer and McLean [1985], Ausubel [1999], Maskin and Dasgupta [2000], Perry and Reny [2002] and Krishna [2003]. Condition (12) implies that if an agent's virtual value is at least zero at a lower type, then his virtual valuation remains stricly positive at all of his higher types. An analogous condition appears in Branco [1996]. Note that (12) follows if $h_i(\cdot, t_i)$ is strictly increasing. With a single object and private values, h_i depends only on t_i and is given by $h_i(t_i) = t_i - \frac{1-F_i(t_i)}{f_i(t_i)}$. Consequently (12) follows if the hazard rate of F_i is nondecreasing.

To prove the claim, define $h^*_{-i}(t) = \max_{j \neq i} h_j(t)$ for every i and t so that

$$u^*_{-i}(A_i, t) = \mu_i(\Omega \backslash A_i) \max\{0, h^*_{-i}(t)\}.$$

Since

$$\mu_i(A_i) - \mu_i(A_i') = \mu_i(\Omega \backslash A_i') - \mu_i(\Omega \backslash A_i),$$

the E-SSCP reduces to the condition: for every i, t_{-i}, $t_i' < t_i$ and $A_i' \subset A_i$

$$\begin{aligned}[\mu_i(A_i) - \mu_i(A_i')] \left[h_i(t_i', t_{-i}) - \max\{0, h^*_{-i}(t_i', t_{-i})\}\right] \geq 0 \\ \Rightarrow [\mu_i(A_i) - \mu_i(A_i')] \left[h_i(t_i, t_{-i}) - \max\{0, h^*_{-i}(t_i, t_{-i})\right]\} > 0.\end{aligned}$$

Since $\mu_i(A') < \mu_i(A)$ whenever $A' \subset A$, it follows that the E-SSCP will be satisfied if

$$h_i(t'_i, t_{-i}) \geq \max\{0, h^*_{-i}(t'_i, t_{-i})\} \Rightarrow h_i(t_i, t_{-i}) > \max\{0, h^*_{-i}(t_i, t_{-i})\}$$

for every $i, t_{-i}, t'_i < t_i$. In particular, E-SSCP is satisfied if conditions (11) and (12) are satisfied.

3. Proposition 3 still works if we replace E-SUPM with quasi-supermodularity of the functions $U^*_i(\cdot, t)$ for every i and t where quasi-supermodularity is defined as in footnote 5. However quasi-supermodularity is not preserved under addition and $U^*_i(\cdot, t)$ may fail to be quasi-supermodular even if $u_i(\cdot, t)$ and $u^*_{-i}(\cdot, t)$ are quasi-supermodular.

It is natural to ask if the conditions of E-SSCP and E-SUPM in Proposition 3 can be weakened. Next, we demonstrate two cases in which such weakenings are indeed possible. First, we show that under private values E-SSCP can be replaced with the condition that all virtual valuations satisfy strictly increasing differences. Next, we show that if there are two objects, E-SUPM can be replaced with the condition that virtual valuations are supermodular on 2^{Ω} at every type vector.

Private Values If valuations are private, then, abusing notation, the E-SSCP condition becomes

$$\begin{aligned}[u_i(A_i, t'_i) - u_i(A'_i, t'_i) \geq u^*_{-i}(A'_i, t_{-i}) - u^*_{-i}(A_i, t_{-i})] \\ \Rightarrow [u_i(A_i, t_i) - u_i(A'_i, t_i) > u^*_{-i}(A'_i, t_{-i}) - u^*_{-i}(A_i, t_{-i})]\end{aligned}$$

for every $i, t'_i < t_i$ and $A'_i \subset A_i$. Note that the right hand side of both of the inequalities is $u^*_{-i}(A'_i, t_{-i}) - u^*_{-i}(A_i, t_{-i})$ and is nonnegative. A sufficient condition for E-SSCP under private values is strictly increasing differences of virtual valuations, i.e., the condition that

$$u_i(A_i, t_i') - u_i(A_i', t_i') < u_i(A_i, t_i) - u_i(A_i', t_i)$$

for every $i, t_i' < t_i$ and $A_i' \subset A_i$. Hence, we have the following result.

Proposition 4 *Suppose that valuations are private. Then the optimal mechanism design problem is regular if*

1. *valuations satisfy nondecreasing differences, and*
2. *virtual valuations satisfy strictly increasing differences and E-SUPM.*

Selling Two Objects (Levin [1997]) Suppose that $\Omega = \{\omega_1, \omega_2\}$ and, for simplicity, that values are private. In this special case E-SUPM follows from the supermodularity of individual virtual valuations.

Lemma 6 *If* $\Omega = \{\omega_1, \omega_2\}$ *and* $u_i(\cdot, t)$ *is supermodular for every* i*, then* $u^*_{-i}(\cdot, t)$ *is supermodular for every* i*. Consequently, as a sum of two supermodular functions* $U^*_i(\cdot, t)$ *is also supermodular.*

Proof. Since $u^*_{-i}(\{\omega_1, \omega_2\}, t\} = 0$, it suffices to show that for every i,

$$u^*_{-i}(\{\omega_1\}, t\} + u^*_{-i}(\{\omega_2\}, t\} \leq u^*_{-i}(\varnothing, t\}.$$

There is nothing to show if $u^*_{-i}(\{\omega_1\}, t\} = 0$ or $u^*_{-i}(\{\omega_2\}, t\} = 0$. Suppose that $u^*_{-i}(\{\omega_1\}, t\} = u_j(\{\omega_2\}, t) > 0$ and $u^*_{-i}(\{\omega_2\}, t\} = u_k(\{\omega_1\}, t) > 0$. If $k = j$, then

$$\begin{aligned} u^*_{-i}(\{\omega_1\}, t\} + u^*_{-i}(\{\omega_2\}, t\} &= u_j(\{\omega_1\}, t) + u_j(\{\omega_2\}, t) \\ &\leq u_j(\{\omega_1, \omega_2\}, t) \\ &\leq \max\{u_l(\{\omega_1, \omega_2\}, t) : l \neq i\} \\ &\leq u^*_{-i}(\varnothing, t\} \end{aligned}$$

and the result follows. If $j \neq k$, the result follows from the observation that the allocation S for which $S_j = \{\omega_2\}$ and $S_k = \{\omega_1\}$ is feasible in the problem defining $u^*_{-i}(\varnothing, t\}$. ■

In the special case of private values, Lemma 6 and Proposition 4 lead to the following corollary.

Corollary 4 *Suppose that valuations are private and that there are only two objects. Then the optimal mechanism design problem is regular if*

1. *valuations satisfy nondecreasing differences, and*
2. *virtual valuations satisfy strictly increasing differences and supermodularity on 2^Ω.*

Unfortunately, Lemma 6 fails when there are three objects or more. In general, the supermodularity of the maps $u^*_{-i}(\cdot, t_i)$ must be explicitly assumed in order to guarantee regularity.

In an interesting paper Levin [1997] considers precisely the environment of Corollary 4 and uses a direct argument tailored to the two object case. Without invoking the machinery of supermodular optimization, he employs an exhaustive analysis of all possible cases to prove that the optimal allocation has the expansion property (3). Levin's assumptions imply, however, that the hypotheses of our Corollary 4 are satisfied.

To see how Levin's framework can be incorporated in our model, let there exist for each agent i nonnegative and differentiable functions $v_{i1}, v_{i2}, \varkappa_i$ such that:

$$v'_{i1}(t_i), v'_{i2}(t_i) > 0, \ \varkappa'_i(t_i) \geq 0$$
$$v''_{i1}(t_i), v''_{i2}(t_i), \varkappa''_i(t_i) \leq 0$$
$$v_i(S, t_i) = \begin{cases} 0 & \text{if } S = \emptyset \\ v_{i1}(t_i) & \text{if } S = \{\omega_1\} \\ v_{i2}(t_i) & \text{if } S = \{\omega_2\} \\ v_{i1}(t_i) + v_{i2}(t_i) + \varkappa_i(t_i) & \text{if } S = \{\omega_1, \omega_2\} \end{cases}$$

Note that v_i satisfies strictly increasing differences. Now assume that the hazard rates of type distributions are nondecreasing (Levin's Assumption 2) so that u_i satisfies strictly increasing differences as well. Finally assume that $\varkappa_i(t_i) - \frac{1-F_i(t_i)}{f_i(t_i)}\varkappa'_i(t_i) \geq 0$ (Levin's Assumption 3) so that $u_i(\cdot, t_i)$ is supermodular for every i and t_i. Now Corollary 4 applies and the problem is regular.

We should emphasize that it is not the complementarity of objects per se that drives Levin's result. Complementarity, in conjunction with the rest of Levin's assumptions, implies that valuations and virtual valuations satisfy the conditions in Corollary 4. As we noted earlier, nondecreasing differences over $2^\Omega \times T_i$ is not nested with supermodularity over 2^Ω.

4.2 Regularity without Supermodularity

In many examples of interest, the supermodularity hypotheses of Section 4.1 are not satisfied. Consider, for example, an environment with two identical objects and with decreasing marginal utilities. Suppose that agents are only interested in the number of units they obtain and that their valuations take the form $v_i(A, t) = \mu(|A|)t_i$ where μ is strictly concave on $\{0, 1, 2\}$. Suppose further that t_i is distributed uniformly over $[0, 1]$ so that $\frac{1-F_i(t_i)}{f_i(t_i)} = 1 - t_i$. Now Corollary 4 can not be used to determine whether the optimal mechanism design problem is regular, since the maps $u_i(\cdot, t_i)$ are not supermodular: for every $i, t_i > \frac{1}{2}$,

$$\begin{aligned} u_i(\{\omega_1, \omega_2\}, t) - u_i(\{\omega_1\}, t) &= [\mu(2) - \mu(1)][2t_i - 1] \\ &< [\mu(1) - \mu(0)][2t_i - 1] \\ &= u_i(\{\omega_2\}, t) - u_i(\emptyset, t). \end{aligned}$$

In order to analyze such problems, we will restrict attention to valuations that satisfy the one-dimensional condensation property of Section 3.2. Assume that for each i, there exist maps $\mu_i : 2^\Omega \to \Re_+$ and $\hat{v}_i : \Re \times T \to \Re_+$ such that $v_i(A, t) = \hat{v}_i(\mu_i(A), t)$ for every A and t, .and define for every $z \in \Re$:

$$\hat{u}_i(z, t) = \hat{v}_i(z, t) - \partial \hat{v}_i(z, t) \frac{1 - F_i(t_i)}{f_i(t_i)}$$

Next, define

$$C_\mu = \{(\mu_1(A_1), ..., \mu_n(A_n)) : \bigcup_i A_i \subseteq \Omega \text{ and } A_i \text{ are disjoint}\}$$

and for every $z \in \Re$, define

$$\hat{u}^*_{-i}(z, t) = \begin{cases} \max\limits_{z:(z, a_{-i}) \in C_\mu} \sum_{j \neq i} \hat{u}_j(a_j, t) & \text{if } z = \mu_i(A) \text{ for some } A \subseteq \Omega \\ 0 & \text{otherwise} \end{cases} .$$

Adapting from Definition 3, we will say that the virtual valuations satisfy the E-SSCP if for every i, t_{-i}, $t'_i < t_i$ and $z_i, z'_i \in \{\mu_i(A) : A \subseteq \Omega\}$ with $z'_i < z_i$, we have

$$\begin{aligned} \hat{u}_i(z_i, t'_i, t_{-i}) - \hat{u}_i(z'_i, t'_i, t_{-i}) &\geq \hat{u}^*_{-i}(z'_i, t'_i, t_{-i}) - \hat{u}^*_{-i}(z_i, t'_i, t_{-i}) \\ &\Rightarrow \hat{u}_i(z_i, t) - \hat{u}_i(z'_i, t) > \hat{u}^*_{-i}(z'_i, t) - \hat{u}^*_{-i}(z_i, t). \end{aligned}$$

Note that the functions $\hat{u}_i(\cdot, t)$ and $\hat{u}^*_{-i}(\cdot, t)$ are defined on the real line. Since the real line is completely ordered, these functions are trivially supermodular. Therefore regularity is obtained without making supermodularity assumptions.

Proposition 5 *Suppose that for every i, t and A, $v_i(A,t) = \hat{v}_i(\mu_i(A), t)$. Then, the optimal mechanism design problem is regular if*

1. *$\hat{v}_i(\cdot,\cdot,t_{-i})$ satisfies nondecreasing differences for every i and t_{-i}, and*

2. *the virtual valuations $\hat{u}_i$, $i \in N$, satisfy the E-SSCP.*

Proof. If the allocation rule S is such that for every t, $(S_1(t), ..., S_n(t))$ solves the optimal partitioning problem (9), then for every i, $\mu_i(S_i(t))$ solves

$$\max_{z_i \in \{\mu_i(A):A \subseteq \Omega\}} [\hat{u}_i(z_i,t) + \hat{u}^*_{-i}(z_i,t)].$$

E-SSCP implies that $(z_i,t_i) \mapsto [\hat{u}_i(z_i,t_i,t_{-i}) + \hat{u}^*_{-i}(z_i,t_i,t_{-i})]$ satisfies SSCP implying that $t_i \mapsto \mu_i(S_i(t_i,t_{-i}))$ is nondecreasing for every i and t_{-i} and condition (4) is satisfied. By Lemma 3 the problem is regular. ■

We note that, analogous to Proposition 4, in the special case of private values the E-SSCP condition can be replaced with the condition that $\hat{u}_i$ satisfies strictly increasing differences for every i.

Proposition 6 *Suppose that for every i, t and A, $v_i(A,t) = \hat{v}_i(\mu_i(A), t_i)$. Then, the optimal mechanism design problem is regular if for every i*

1. *$\hat{v}_i(\cdot,\cdot)$ satisfies nondecreasing differences on $\Re \times T_i$ and*

2. *$\hat{u}_i$ satisfies strictly increasing differences.on $\Re \times T_i$*

As an application of Proposition 6, consider a model with private values in which for every i,

$$v_i(A,t_i) = \begin{cases} t_i & \text{if } A_i^* \subseteq A \\ 0 & \text{otherwise} \end{cases}$$

for some set A_i^*. This is precisely the problem analyzed in Ledyard [2007]. The interpretation is that every agent is "single-minded" in the sense of being interested in obtaining only one set. Let $\mu_i(A) = 1$ if $A_i^* \subseteq A$ and $\mu_i(A) = 0$ otherwise so that we can write $v_i(A, t_i) = \mu_i(A)t_i$. Note that the condensed valuations take the form $\hat{v}_i(a, t_i) = at_i$ and nondecreasing differences condition is automatically satisfied. The virtual valuations are given by

$$u_i(A, t_i) = \begin{cases} t_i - \frac{1-F_i(t_i)}{f_i(t_i)} & \text{if } A_i^* \subseteq A \\ 0 & \text{otherwise} \end{cases}$$

and they can be written as $u_i(A, t_i) = \mu_i(A)(t_i - \frac{1-F_i(t_i)}{f_i(t_i)})$. Let the condensed virtual valuation be $\hat{u}_i(a, t_i) = a(t_i - \frac{1-F_i(t_i)}{f_i(t_i)})$ so that $u_i(A, t_i) = \hat{u}_i(\mu_i(A), t_i)$ and observe that $\hat{u}_i$ satisfies strictly increasing differences if the hazard rate of i's type is nondecreasing. Hence the sufficient condition in this example is precisely the sufficient condition for regularity in Myerson [1981]. [6]

Identical Objects In problems with identical objects, agents care about the cardinality of the set of objects that they receive. Hence, the one-dimensional condensation property is satisfied with $\mu_i(A) = |A|$, i.e., for every A and t,

$$v_i(A, t) = \hat{v}_i(|A|, t)$$

for some condensed valuations $\hat{v}_i : \Re \times T \to \Re_+$. Define the maps $w_{ik} : T \to \Re$ and $u_{ik} : T \to \Re$ by

$$\begin{aligned} w_{i0}(t) &= \hat{v}_i(0, t), \\ u_{i0}(t) &= u_i(0, t) \\ w_{ik}(t) &= \hat{v}_i(k, t) - \hat{v}_i(k-1, t) \text{ if } k = 1, ..., m, \text{ and} \\ u_{ik}(t) &= u_i(k, t) - u_i(k-1, t) \text{ if } k = 1, ..., m. \end{aligned}$$

[6] In any example with private values in which the one-dimensional condensation is satisfied, the functions μ_i are monotone and the functions $\hat{v}_i$ are linear in t_i, regularity follows if types have nondecreasing hazard rates.

so that $\hat{v}_i(a,t) = \sum_{k=1}^{a} w_{ik}(t)$ and $\hat{u}_i(a,t) = \sum_{k=1}^{a} u_{ik}(t)$. Sufficient conditions for regularity can now be obtained by resorting to Proposition 5 (See also Branco [1996].) However we can identify a different set of conditions under which regularity is obtained directly without applying monotone comparative static arguments. These conditions make heavy use of concavity of $\hat{u}_i(\cdot,t)$ which is not required in Proposition 5.

Proposition 7 *Suppose that the one-dimensional condensation property is satisfied with $\mu_i(A) = |A|$ for every i and define the maps w_{ik} and u_{ik} as above. Then the optimal mechanism design problem is regular if*

1. $w_{ik}(\cdot, t_{-i})$ *is nondecreasing for each* i, k *and* t_{-i}*,*

2a. $u_{ik}(t) \geq u_{ik+1}(t)$ *for each* i, k *and* t

2b. $u_{ik}(t_i', t_{-i}) \geq 0 \Rightarrow u_{ik}(t_i, t_{-i}) > 0$ *for each* $i, k, t_i' < t_i$ *and* t_{-i}

2c. $u_{ik}(t_i', t_{-i}) \geq u_{jk'}(t_i', t_{-i}) \Rightarrow u_{ik}(t_i, t_{-i}) > u_{jk'}(t_i, t_{-i})$ *for each* $i, j, k, k', t_i' < t_i$ *and* t_{-i}*.*

The first condition in Proposition 7 is precisely nondecreasing differences, while 2a, 2b and 2c replace the E-SSCP condition. Condition 2a implies that virtual valuations are concave in the number of objects received. Condition 2b is essentially a form of single crossing, which follows if $\hat{u}_i(\cdot,\cdot,t_{-i})$ satisfies the strict single crossing property. Condition 2c is an inter-agent comparison condition which captures the idea that for any pair of agents i and j, t_i has a larger effect on i's virtual valuation than on j's virtual valuation. Note that E-SSCP is not implied by 2a-2c.

Proof. For every t, let $u^{(1)}(t) \geq u^{(2)}(t) \geq \cdots \geq u^{(m)}(t)$ be the first m highest elements of $\{u_{ik}(t) : i = 1, ..., n \text{ and } k = 1, ...m\}$. Define

$$\begin{aligned} W(t) &= \{u^{(1)}(t), ..., u^{(m)}(t)\} \cap \Re_+ \\ W_i(t) &= W(t) \cap \{u_{ik}(t) : k = 1, ..., m\} \end{aligned}$$

be, respectively, the set of "winning bids" and the set of i's winning bids. For this identical units problem, we have

$$C_\mu = \{(a_1, ..., a_n) : \sum a_i \leq m \text{ and } a_i \text{ is a nonnegative integer}\}.$$

The optimal partitioning problem becomes

$$\max_{(A_1,...,A_n)\in C} \sum_{i=1}^{n} \sum_{k=1}^{|A_i|} u_{ik}(t)$$

and if $S(t)$ is a solution to this problem, then $(|S_1(t)|, ..., |S_n(t)|)$ solves

$$\max_{(a_1,...,a_n)\in C_\mu} \sum_{i=1}^{n} \sum_{k=1}^{a_i} u_{ik}(t)$$

implying that $|S_i(t)| = |W_i(t)|$.

Fix an agent i and types $t'_i < t_i$ and t_{-i}. Suppose that $S(t'_i, t_{-i})$ and $S(t_i, t_{-i})$ solve the optimal partitioning problem at the corresponding type vectors. We will first show that i does not receive fewer units at the type vector (t'_i, t_{-i}). Let $|S_i(t'_i, t_{-i})| = k > 0$. Then $u_{ik}(t'_i, t_{-i})$ is a winning bid and is therefore nonnegative. Suppose that $|S_i(t_i, t_{-i})| < k$ so that $u_{ik}(t_i, t_{-i})$ is not a winning bid. Nevertheless $u_{ik}(t_i, t_{-i})$ is strictly positive by condition 2b and we must have $\sum_{j\in N} |S_j(t_i, t_{-i})| = m$ since otherwise $u_{ik}(t_i, t_{-i})$ would have to be a winning bid. Since $|S_i(t'_i, t_{-i})| > |S_i(t_i, t_{-i})|$ and $\sum_{j\in N} |S_j(t'_i, t_{-i})| \leq m = \sum_{j\in N} |S_j(t_i, t_{-i})|$, there must exist some agent who receives more units at the type vector (t_i, t_{-i}) compared to (t'_i, t_{-i}). In other words, there exists $j \neq i$ and an integer k' such that $u_{jk'}(t_i, t_{-i})$ is a winning bid but $u_{jk'}(t'_i, t_{-i})$ is not. It follows that

$$u_{ik}(t'_i, t_{-i}) \geq u_{jk'}(t'_i, t_{-i}) \text{ and } u_{ik}(t_i, t_{-i}) \leq u_{jk'}(t_i, t_{-i})$$

contradicting condition 2c. Thus $|S_i(t_i, t_{-i})| \geq k$. Condition 1 implies that $t \mapsto S(t)$ satisfies M by Lemma 3, and regularity is obtained. ■

5 Conclusion and Extensions

We formulated a notion of regularity for optimal mechanism design problems with multiple heterogenous objects and identified conditions that imply regularity. If the optimal mechanism design problem is regular, then a recipe for the optimal mechanism is readily available: the principal must allocate objects in a way to maximize the sum of virtual valuations and determine payments according to a fixed formula, which depends on the allocation. Among the sufficiency conditions that we identify, of particular importance are supermodularity-related conditions which are automatically satisfied in environments with one-dimensional condensation. In such environments valuations depend on scalars associated with sets rather than the sets themselves and supermodularity conditions are moot. Since most work on mechanism design implicitly or explicitly assumes condensation, the role of supermodularity in mechanism design has largely been neglected. One important exception is the work of Levin [1997] on optimal mechanism design with two complementary objects whose analysis depends on supermodularity in a subtle way. Our approach highlights the role of supermodularity and allows us to extend Levin's results in a number of directions, including to an arbitrary number of objects and to interdependent values.

There exist several important extensions of the present model in which our results continue to hold with appropriate modifications. We conclude by considering some of these extensions.

Interim Incentive Constraints As we discussed in the introduction, imposing ex post incentive constraints in mechanism design is desirable from a robustness viewpoint. However ex post constraints are stronger than interim constraints and their use raises an important question as to whether or not the value of the optimal mechanism design problem decreases significantly as a result. Here we will argue that under any set of sufficient conditions for regularity which we identified, the principal does not lose expected revenue by restricting the set of feasible mechanisms to those that satisfy ex post rather than interim incentive constraints.

Let $\mathcal{G}$ be the class of mechanisms which satisfy Bayesian incentive compatibility and interim individual rationality. To be precise $\mathcal{G}$ consists of mechanisms (S, x) such that

$$\mathbb{E}_{-i}[v_i(S_i(t_i, \tilde{t}_{-i}), t_i, \tilde{t}_{-i}) - x_i(t_i, \tilde{t}_{-i})] \geq \max\{0, \mathbb{E}_{-i}[v_i(S_i(t_i', \tilde{t}_{-i}), t_i, \tilde{t}_{-i}) - x_i(t_i', \tilde{t}_{-i})]\}$$

for every i, t_i and $t_i' \neq t_i$. Now consider the optimal mechanism design problem with interim constraints given by:

$$\max_{(S,x)\in\mathcal{G}} \mathbb{E} \sum_{i\in N} x_i(\tilde{t}) \tag{5$'$}$$

Clearly $\mathcal{F} \subseteq \mathcal{G}$ and the value of problem (5′) is at least as large as the value of problem (5). The analysis of (5′) is parallel to the analysis of (5). One characterizes incentives, derives a reformulation, defines regularity and identifies sufficient conditions for regularity.

The monotonicity condition that we need to characterize interim incentives is:

Interim Monotonicity (M′): For every i, t_i and $t_i' \neq t_i$,

$$\mathbb{E}_{-i} v_i(S_i(t_i, \tilde{t}_{-i}), t_i, \tilde{t}_{-i}) \geq \mathbb{E}_{-i}[v_i(S_i(t_i, \tilde{t}_{-i}), t_i', \tilde{t}_{-i}) + \int_{t_i'}^{t_i} \partial v_i(S_i(y, \tilde{t}_{-i}), y, \tilde{t}_{-i}) dy]. \tag{M$'$}$$

Note that, as we may have expected, M′ is precisely the interim version of M and is therefore a weaker condition. For any allocation rule S, it can be shown that there exists x such that $(S, x) \in \mathcal{G}$ if and only if S satisfies M′.

Now we can reformulate (5′) with an exact analogue of Proposition 1 using a problem in which the expected sum of virtual valuations is maximized by a choice of an allocation rule that satisfies M′. The definition of regularity in this case also remains almost identical to Definition 2, except for the use of M′ instead of M. In particular, the optimal mechanism design problem (5′) is *regular* if every allocation rule obtained by solving (9) at every type vector satisfies M′. Hence, the difference between working with ex post versus interim incentives is precisely the difference between analyzing solutions to the optimal partitioning problem (9) and asking when these solutions will satisfy condition M versus asking when they will satisfy condition M′. To be sure, under any set of conditions from which M follows, M′ also follows as it is weaker. Hence sufficient conditions for regularity of (5) are also sufficient for the regularity of (5′) and, as a consequence, exactly the same recipe for the optimal mechanism applies in both problems. Put differently, the sufficient conditions for the regularity of (5) are also conditions under which the values of problems (5) and (5′) are the same.

In principle, it is reasonable to expect that conditions that are weaker than the ones in, say, Proposition 3 can be identified under which a solution to (9) satisfies M′ but not M. The identification of such weaker conditions is an open problem.

Efficient Mechanism Design Consider the *efficient* mechanism design problem with ex post incentive constraints

$$\max_{(S,x)\in\mathcal{F}} \mathbb{E}\sum_{i\in N} v_i(\tilde{t}) \tag{13}$$

and define the efficient partitioning problem at type vector t to be

$$\max_{(A_1,\ldots,A_n)\in C} \sum_{i\in N} v_i(A_i, t). \tag{14}$$

Suppose that every allocation rule obtained by solving (14) at every t satisfies condition M. Then a solution to (13) can be obtained by solving (14) at every t and tagging onto this allocation rule, the payments in (8). In order to solve (13), then, we need to identify conditions under which the pointwise solution to (14) satisfies M. But this is an exact analog of finding sufficient conditions for regularity of the optimal mechanism design problem, with virtual valuations replaced by actual valuations. Hence we have the following analog of Proposition 3.

Proposition 8 *A solution to the efficient mechanism design problem (13) is obtained by allocating the objects in a way to solve (14) at every t and by determining payments using this allocation rule and equation (8), if the following conditions are satisfied:*

1. $v_i(\cdot, \cdot, t_{-i})$ *satisfies nondecreasing differences for every i and t_{-i}, and*
2. *valuations satisfy E-SSCP and E-SUPM.*

Note that all conditions here are on valuations. Hence, efficient mechanism design leads to a neater set of conditions under which a recipe for an efficient mechanism can be obtained. In particular no assumptions need to be made on virtual valuations or on type distributions.

Optimal mechanism design and efficient mechanism design are at the opposite extremes of a range of mechanism design problems in which the objective is to maximize a weighted sum of revenue and agents' welfare. Similar regularity conditions can be obtained in such problems by appropriately changing the objective function of the partitioning problem to the weighted sum in consideration and requiring the E-SSCP and E-SUPM conditions to be satisfied by these weighted sums.

Myerson's Revision Effects An important difference between Myerson's original formulation of the mechanism design problem and the present paper lies in the principal's valuation structure. In Myerson's model an agent's type affects the valuations all other agents *and* the principal linearly and in exactly the same way through "revision effects."

Definition 5 Valuations exhibit *revision effects* if for each i there exist maps $e_i : 2^\Omega \times T_i \to \Re$ and $g_i : 2^\Omega \times T_i \to \Re$ such that

1. $e_i(\cdot, t_i)$ is additive, i.e., if $A \cap A' = \emptyset$, then $e_i(A \cup A', t_i) = e_i(A, t_i) + e_i(A', t_i)$,
2. $v_i(A, t) = g_i(A, t_i) + \sum_{j \neq i} e_j(A, t_j)$, and
3. the principal's valuation is given by $v_0(A, t) = \sum_{j \in N} e_j(A, t_j)$.

Since the principal may have nonzero valuations for different sets, his objective is now to maximize his net revenue and the optimal mechanism design problem becomes

$$\max_{(S,x)\in\mathcal{F}} \mathbb{E} \sum_{i\in N} [x_i(\tilde{t}) - v_0(S_i(\tilde{t}), \tilde{t})]. \tag{15}$$

As a result, simple changes in the arguments show that the reformulated problem becomes

$$\max_{S\in\mathcal{M}} \mathbb{E} \sum_{i\in N} \sigma_i(S_i(\tilde{t}), \tilde{t}_i)$$

where $\sigma_i : 2^\Omega \times T_i \to \Re$ is defined by

$$\sigma_i(A, t_i) = g_i(A_i, t_i) - e_i(A_i, t_i) - \partial g_i(A_i, t_i) \frac{1 - F_i(t_i)}{f_i(t_i)}.$$

Quite remarkably, σ_i does not depend on t_{-i} even though agents have interdependent valuations. Hence, because of the special structure of revision effects, sufficient conditions for regularity of (15) do not include any interpersonal comparison condition like E-SSCP or condition 2c in Proposition 7.

Proposition 9 *Suppose that valuations satisfy revision effects and define σ_i as above. The optimal mechanism design problem (15) is regular if*

1. *g_i satisfies nondecreasing differences for every i, and*

2. *σ_i has SSCP and $\sigma_i(\cdot, t_i)$ is supermodular.*

Fluid Models In important work, Maskin and Riley [1989] and Ausubel and Cramton [1999] analyze the multiunit optimal mechanism design problem in a slightly different environment than ours. They analyze a problem in which the object is *fluid*, i.e., perfectly divisible, with a fixed supply of q_0 units and valuations take the form $v_i : [0, q_0] \times T \to \Re$. In particular they hypothesize that $v_i(q, t) = \int_0^q p_i(y, t)dy$ for some demand function $p_i : \Re_+ \times T \to \Re_+$.

The techniques of supermodular optimization can be employed in more general fluid models. Suppose there are m fluid objects and the supply constraints are given by q_0^k, $k = 1, ..., m$. A feasible allocation is a vector $q = (q_1, ..., q_n)$ where $q_i = (q_i^k)_{k=1}^m$ and $\sum_i q_i^k \leq q_0^k$. Let $Q = [0, q_0^1] \times \cdot \cdot \cdot \times [0, q_0^m]$. Note that Q is a lattice ordered with the partial order $\leq$ given by $\bar{q}_i \leq q_i$ if $\bar{q}_i^k \leq q_i^k$ for every k. Suppose that valuations take the form $v_i : Q \times T \to \Re$ where $v_i(q_i, \cdot, t_{-i})$ is differentiable and increasing and define $u_i(q_i, t) = v_i(q_i, t) - \partial v_i(q_i, t)\frac{1-F_i(t_i)}{f_i(t_i)}$. Now by appropriately modifying the proof of Proposition 3, we can show that if (1) each v_i satisfies nondecreasing differences on $Q \times T_i$, and (2) virtual valuations satisfy the appropriate modifications of conditions E-SSCP and E-SUPM, then the mechanism design problem is regular. In Maskin and Riley [1989] and Ausubel and Cramton [1999], $m = 1$ and, not surprisingly, the conditions they identify for regularity do not require supermodularity.

More general problems The methods we employ apply to more general settings as long as the lattice structure is preserved. As an example, consider a single agent mechanism design problem in which the outcome space is a lattice L_1 and the valuation of the agent is a map $v : L_1 \times [a, b] \to \Re$ given by $v(q, t) = g(w(q), t)$ where $w : L_1 \to L_2$ is isotone, L_2 is a lattice and $g : L_2 \times T \to \Re$. If $L_1 = L_2 = 2^\Omega$ for some finite set Ω and if w is the identity map, we specialize to the environment considered in Lemma 2 and Section 4.1. If $L_1 = 2^\Omega$, $L_2 = \Re$ and w is a set function we specialize to the environment considered in Lemma 3 and Section 4.2.

References

[1] L. Ausubel, A Generalized Vickrey Auction, University of Maryland working paper, 1999.

[2] L. Ausubel and P. Cramton, The Optimality of Being Efficient, University of Maryland working paper, 1999.

[3] F. Branco, Multiple Unit Auctions of an Indivisible Good, Economic Theory 8, 1996, 77-101.

[4] J. Cremer and R. P. McLean, Optimal Selling Strategies for a Discriminating Monopolist When Demands are Interdependent, Econometrica, 53, 1985, 345-361.

[5] J. J. Koliha, Fundamental Theorem of Calculus for Lebesgue Integration, American Mathematical Monthly 113,.2006, pp. 551-555.

[6] V. Krishna, Asymmetric English Auctions, Journal of Economic Theory 112, 2003, 261-288.

[7] V. Krishna and E. Maenner, Convex Potentials with an Application to Mechanism Design, Econometrica 69, 2001, 1113-1119.

[8] J. Ledyard, Optimal Combinatoric Auctions with Single-Minded Bidders, working paper 2007.

[9] J. Levin, An Optimal Auction for Complements, Games and Economic Behavior 18, 1997, 176-192.

[10] E. Maskin and P. Dasgupta, Efficient Auctions, Quarterly Journal of Economics, May 2000, 341-388.

[11] E. Maskin and J. Riley, Monopoly with Incomplete Information, Rand Journal of Economics, 15:2, 1984, 171-196.

[12] E. Maskin, and J. G. Riley, Optimal Multi-unit Auctions, in The Economics of Missing Markets, Information, and Games (F. Hahn, Ed.), 1989, Oxford: Oxford University Press.

[13] P. Milgrom and I. Segal, Envelope Theorems for Arbitrary Choice Sets, Econometrica 70, 2002, 583-601.

[14] P. Milgrom and C. Shannon, Monotone Comparative Statics, Econometrica 62, 1994, 157-180.

[15] P. K. Monteiro, Optimal Auctions in a General Model of Identical Goods, Journal of Mathematical Economics 37:1, 2002, 71-79.

[16] D. Mookherjee and S. Reichelstein, Dominant Strategy Implementation of Bayesian Incentive Compatible Allocation Rules, Journal of Economic Theory, 56, 2, 1992, 378-399.

[17] R. Myerson, Optimal Auction Design, Mathematics of Operations Research 6, 1981, 58-73.

[18] P. Reny and M. Perry, An Efficient Auction, Econometrica 70, 2002, 1199-1212.

[19] J. C. Rochet, A Necessary and Sufficient Condition for Rationalizability in a Quasi-Linear Context, Journal of Mathematical Economics 16, 2, 1987, 191-200.

[20] D. Topkis, Supermodularity and Complementarity, 1998, Princeton University Press.