INSTITUTO TECNOLÓGICO AUTÓNOMO DE MÉXICO

CENTRO DE INVESTIGACIÓN ECONÓMICA

WORKING PAPER SERIES

Nonmonotone Mechanism Design

Levent Ülkü
Instituto Tecnológico Autónomo de México

June/2012
Working Paper 12-02

Av. Camino Santa Teresa #930
México D.F., 10700
Tel. +52(55)5628-4197

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Levent Ülkü[2]

ITAM
June 2012

Abstract: I characterize the set of implementable allocation functions in the standard one dimensional mechanism design environment where the relationship between private information and payoffs is possibly nonmonotone. The characterization is useful in two aspects. First it leads to a rather mild condition under which individual rationality follows directly from incentive compatibility. Second, it can be conveniently used to determine the implementability of allocation functions in certain novel applications. In particular I show that neither monotonicity of allocations, nor the monotone differences property on values is necessary for implementation. In an application, I study a buyer-seller relationship where the buyer's value displays habit formation, which enters into his payoff through a commonly known parameter. Habit implies that the agent's value is a nonmonotone function of his type and that monotone differences condition can not be satisfied for all parameters. For a set of parameters, the seller-optimal mechanism is nonmonotone: the seller screens out low *and* high types.
Keywords: Implementation, Monotonicity, Monotone differences, Habits
JEL classification: D42, D61, D82

1 Introduction

In this paper I study conditions for implementation of allocation functions in a standard quasilinear model with one dimensional types. The novelty here is to allow agents' types to affect their payoffs nonmonotonically. Such nonmonotonicities arise naturally in many economic applications and they create certain complications. In particular they may lead to the failure of the monotone differences condition, a fundamental workhorse of mechanism design theory, which loosely stipulates that

[1]I would like to acknowledge the very helpful comments of Rich McLean, Nicolas Melissas, Debasis Mishra, Tymofiy Mylovanov and Tridib Sharma.
[2]Contact: levent.ulku@itam.mx

larger types should value larger allocations more at the margin. It is well known that if this condition holds, then monotone allocation functions are implementable. I show that even in the absence of monotone differences, a convenient characterization for implementation can be obtained using envelope results associated with incentive compatibility. In this case monotone allocation functions need not be implementable and implementable allocation functions need not be monotone. I study an application where a seller interacts with a buyer whose preferences exhibit habit formation. Habit implies that the agent's value is a nonmonotone function of his type and that monotone differences condition may fail. Using my characterization, I show that for some habit parameters, the seller-optimal mechanism is nonmonotone: the seller screens out low *and* high types.

The paper fits in the literature on mechanism design dealing with the problem of characterizing implementation when payoffs are quasilinear. The seminal paper in this area is Rochet (1987) which showed that a cycle monotonicity condition is necessary and sufficient for implementation in general. Despite its elegance, this condition is difficult to use in practice as it requires one to check that a certain type graph has no cycles (of any finite number of nodes) with negative length. Hence it is of interest to identify models where simpler conditions characterize implementation. Bikchandani et al. (2006) show that in various multidimensional problems with convex type spaces, implementability is equivalent to 2-cycle monotonicity (also called weak monotonicity) which requires that no cycle of any two nodes in the type graph should have negative length. Saks and Yu (2005) show that 2-cycle monotonicity characterizes implementability in all environments with convex type spaces.

Extending a classical result of Myerson (1981), Vohra (2011) shows that in models where both outcomes and types are one dimensional and where the monotone difference condition is satisfied strictly, an allocation function is implementable if and only if it assigns larger types larger allocations, i.e., if and only if it is monotone. In light of Vohra's result, the contribution of the current paper can be summarized as follows:

- If allocations are not one dimensional (if they belong to an arbitrary unordered set) then it is not clear what monotone differences of payoffs and monotonicity of allocation functions would have to mean. Proposition 1 states that if allocations can be partially ordered in such a way that these conditions are satisfied, then implementation follows.

- There are environments where no binary relation exists which ren-

> ders an allocation function of interest monotone while at the same time ensuring that the environment satisfies the monotone differences condition. Proposition 2 states that a tractable characterization for implementation can nevertheless be obtained without using any order-theoretic notion, via the standard envelope theorem. The condition given in Propositon 2 is both tractable enough to be put to use in applications, and it leads to a simple condition under which individual rationality follows from implementation even when payoffs are nonmonotone in types (Proposition 3.) The characterizing condition is stronger than 2-cycle monotonicity, which is equivalent to monotonicity *and* implementation under monotone differences.

Proposition 1 has important precedents. In particular, Jehiel and Moldovanu (2001) and Bergemann and Välimäki (2002) show that in a multiagent model with interdependent values, the efficient allocation rule is implementable if there is an order on allocations which renders it monotone and with respect to which the monotone differences condition holds. My result indicates that efficiency plays no role in these results.

I should note that the value of my characterization in Proposition 2 largely depends on the breadth of economically relevant mechanism design environments where Proposition 1 fails to apply. This happens, most notably, but not exclusively, when the outcome is either a null allocation giving the agent a payoff of zero regardless of his type, or some other allocation giving him a value which is nonmonotone in his type. In the next section, I present one such example. In the concluding section I describe a methodology to construct various nonmonotone environments by taking the agent's value to be his expectation of an unobservable state conditional on an informative signal. It turns out that for a class of joint densities for the state and the signal, the resulting conditional expectation is a nonmonotone function of the signal.

I study in Section 4 the problem faced by a seller interacting with a privately informed buyer whose values exhibit an intertemporal allocation externality in the form of habit persistence: current consumption generates disutility in the future. The existence of habit renders the agent's value nonmonotone in type. This implies that for certain habit parameters there is no nontrivial order on allocations which leads to monotone differences. Using the classical approach of Myerson (1981) I identify the seller-optimal mechanism and show that it is nonmonotone for a range of habit parameter values. This means that as part of an optimal mechanism, the seller screens out high as well as low types. This phenomenon occurs if habit is strong, i.e., if future disutility from current consumption is high.

The possibility of nonmonotone optimal mechanisms has been pointed out in the literature. With single dimensional types and multidimensional outcomes, Matthews and Moore (1987) show that a dimension of the outcome may be nonmonotone at a seller-optimum. Focusing on the class of differentiable mechanisms, García (2005) provides conditions under which the optimal mechanism is nonmonotone when the outcome space is multidimensional. I provide an example where a nonmonotone optimum exists (for a range of parameters) with a single dimensional outcome. I also analyze a two-period extension of my application in which the seller can provide the agent with the commodity in both periods in exchange for a one-shot payment. In this extension monotone differences property continues to fail, however a monotone mechanism is optimal for all habit parameters.

I consider a single-agent setting for brevity. All proofs as well as an outline of an extension to multiagent models with interdependent values are collected in the appendices.

2 Environment

Consider a standard mechanism design environment with a single agent. All results extend to a multiagent model with interdependent values where implementation is in ex post Nash equilibrium. Let A be a set of alternatives. An *outcome* is a pair (a, x) where $a \in A$ is an alternative chosen for the agent in return for a payment $x \in \Re$. The agent's resulting utility is $v(a, t) - x$ where $t \in T = [0, 1]$ is his type. To make use of an envelope result which characterizes implementability, I will assume that $v(a, \cdot)$ is absolutely continuous and differentiable for all a, and I will denote its derivative at t by $v_2(a, t)$.[3] Throughout the text (s, t) refers to a pair of types, not necessarily distinct.

Importantly, v need not be linear or even monotone in t. I will analyze an example of this nature at length in Section 4. Let me note here another example where nonmonotonicity occurs naturally.

Example 1: (Nonmonotonicity in type) Consider a trader who fixes his position z in the face of an unknown state θ to maximize the expectation of utility $1 - (\theta - z)^2$. The state θ takes two values, $\theta = -1$ and $\theta = 1$, and the trader's type t is his perceived probability of the state

[3] Milgrom and Segal (2001) show that under these assumptions the envelope theorem applies in characterizing incentives. Example 1 below and the application in Section 4 satisfy these conditions. In the concluding section I sketch a different application where v is convex but not everywhere differentiable in t. In this case, Proposition 1 in Krishna and Maenner (2002) can be used to get the envelope result of interest.

$\theta = 1$. Hence the trader chooses z to solve

$$\max_{z \in \Re} (1-t)(1-(-1-z)^2) + t(1-(1-z)^2).$$

It can be checked that the value of this problem is $4t^2 - 4t + 1$, which is precisely $1 - var(\theta)$. Now imagine selling to this agent the right to trade. If he does not have the right to trade, set $a = 0$ and $v(a,t) = 0$. If he has the right, set $a = 1$ and $v(a,t) = 4t^2 - 4t + 1$. Note that $v(1,t)$ is nonmonotone. ▲

An *allocation function* is a function $q : [0,1] \to A$ mapping types into alternatives. A *mechanism* is a pair (q,x) where q is an allocation function and $x : [0,1] \to \Re$ determines monetary transfers. An allocation function $q(\cdot)$ is *implementable* if there exists $x(\cdot)$ such that the agent can not gain by misreporting his type to the mechanism (q,x), i.e., $v(q(t),t) - x(t) \geq v(q(s),t) - x(s)$ for every (s,t). If this is the case, the mechanism (q,x) is said to be *incentive compatible*, or payments x are said to implement q.[4]

3 Implementability

The literature contains several related results which establish sufficient conditions for implementation of allocation functions. A typical such result indicates:

Proposition 1 *An allocation function $q(\cdot)$ is implementable if there exists a binary relation $\preceq$ on A such that*

1. *$s \leq t$ implies $q(s) \preceq q(t)$, and*
2. *$a' \preceq a$ implies $v_2(a',t) \leq v_2(a,t)$ for every t.*

Note that a binary relation on A satisfying the conditions of Proposition 1 is necessarily non-empty. Furthermore its restriction to the image of q is complete and transitive. The first condition orders A in such a way that $q(\cdot)$ is monotone, and the second condition requires that $v(\cdot,\cdot)$ satisfy monotone differences with respect to this order on A and the usual less-than-or-equal-to order on $[0,1]$.

Note on the literature: Proposition 1 has many precedents in various related environments. Earlier results such as Lemma 2 of Crémer

[4] The focus here is on incentive compatibility. If an allocation function is implementable, then it is part of an incentive compatible and individually rational mechanism under a rather mild condition. See Proposition 3 in the next section.

and McLean (1985) and Proposition 1 in Rochet (1987) present implementation results for arbitrary allocation functions when A is an interval and comes readily ordered by the less-than-or-equal-to relation and v is continuous in a. More recent results such as Proposition 3 in Bergemann and Välimäki (2002) and Theorem 5.1 in Jehiel and Moldovanu (2001) focus on efficient allocations in more abstract social choice problems, where an order on A needs to be constructed in order to attain monotonicity and monotone differences.[5] Proposition 1 above can be seen as an extension of the earlier results to arbitrary sets of alternatives, or as an extension of the more recent results that eliminates the condition of efficiency.

The payoff environment of Example 1 indicates that the sufficient conditions of Proposition 1 may be too strong.

Example 1 continued: Recall that $A = \{0,1\}$, $v(0,t) = 0$ and $v(1,t) = 4t^2 - 4t + 1$. Note that $v_2(0,t) = 0$ and $v_2(1,t) = 8t - 4$. Clearly no nontrivial order exists on A satisfying the second condition of Proposition 1 as $v_2(0,t)$ and $v_2(1,t)$ can not be ranked independently of t. ▲

The sufficient condition of Proposition 1 is not necessary for implementability and I will present an example to this effect in Section 4. (See Observation 5 therein.) This raises the question of whether there exists a tractable sufficient *and* necessary condition for implementability. In the following result the integrals are Lebesgue.

Proposition 2 *The following statements are equivalent:*

1. *The allocation function $q(\cdot)$ satisfies*

$$\int_s^t v_2(q(y), y)dy \leq v(q(t), t) - v(q(t), s) \text{ for every } (s,t). \tag{1}$$

2. *The mechanism $(q, \hat{x}_q)$ is incentive compatible where*

$$\hat{x}_q(t) = v(q(t), t) - \int_0^t v_2(q(y), y)dy \text{ for every } t. \tag{2}$$

3. *The allocation function $q(\cdot)$ is implementable.*

[5]In general, as the multiagent efficient allocation function solves an ex post maximization problem parametrized by different realizations of type vectors, the techniques of monotone comparative statics can be employed to determine conditions under which it is monotone. Jehiel and Moldovanu (2001) work with such conditions rather than assuming monotonicity directly.

The implementability condition (1) may look unappealing, but it is a suitable generalization of well known monotonicity conditions.[6] For example if $A = [0,1]$ and $v(q,t) = qt$, then (1) is equivalent to the condition that $q(\cdot)$ is nondecreasing. Furthermore, (1) gives rise to the usual revenue equivalence result: statements 2 and 3 of the proposition are equivalent. Suppose that some payment function x implements q and define $\hat{x}_q$ as in (2). Using the assumption of absolute continuity and differentiability, we can apply an envelope result (Milgrom and Segal [2001]) and deduce that $\hat{x}_q(s) - x(s) = \hat{x}_q(t) - x(t)$ for every (s,t). Hence $(q, \hat{x}_q)$ is incentive compatible as well. On the other hand if $(q, \hat{x}_q)$ is not incentive compatible, then q is not implementable. For if q is implemented by some x, then, again by an envelope argument, x and $\hat{x}_q$ can only differ by a constant and $(q, \hat{x}_q)$ would have to be incentive compatible. Thus, for implementation purposes, restricting attention to implementability by $\hat{x}_q$ is without loss of generality. In what follows, I will use the implementability of an allocation function q and the incentive compatibility of the particular mechanism $(q, \hat{x}_q)$ interchangeably.

In order to economize on space, I introduce the following expressions. For any allocation function q, and any pair (s,t) of types,

$$\begin{aligned} L_q(s,t) &:= \int_s^t v_2(q(y), y)dy, \text{ and} \\ R_q(s,t) &:= v(q(t),t) - v(q(t),s) \end{aligned}$$

are the left- and right-hand sides of (1). Note that $L_q(s,t) = -L_q(t,s)$ and (1) is equivalent to the statement

$$L_q(t,s) \geq -R_q(s,t) \text{ for all } (s,t).$$

Now, to make sense of (1), suppose that for some allocation function q, the mechanism $(q, \hat{x}_q)$ is **not** incentive compatible. Then there is a pair (s,t) such that type s is better off reporting type t, i.e.,

$$\begin{aligned} & v(q(s),s) - v(q(t),s) < \hat{x}_q(s) - \hat{x}_q(t) \\ & \Leftrightarrow \\ & v(q(s),s) - v(q(t),s) < v(q(s),s) - L_q(0,s) - v(q(t),t) + L_q(0,t) \\ & \Leftrightarrow \\ & R_q(s,t) < L_q(s,t). \end{aligned}$$

This is precisely the converse of the implementability condition (1).

[6]Condition (1) appears in Mookherjee and Reichelstein (1992) in an abstract social choice model, in Branco (1996) in a multiunit auction problem and in Ülkü (forthcoming) in a combinatorial auction problem.

2-cycle monotonicity An allocation function $q : [0,1] \to A$ satisfies *2-cycle monotonicity* (or *weak monotonicity*) if $v(q(t), s) - v(q(s), s) \leq v(q(t), t) - v(q(s), t)$ for every (s, t). Vohra (2011; Theorem 4.2.5) shows that if $A \subseteq \Re$ and if v satisfies the strict monotone differences condition $v_2(a', t) < v_2(a, t)$ for every t and every $a' < a$, then 2-cycle monotonicity characterizes implementability. In my model, the monotone differences condition need not hold. Consequently, 2-cycle monotonicity does not imply implementation and Section 4.2 gives an example of an allocation function which satisfies 2-cycle monotonicity but fails the implementability condition (1). However, 2-cycle monotonicity remains a necessary condition for implementation. To see this simply, note that if q satisfies (1), then for every (s, t)

$$-R_q(t, s) \leq L_q(s, t) \leq R_q(s, t)$$

giving $v(q(s), t) - v(q(s), s) \leq v(q(t), t) - v(q(t), s)$, which is equivalent to 2-cycle monotonicity.[7]

Individual rationality An important consideration in mechanism design is participation, i.e., the issue of whether the truth-telling equilibrium of a mechanism gives the agent a payoff at least as large as his outside option. In standard models, the outside option is taken to be independent of the agent's type, and it is normalized to zero. Suppose this is the case. I will say that an allocation function q is *individually rationally implementable* if there exists a payment rule $x : T \to \Re$ such that for every (s, t), $v(q(t), t) - x(t) \geq \max\{0, v(q(s), t) - x(s)\}$. In general an implementable q is not necessarily individually rationally implementable.

Proposition 3 *Suppose that an allocation function q is implementable. Then it is individually rationally implementable if $R_q(s, 0) \leq 0$ for all s.*

It is common in models of mechanism design to assume that values are increasing in types. Under this assumption the sufficient condition in Proposition 3 trivially follows: for all s, $0 \leq v(q(0), s) - v(q(0), 0) = -R_q(s, 0)$. In problems where v is not monotone, Proposition 3 indicates that an incentive compatible mechanism $(q, \hat{x}_q)$ is also individually rational if all types of the agent receive the same payoff from the alternative chosen for the smallest type. The optimal mechanisms in the environment of the next section will satisfy this condition.

[7] Another way to see this is to recall that implementability $\Leftrightarrow$ cycle monotonicity $\Rightarrow$ 2-cycle monotonicity.

4 Application: Optimal mechanism design

In this section, I will find the optimal mechanism for a seller in a simple parametrized environment which introduces habit formation in an otherwise standard framework. The optimal mechanism is, of course, incentive compatible and in order to solve for it, I will first need to identify implementable allocation functions. The environment fails monotone differences for a large set of parameter values. Hence in order to characterize incentives for these parameters, Proposition 1 is not of much help. (To be precise, Proposition 1 only identifies constant allocation functions as implementable at these parameter values.) I will show that for a subset of these parameter values, the optimal mechanism is nonmonotone. This means that Proposition 1 would not identify the optimal allocation function as implementable and highlights the utility of Proposition 2 as it applies in characterizing incentives for all parameter values.

Let $A = \{0, 1\}$ be the set of allocations and

$$v(a, t) = at - \beta a t^2$$

give the agent's values for every $(a, t) \in \{0, 1\} \times [0, 1]$. The number β is common knowledge and it parametrizes the environment. The interpretation is as follows. The agent is purchasing an indivisible object from a seller. The allocation $a = 1$ corresponds to sale and $a = 0$ indicates no sale. The payment x takes place at time zero. Consumption takes place at time 1 but it generates utilities in two periods: a at time 1 and $-\beta a$ at time 2. The agent's type t is his discount rate and his value $v(a, t)$ is the discounted sum of utilities. Thus if $\beta \neq 0$, the problem exhibits intertemporal allocation externalities in the form of habit formation: consumption at time 1 generates a disutility of $-\beta$ at time 2. I will refer to β as the habit parameter and maintain the following assumption.

Assumption: $\beta \in (0, 1)$.

Note that v is continuously differentiable (and therefore absolutely continuous) and nonmonotone in t. In particular, if $\beta > \frac{1}{2}$, then there is no nontrivial order on A that gives monotone differences since $v_2(0, t) = 0$ can be less than, equal to, or greater than $v_2(1, t) = 1 - 2\beta t$ depending on the value of t. Hence, in order to guarantee monotone differences, one is restricted to use a binary relation which puts $0 \sim 0$ and/or $1 \sim 1$ only.

The problem of optimal mechanism design is that of finding an incentive compatible and individually rational mechanism (q, x) such that the expectation of $x(\cdot)$ is at a maximum. The formulation and solution of this problem require a statistical distribution assumption on t. To keep matters as simple as possible I put

Assumption: t is uniform on $[0, 1]$.

Suppose that the seller has no costs and the buyer's outside option is zero regardless of his type so that the optimal mechanism design problem is:

$$\max_{(q,x)} \int_0^1 x(y)dy$$
$$\text{s.t.} \begin{cases} q(s)(s - \beta s^2) - x(s) \geq q(t)(s - \beta s^2) - x(t) \text{ for all } (s, t), \\ q(s)(s - \beta s^2) - x(s) \geq 0 \text{ for all } s. \end{cases}$$

The next proposition identifies the optimal mechanism for every β.

Proposition 4 *The seller's optimal mechanism* $(q^*, \hat{x}_{q^*})$ *is such that*

$$q^*(t) = 1 \textit{ if and only if } \underline{\tau}(\beta) \leq t \leq \overline{\tau}(\beta)$$

where the cutoff types are given by

$$[\underline{\tau}(\beta), \overline{\tau}(\beta)] = \begin{cases} \left[\frac{1+\beta-\sqrt{\beta^2-\beta+1}}{3\beta}, 1\right] & \textit{if} \quad 0 < \beta \leq \frac{5+\sqrt{5}}{10} \\ \left[\frac{1-\beta}{\beta}, 1\right] & \textit{if} \quad \frac{5+\sqrt{5}}{10} < \beta \leq \frac{3+\sqrt{3}}{6} \\ \left[\frac{3-\sqrt{3}}{6\beta}, \frac{3+\sqrt{3}}{6\beta}\right] & \textit{if} \quad \frac{3+\sqrt{3}}{6} < \beta < 1 \end{cases}$$

and the payments $\hat{x}_{q^*}$ *are defined as in (2).*

The following picture is meant to clarify the optimal mechanisms.[8] For every habit parameter β, measured on the horizontal axis, the corresponding vertical slice of the shaded area gives the set of types, measured on the vertical axis, that are optimally allocated the object. This slice is given for β_0 in the figure. A pair (β, t) is outside the shaded region if and only if the optimal mechanism of Proposition 4 does not allocate the object to type t when the habit parameter is β. Note that if $\beta > (3 + \sqrt{3})/6$, then the optimal mechanism is nonmonotone: low *and* high types are screened out.

[8] The boundaries $\underline{\tau}(\beta)$ and $\overline{\tau}(\beta)$ of the shaded region are plotted as constant slope functions even though they clearly are not, with the exception $\overline{\tau}(\beta) = 1$ if $0 < \beta < \frac{3+\sqrt{3}}{6}$. However the slopes do not change significantly in the domain $(0, 1)$. Thus, even though the figure lacks mathematical precision, it gives the right idea about how optimal mechanisms change as a function of β.

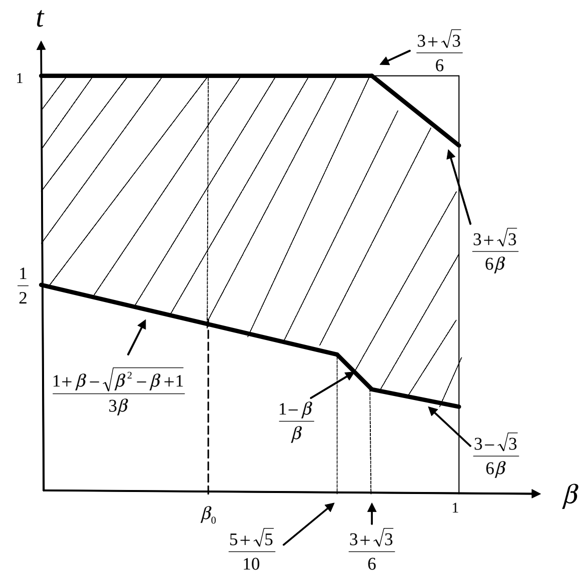

FIGURE 1: OPTIMAL MECHANISMS IN PROPOSITION 4

I follow with several observations regarding the optimal mechanisms.

Observation 1: Since the lowest type $t = 0$ is screened out for every habit parameter value, i.e., $q^*(0) = 0$ for every β, it follows that $v(q^*(0), t) = 0$ for all t. Now by Proposition 3, $(q^*, \hat{x}_{q^*})$ is individually rational. In other words, the individual rationality constraint of the optimal mechanism design problem is not binding, despite the non-monotonicity of v in t.

Observation 2: Using L'Hospital's rule gives $\lim_{\beta\to 0} \underline{\tau}(\beta) = \frac{1}{2}$, which is precisely the optimal cutoff type that optimally receives the object in the standard model with no habit formation under the uniform distribution assumption.

Observation 3: Let us define the agent's *virtual* valuation by

$$\begin{aligned} u(a,t) &= v(a,t) - (1-t)v_2(a,t) \\ &= a\left[(t-\beta t^2) - (1-t)(1-2\beta t)\right] \\ &= a(-3\beta t^2 + 2(1+\beta)t - 1). \end{aligned}$$

The maximization of $u(a,t)$ yields

$$\bar{q}(t) = \begin{cases} 1 & \text{if} \quad \frac{1+\beta-\sqrt{\beta^2-\beta+1}}{3\beta} \le t \le 1, \text{ and} \\ 0 & \text{if} \quad 0 \le t < \frac{1+\beta-\sqrt{\beta^2-\beta+1}}{3\beta}. \end{cases} \tag{3}$$

The methods of Myerson (1981) can be used to show that if $\bar{q}$ is individually rationally implementable, then it is part of an optimal mechanism.

Observation 4: Suppose that $\beta \in (0, \frac{1}{2}]$ so that values satisfy monotone differences when A is ordered by $0 \preceq 1$: $v_2(0,t) = 0 \leq 1 - 2\beta = v_2(1,t)$ for all t. Now Proposition 1 indicates that all monotone allocation functions are implementable. Consequently $\bar{q}$ is part of the optimal mechanism as it is monotone.

Observation 5: If $\frac{1}{2} < \beta$, then the monotone differences property fails. Yet, as the proof of the proposition indicates, if $\beta \in (\frac{1}{2}, \frac{5+\sqrt{5}}{10}]$, the allocation function $\bar{q}$ in (3) nevertheless satisfies the implementability condition (1) and is part of an optimal mechanism. In other words, standard methods can yield a monotone optimal mechanism even in the absence of monotone differences.

Observation 6: If $\frac{5+\sqrt{5}}{10} < \beta$, on the other hand, the allocation function $\bar{q}$ in (3) fails the implementability condition (1). To see this note that

$$L_{\bar{q}}(1,0) = \underline{\tau}(\beta) - \beta\underline{\tau}^2(\beta) - 1 + \beta > 0 = R_{\bar{q}}(1,0),$$

in other words, type $s = 1$ has a strict incentive to report $t = 0$ to the mechanism $(\bar{q}, \hat{x}_{\bar{q}})$. The optimal mechanism for these habit parameters can be solved for using the observation that there is a one-to-one relationship between implementable allocation functions and allocation functions supported by take-it-or-leave-it offers at some price. Solving for the optimal take-it-or-leave-it price, then, gives rise to an equivalent direct revelation mechanism which is optimal. If β is high enough, namely if $\frac{3+\sqrt{3}}{6} < \beta$, then the optimal mechanism is non-monotone, low *and* high values of t are screened out by the seller. This is intuitive: high types discount future habit cost at a higher rate, leading to a lower overall value for the allocation. If this cost is large enough, then the seller is better off excluding high types in favor of types with higher discounted values.

4.1 An extension

In this subsection I will discuss an extension of the model with habits that enriches the set of alternatives. As before, the optimal mechanism can be constructed despite the failure of monotone differences for a large set of parameters. However, the optimal mechanism is monotone for every parameter.

Suppose that the seller can provide the object to the agent at times 1 and 2, as opposed to only at time 1. Let $A = \{(0,0), (0,1), (1,0), (1,1)\}$

where an alternative $a = (a_1, a_2) \in A$ determines the quantity a_1 allocated to the agent at time 1 and a_2 allocated at time 2. Consider the valuation

$$v(a,t) = a_1 t + (a_2 - \beta a_1)t^2$$

where, as before, $\beta \in (0,1)$ is the commonly known habit parameter and t is the agent's privately known discount factor which is uniform on $[0,1]$. As before a_1 generates utilities in two periods. For simplicity, I focus on a two period problem where no habits are formed based on the consumption in the last period.

To proceed with Myerson's methods, note that the associated virtual utility is

$$u(a,t) = a_1[-3\beta t^2 + (2+2\beta)t + 1] + a_2[3t^2 - 2t].$$

Ignoring the incentive constraints and solving $\max_{a \in A} u(a,t)$ at every t yields the allocation

$$\bar{q}(t) = \begin{cases} (0,0) & \text{if } t < r(\beta) \\ (1,0) & \text{if } r(\beta) \leq t < \frac{2}{3} \\ (1,1) & \text{if } \frac{2}{3} \leq t \leq 1 \end{cases}$$

where

$$r(\beta) = \frac{\beta + 1 - \sqrt{\beta^2 - \beta + 1}}{3\beta}$$

is the small root of $-3\beta t^2 + (2+2\beta)t + 1$, the coefficient of a_1 in $u(a,t)$. Note that $r : (0,1) \to \Re$ is a monotone decreasing function with $\lim_{\beta \to 0} r(\beta) = \frac{1}{2}$ and $\lim_{\beta \to 1} r(\beta) = \frac{1}{3}$. In particular, $r(\beta) < \frac{1}{2\beta}$ for all β.

As usual, $\bar{q}$ is part of the optimal mechanism if it is individually rationally implementable. Note that $\bar{q}(0) = (0,0)$ and $v(\bar{q}(0),t) = 0$ for all t. Thus I need only check for the implementability of $\bar{q}$ as individual rationality follows automatically from incentive compatibility by Proposition 3.

If $\bar{q}$ is to be monotone with respect to an order $\prec$ on A, it must be that $(0,0) \prec (1,0) \prec (1,1)$. (Note that this binary relation is not complete.) To check whether or not the monotone differences property is satisfied with respect to this binary relation, note that

$$\begin{aligned} v_2((0,0),t) &= 0, \\ v_2((1,0),t) &= 1 - 2\beta t, \text{ and} \\ v_2((1,1),t) &= 1 - 2\beta t + 2t. \end{aligned}$$

Now if $\beta \leq \frac{1}{2}$, then for all t, $v_2((0,0),t) \leq v_2((1,0),t) \leq v_2((1,1),t)$. In other words, for all t, $v_2(\cdot,t)$ is nondecreasing on A when A is ordered by $\prec$. This gives us monotone differences, and, by Proposition 1, implementability of $\bar{q}$.

If $\beta > \frac{1}{2}$, however, it is no longer true that $v_2((0,0),t) \leq v_2((1,0),t)$ for all t. Yet, we can use Proposition 2 to conclude that $\bar{q}$ is implementable as follows.

Suppose that $\beta > \frac{1}{2}$ and take any (s,t). Denote $r(\beta)$ by r for brevity. Clearly if $\bar{q}(s) = \bar{q}(t)$, then $L_{\bar{q}}(s,t) = R_{\bar{q}}(s,t)$. Thus I need only consider pairs (s,t) such that $\bar{q}(s) \neq \bar{q}(t)$. There are six cases to consider. The first three correspond to $s < t$ and the last three to $t < s$.

Case 1: If $s < r \leq t < \frac{2}{3}$, then

$$\begin{aligned} L_{\bar{q}}(s,t) &= \int_r^t (1-2\beta y)dy = t - \beta t^2 - (r - \beta r^2), \\ R_{\bar{q}}(s,t) &= t - \beta t^2 - (s - \beta s^2). \end{aligned}$$

Since $r \leq \frac{1}{2\beta}$, the map $y \mapsto y - \beta y^2$ is increasing on $[0,r]$. Therefore $s - \beta s^2 \leq r - \beta r^2$. This gives $L_{\bar{q}}(s,t) \leq R_{\bar{q}}(s,t)$.

Case 2: If $s < r$ and $\frac{2}{3} \leq t$, then

$$\begin{aligned} L_{\bar{q}}(s,t) &= \int_r^{\frac{2}{3}} (1-2\beta y)dy + \int_{\frac{2}{3}}^t (1-2\beta y+2y)dy = t - \beta t^2 - (r - \beta r^2) + t^2 - \frac{4}{9}, \\ R_{\bar{q}}(s,t) &= t + (1-\beta)t^2 - (s + (1-\beta)s^2). \end{aligned}$$

To establish $L_{\bar{q}}(s,t) \leq R_{\bar{q}}(s,t)$, I need only show $s - \beta s^2 + s^2 \leq r - \beta r^2 + \frac{4}{9}$. This follows since, as in Case 1, $s - \beta s^2 \leq r - \beta r^2$ and $s^2 \leq \frac{4}{9}$.

Case 3: If $r \leq s < \frac{2}{3} \leq t$, then

$$\begin{aligned} L_{\bar{q}}(s,t) &= \int_s^{\frac{2}{3}} (1-2\beta y)dy + \int_{\frac{2}{3}}^t (1-2\beta y+2y)dy = t - \beta t^2 - (s - \beta s^2) + t^2 - \frac{4}{9}, \\ R_{\bar{q}}(s,t) &= t + (1-\beta)t^2 - (s + (1-\beta)s^2). \end{aligned}$$

Since $s^2 \leq \frac{4}{9}$, $L_{\bar{q}}(s,t) \leq R_{\bar{q}}(s,t)$.

Case 4: If $t < r \leq s < \frac{2}{3}$, then

$$\begin{aligned} L_{\bar{q}}(s,t) &= \int_s^r (1-2\beta y)dy = r - \beta r^2 - (s - \beta s^2), \\ R_{\bar{q}}(s,t) &= 0. \end{aligned}$$

I need to show, then, that $r-\beta r^2 \leq \min_{s\in[r,\frac{2}{3}]}(s-\beta s^2)$. Since $s \mapsto s-\beta s^2$ is a concave function, the minimum is achieved at an extreme point. Thus, all we need to show is that $r - \beta r^2 \leq \frac{2}{3} - \frac{4}{9}\beta$. Direct computation yields this result.

Case 5: If $t < r$ and $\frac{2}{3} < s$, then

$$L_{\bar{q}}(s,t) = \int_{\frac{2}{3}}^{r}(1-2\beta y)dy + \int_{s}^{\frac{2}{3}}(1-2\beta y+2y)dy = r - \beta r^2 - (s-\beta s^2) + \frac{4}{9} - s^2,$$
$$R_{\bar{q}}(s,t) = 0.$$

I need to show that $r-\beta r^2+\frac{4}{9} \leq s+(1-\beta)s^2$. Since $s \mapsto s+(1-\beta)s^2$ is increasing on $[0,\infty)$, I need $r-\beta r^2+\frac{4}{9} \leq \frac{2}{3}+(1-\beta)\frac{4}{9}$, or $r-\beta r^2 \leq \frac{2}{3}-\frac{4}{9}\beta$. As in case 5, this follows from direct computation.

Case 6: If $r \leq t < \frac{2}{3} \leq s$, then

$$L_{\bar{q}}(s,t) = \int_{\frac{2}{3}}^{t}(1-2\beta y)dy + \int_{s}^{\frac{2}{3}}(1-2\beta y+2y)dy = t - \beta t^2 - (s-\beta s^2) + \frac{4}{9} - s^2,$$
$$R_{\bar{q}}(s,t) = 0.$$

Direct computation yields $t - \beta t^2 + \frac{4}{9} \leq s + (1-\beta)s^2$, which gives the desired result.

Thus I conclude that $\bar{q}$ satisfies (1) and is part of the optimal mechanism.

4.2 2-cycle monotonicity is not enough for implementation

As I noted earlier, 2-cycle monotonicity is weaker than the implementation condition (1) in my model. Next I present an example of a weakly monotone allocation function which is not implementable.

Example 2: Take $\beta = \frac{2}{3}$ in the model of Section 4.1. For some positive but sufficiently small ε, consider the allocation function

$$(q_1(t), q_2(t)) = \begin{cases} (0,1) & \text{if } t \in [0,\varepsilon) \\ (1,0) & \text{if } t \in [\varepsilon, \frac{3}{5}-\varepsilon) \\ (1,1) & \text{if } t \in [\frac{3}{5}-\varepsilon, 1]. \end{cases}$$

Claim 1: q satisfies 2-cycle monotonicity, i.e., for all (s,t)

$$q_1(s)(t-s)+(q_2(s)-\frac{2}{3}q_1(s))(t^2-s^2) \leq q_1(t)(t-s)+(q_2(t)-\frac{2}{3}q_1(t))(t^2-s^2).$$

Let $l(s,t)$ and $r(s,t)$ denote the left and right hand sides of this inequality. It will also be convenient to define the functions

$$\begin{aligned} f(x) &= x - \frac{5}{3}x^2, \text{ and} \\ h(x) &= x - \frac{2}{3}x^2. \end{aligned}$$

Note that $f(\varepsilon) = f(\frac{3}{5} - \varepsilon)$.

To prove the claim, I consider the following cases.

Case 1: Suppose that $s < \varepsilon \leq t < \frac{3}{5} - \varepsilon$. Then $q(s) = (0,1)$, $q(t) = (1,0)$ and therefore

$$\begin{aligned} l(s,t) &= t^2 - s^2, \text{ and} \\ r(s,t) &= t - s - \frac{2}{3}(t^2 - s^2). \end{aligned}$$

We have $l(s,t) \leq r(s,t)$ iff $f(s) \leq f(t)$. This follows because $f(s) < f(\varepsilon) \leq f(t)$.

Case 2: Suppoe that $s < \varepsilon$ and $\frac{3}{5} - \varepsilon \leq t$. Then $q(s) = (0,1)$, $q(t) = (1,1)$ and therefore

$$\begin{aligned} l(s,t) &= t^2 - s^2, \text{ and} \\ r(s,t) &= t - s + \frac{1}{3}(t^2 - s^2). \end{aligned}$$

Consequently $l(s,t) \leq r(s,t)$ iff $h(s) \leq h(t)$. This follows because $\sup_{s\in[0,\varepsilon)} h(s) = h(\varepsilon) < h(1) = \inf_{t\in[\frac{3}{5}-\varepsilon,1]} h(t)$.

Case 3: Suppose that $\varepsilon \leq s < \frac{3}{5} - \varepsilon \leq t$. Then $q(s) = (1,0)$, $q(t) = (1,1)$ and therefore

$$\begin{aligned} l(s,t) &= t - s - \frac{2}{3}(t^2 - s^2), \text{ and} \\ r(s,t) &= t - s + \frac{1}{3}(t^2 - s^2). \end{aligned}$$

Now $l(s,t) \leq r(s,t)$ iff $s^2 \leq t^2$ which follows since $s < t$.

Case 4: Suppose that $t < \varepsilon \leq s < \frac{3}{5} - \varepsilon$. Then $q(s) = (1,0)$, $q(t) = (0,1)$ and therefore

$$\begin{aligned} l(s,t) &= t - s - \frac{2}{3}(t^2 - s^2), \text{ and} \\ r(s,t) &= t^2 - s^2. \end{aligned}$$

Now $l(s,t) \leq r(s,t)$ iff $f(t) \leq f(s)$. This follows since $\sup_{t\in[0,\varepsilon)} f(t) = f(\varepsilon) = \inf_{s\in[\varepsilon,\frac{3}{5}-\varepsilon)} f(s)$.

Case 5: Suppose that $t < \varepsilon$ and $\frac{3}{5} - \varepsilon \leq s$. Then $q(s) = (1,1)$, $q(t) = (0,1)$ and therefore

$$\begin{aligned} l(s,t) &= t - s + \frac{1}{3}(t^2 - s^2), \text{ and} \\ r(s,t) &= t^2 - s^2. \end{aligned}$$

Now $l(s,t) \leq r(s,t)$ iff $h(t) \leq h(s)$, which follows since $\sup_{t\in[0,\varepsilon)} h(t) = h(\varepsilon) < h(1) = \inf_{s\in[\frac{3}{5}-\varepsilon,1)} h(s)$.

Case 6: Suppose that $\varepsilon \leq t < \frac{3}{5} - \varepsilon \leq s$. Then $q(s) = (1,1)$, $q(t) = (1,0)$ and therefore

$$\begin{aligned} l(s,t) &= t - s + \frac{1}{3}(t^2 - s^2), \text{ and} \\ r(s,t) &= t - s - \frac{2}{3}(t^2 - s^2). \end{aligned}$$

It follows that $l(s,t) \leq r(s,t)$ as $t^2 \leq s^2$.

This shows that q satisfies 2-cycle monotonicity.

Claim 2: q fails the implementability condition, i.e., for some (s,t),

$$\int_s^t [q_1(y) + 2(q_2(y) - \frac{2}{3}q_1(y))y]dy > r(s,t).$$

To see this, take $\varepsilon = \frac{1}{20}$, any $t < \varepsilon$ and $s = 1$, then

$$\begin{aligned} &\int_1^t [q_1(y) + 2(q_2(y) - \frac{2}{3}q_1(y))y]dy \\ &= \int_1^{\frac{3}{5}-\frac{1}{20}} [1 + \frac{2}{3}y]dy + \int_{\frac{3}{5}-\frac{1}{20}}^{\frac{1}{20}} [1 - \frac{4}{3}y]dy + \int_{\frac{1}{20}}^t 2ydy \\ &= t^2 - \frac{197}{200} \\ &> t^2 - 1 \\ &= r(s,t). \end{aligned}$$

5 Conclusion

In this paper, I give a characterization of incentive compatibility that is especially useful when values are not monotone in types and put it to use in a buyer-seller example featuring habit formation. The characterization builds on the usual envelope condition and is independent of monotonicity and monotone differences conditions commonly assumed in the literature. The characterization result can be used to establish individual rationality of mechanisms in many applications under a mild condition. It is also useful, as in the parametrized example of Section 4, to show that standard methods of mechanism design may apply when values fail monotone differences and that the resulting optimal mechanisms can be nonmonotone.

There exist other interesting nonmonotone environments one may wish to consider. I will finish by describing one of them. Suppose that the agent's value for an object is some unobservable state θ which takes values in $[0,1]$. The type $t \in [0,1]$ is an informative signal regarding θ. Consider the following joint density for (θ, t):

$$f(\theta, t) = \begin{cases} 2 & \text{if } (\theta, t) \in [0,1]^2 \text{ and } \theta \geq \max\{1-2t, 2t-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

The signal technology can be interpreted as follows. The conditional density $f(t|\theta)$ is uniform on $[\frac{1-\theta}{2}, \frac{1+\theta}{2}]$. Thus higher values of the state induce less precise signals while keeping the conditional expectation of the signal always the same at $\frac{1}{2}$.

Now note that the conditional density $f(\theta|t)$ is uniform on $[1-2t, 1]$ if $t \in [0, \frac{1}{2}]$ and it is uniform on $[2t-1, 1]$ if $t \in [\frac{1}{2}, 1]$. Consequently the agent's conditional expectation of state θ given his signal t is

$$E[\theta|t] = \begin{cases} 1-t & \text{if } t \in [0, \frac{1}{2}] \\ t & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

Now we can take the agent's value for the object at type t to be $v(t) = E[\theta|t]$, which is nonmonotone. Note that v not differentiable at $\frac{1}{2}$, but it is convex. Using the envelope result in Krishna and Maenner (2001, Proposition 1), the implementability condition (1) characterizes implementability.

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Appendix 1: Proofs

Proof of Proposition 1 Using Proposition 2 -whose proof is to follow- I need only show that if there is an order $\preceq$ on A such that (i) $s \leq t$ implies $q(s) \preceq q(t)$, and (ii) $a' \preceq a$ implies $v_2(a', t) \leq v_2(a, t)$ for every t, then the implementability condition (1) follows. Suppose such $\preceq$ exists and take $s < t$. Then

$$\int_s^t v_2(q(y), y)dy \leq \int_s^t v_2(q(t), y)dy = v(q(t), t) - v(q(t), s).$$

Similarly if $t < s$. Thus q satisfies (1) and the proof is complete. ■

Proof of Proposition 2 (1 $\Rightarrow$ 2) Suppose that q satisfies (1). Then the mechanism $(q, \hat{x}_q)$ is incentive compatible as for every (s, t)

$$\begin{aligned}\hat{x}_q(t) - \hat{x}_q(s) &= v(q(t), t) - v(q(s), s) - \int_s^t v_2(q(y), y)dy \\ &\geq v(q(t), s) - v(q(s), s).\end{aligned}$$

(2 $\Rightarrow$ 3) This trivially follows from definitions.

(3 $\Rightarrow$ 1) To see that (1) is necessary for implementation, suppose that for some payment function x, (q, x) is incentive compatible. Since v is absolutely continuous and differentiable in types, the envelope theorem of Milgrom and Segal (2001) applies and

$$v(q(t), t) - x(t) = v(q(s), s) - x(s) + \int_s^t v_2(q(y), y)dy$$

for every (s, t). Rearranging and using incentive compatibility,

$$\begin{aligned}v(q(t), t) &= v(q(s), s) - x(s) + x(t) + \int_s^t v_2(q(y), y)dy \\ &\geq v(q(t), s) + \int_s^t v_2(q(y), y)dy\end{aligned}$$

and (1) follows. ■

Proof of Proposition 3 Suppose that q is implementable and $v(q(0), t) \geq v(q(0), 0)$ for all t. Since $(q, \hat{x}_q)$ is incentive compatible (Proposition 2), it follows that

$$v(q(t), t) - \hat{x}_q(t) = L_q(0, t) \geq -R_q(t, 0) \geq 0$$

where the equality follows from the definition of payments $\hat{x}_q$ in (2), the first inequality follows from the implementation condition (1) and the second from hypothesis. ■

Proof of Proposition 4 I will show that q^* defined in the proposition is implementable in three parts depending on the value of the habit parameter β. If q^* is implementable, then $(q^*, \hat{x}_{q^*})$ is incentive compatible by Proposition 2 and individually rational by Proposition 3 since $q^*(0) = 0$ for all β. Thus $(q^*, \hat{x}_{q^*})$ is feasible in the optimal mechanism design problem. Optimality of $(q^*, \hat{x}_{q^*})$ will follow from the observation that q^* solves the virtual utility problem (i.e., it coincides with $\bar{q}$ in (3) in Parts 1 and 2. In Part 3, when β is sufficiently high, $q^* \neq \bar{q}$ and I establish the optimality of q^* by directly solving an optimal pricing problem.

Part 1 Suppose that $\beta \in (0, 1/2]$. As noted in the main text, monotone differences property is satisfied when $0 \preceq 1$, as $v_2(0,t) = 0 \leq 1 - 2\beta t = v_2(1,t)$ for all t. Now the allocation function (3) defined in the main text is nondecreasing and therefore implementable by Proposition 1. To see that $(q^*, \hat{x}_{q^*})$ is feasible in the optimal mechanism design problem, note that the allocation functions in Proposition 3 and in equation (3) coincide for this parameter range. Thus q^* maximizes virtual utility. Optimality now follows from the definition (2) of payments $\hat{x}_{q^*}$.

Part 2 Suppose that $\beta \in (1/2, (5+\sqrt{5})/10]$. Note that q^* maximizes virtual utility for this parameter range as well. Hence q^* is part of an optimal mechanism if it is implementable. I will show that q^* satisfies (1) and is therefore implementable.

To begin, note that

$$0 < \underline{\tau}(\beta) < 1/2\beta < 1 < 1/\beta.$$

In order to apply the characterization result, first take (s,t) such that $s < \underline{\tau}(\beta) \leq t$. It follows that

$$\begin{aligned} L_{q^*}(s,t) &= \int_{\underline{\tau}(\beta)}^{t} (1 - 2\beta y)dy \\ &= t - \underline{\tau}(\beta) - \beta t^2 + \beta[\underline{\tau}(\beta)]^2 \\ &< t - s - \beta t^2 + \beta s^2 \\ &= R_{q^*}(s,t) \end{aligned}$$

where the inequality follows because $s < \underline{\tau}(\beta) < -1/2\beta$.

Now take (s,t) such that $t < \tau(\beta) \leq s$ so that

$$\begin{aligned} L_{q^*}(s,t) &= \int_{s}^{\underline{\tau}(\beta)} (1 - 2\beta y)dy \\ &= \underline{\tau}(\beta) - s - \beta[\underline{\tau}(\beta)]^2 + \beta s^2 \\ &\leq 0 \\ &= R_{q^*}(s,t). \end{aligned}$$

The weak inequality follows from the following observations: let $f(y) = y - \beta y^2$ on $[0,1]$. I need to show $f(\underline{\tau}(\beta)) \leq f(s)$. Note that f is concave and maximized at $1/2\beta$. Hence if $\underline{\tau}(\beta) \leq s \leq 1/2\beta$, the inequality follows trivially. Otherwise, note that

$$\min_{\frac{1}{2\beta} < y \leq 1} f(y) = f(1) = 1 - \beta.$$

Thus, it suffices to show that $f(\underline{\tau}(\beta)) \leq f(1)$. But this follows as $\beta \leq (5+\sqrt{5})/10$. Thus q^* satisfies (1) and is therefore implementable.

Part 3 If $\beta > (5+\sqrt{5})/10$, then the allocation function in equation (3) is no longer implementable. I will derive the optimal mechanism by solving an optimal pricing problem. I need some preparations.

Let $p \geq 0$ be a price for the object. If the seller makes a take-it-or-leave-it offer at p, then any type whose value weakly exceeds p will purchase the object. (See Figures 2 and 3.) Hence each p gives rise to an allocation function q^p defined by

$$q^p(t) = \begin{cases} 1 & \text{if} \quad t - \beta t^2 \geq p, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, if $p > \frac{1}{4\beta} = \max_t t - \beta t^2$, then $q^p(t) = 0$ for all p. Such prices earn the seller a revenue of zero. I will concentrate on prices $p \in [0, \frac{1}{4\beta}]$. In this case q^p becomes

$$q^p(t) = \begin{cases} 1 & \text{if} \quad \frac{1-\sqrt{1-4\beta p}}{2\beta} \leq t \leq \frac{1+\sqrt{1-4\beta p}}{2\beta}, \\ 0 & \text{otherwise.} \end{cases}$$

Calculating the payments $\hat{x}_{q^p}$ as in (2), I get

$$\hat{x}_{q^p}(t) = \begin{cases} p & \text{if} \quad \frac{1-\sqrt{1-4\beta p}}{2\beta} \leq t \leq \frac{1+\sqrt{1-4\beta p}}{2\beta}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the indirect mechanism of a take-it-or-leave-it price at p is identifiable with the direct revelation mechanism $(q^p, \hat{x}_{q^p})$. On the other hand, if a mechanism $(q, \hat{x}_q)$ is implementable, then it satisfies the following monotonicity property:

$$\text{if } q(s) = 1 \text{ and } t - \beta t^2 > s - \beta s^2 \text{ then } q(t) = 1. \tag{*}$$

For otherwise for some (s,t) such that $q(s) = 1$, $q(t) = 0$ and $t - \beta t^2 > s - \beta s^2$, we would have

$$L_q(s,t) \leq R_q(s,t) = 0$$

and

$$0 \leq -L_q(s,t) = L_q(t,s) \leq R_q(t,s) = s - \beta s^2 - t + \beta t^2 < 0,$$

a contradiction to implementability of q. The monotonicity property (*) in turn implies that the allocation function q is supported by a take-it-or-leave-it price at $p = t_0 + \beta t_0^2$ where $t_0 = \inf\{t : q(t) = 1\}$. But $t_0 + \beta t_0^2$ is precisely $\hat{x}_q(t)$ for any t such that $q(t) = 1$. In other words, any incentive compatible mechanism $(q, \hat{x}_q)$ is virtually a take-it-or-leave-it offer.

In order to solve for the optimal mechanism, then, I need only solve for the optimal price p for the seller and then convert it to the equivalent direct mechanism. In order to formulate the optimal pricing problem, first fix any habit parameter $\beta \in (\frac{\sqrt{5}+5}{10}, 1)$. Using the uniformity of t, for every price $p \geq 0$, the expected revenue becomes

$$\begin{aligned}\pi(p|\beta) &= p \Pr\{t - \beta t^2 \geq p\}\\ &= \begin{cases} p\left(1 - \frac{1-\sqrt{1-4\beta p}}{2\beta}\right) & \text{if} \quad 0 \leq p \leq 1-\beta, \\ \frac{p\sqrt{1-4\beta p}}{\beta} & \text{if} \quad 1-\beta < p \leq \frac{1}{4\beta}, \\ 0 & \text{if} \quad \frac{1}{4\beta} < p. \end{cases}\end{aligned}$$

Note that $0 < 1 - \beta < \frac{1}{4\beta}$ so the expected revenue function is well defined. Also note that $\pi(\cdot|\beta)$ is continuos. To show this I need only check continuity at the break points $p = 1 - \beta$ and $p = \frac{1}{4\beta}$. To check continuity at $p = 1 - \beta$ fix any β. Continuity from the left is obvious. Note

$$\begin{aligned}\lim_{p \downarrow 1-\beta} \pi(p|\beta) &= \lim_{p \downarrow 1-\beta} \frac{p\sqrt{1-4\beta p}}{\beta}\\ &= \frac{(1-\beta)\sqrt{1-4\beta(1-\beta)}}{\beta}\\ &= \frac{(1-\beta)(2\beta-1)}{\beta}\\ &= \pi(1-\beta|\beta).\end{aligned}$$

$\pi(p|\beta)$ is also continuous at $p = \frac{1}{4\beta}$ as $\pi(\frac{1}{4\beta}|\beta) = 0 = \pi(p|\beta)$ for all $p > \frac{1}{4\beta}$.

Now I will solve

$$\max_p \pi(p|\beta)$$

piece-by-piece, first on $[0, 1-\beta]$, and then on $(1-\beta, \frac{1}{4\beta}]$.

I claim that

$$\{1-\beta\} = \arg \max_{0 \leq p \leq 1-\beta} \pi(p|\beta).$$

To see this, first note that $\pi(0|\beta) = 0$. Moreover for every $p \in [0, 1-\beta]$, the first and second derivatives of expected revenue with respect to price are

$$\pi'(p|\beta) = 1 + \frac{1}{2\beta}\left(\sqrt{1-4p\beta} - 1\right) - \frac{p}{\sqrt{1-4p\beta}}$$
$$\pi''(p|\beta) = -2p\frac{\beta}{(1-4p\beta)^{\frac{3}{2}}} - \frac{2}{\sqrt{1-4p\beta}} = -2 < 0$$

Thus $\pi(\cdot|\beta)$ is strictly concave on $[0, 1-\beta]$. The first derivative evaluated at $p = 0$ is $\pi'(0|\beta) = 1 > 0$, and therefore the expected revenue is increasing around 0. Finally the first derivative evaluated at $p = 1-\beta$ is

$$\pi'(1-\beta|\beta) = \frac{\beta - 1}{\beta} - \frac{1-\beta}{2\beta - 1} + 1 > 0.$$

(In fact $\pi'(1-\beta|\beta) = 0$ if $\beta = \frac{\sqrt{5}+5}{10}$.) Thus $\pi(\cdot|\beta)$ is strictly increasing on $[0, 1-\beta]$ and is maximized at $p = 1-\beta$.

Next, I note that

$$\beta \in \left(\frac{\sqrt{3}+3}{6}, 1\right) \Rightarrow \arg \max_{1-\beta < p \leq \frac{1}{4\beta}} \pi(p|\beta) = \left\{\frac{1}{6\beta}\right\}, \text{ and}$$
$$\beta \in \left(\frac{\sqrt{5}+5}{10}, \frac{\sqrt{3}+3}{6}\right] \Rightarrow \pi(1-\beta|\beta) = \sup_{1-\beta < p \leq \frac{1}{4\beta}} \pi(p|\beta).$$

Consequently the expected revenue maximizing price is

$$p^*(\beta) = \begin{cases} \frac{1}{6\beta} & \text{if} \quad \frac{\sqrt{3}+3}{6} < \beta < 1, \text{ and} \\ -\beta + 1 & \text{if} \quad \frac{\sqrt{5}+5}{10} < \beta \leq \frac{\sqrt{3}+3}{6}. \end{cases}$$

The optimal price and the allocation function it supports is presented in the next two figures.

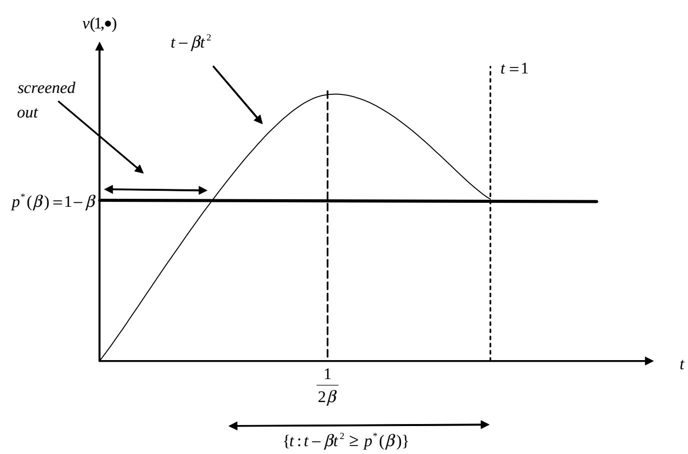

FIGURE 2: OPTIMAL PRICE when $\frac{\sqrt{5}+5}{10} < \beta \leq \frac{\sqrt{3}+3}{6}$

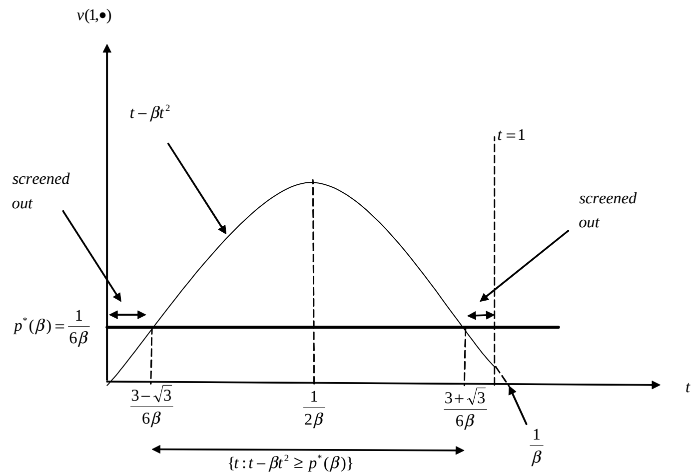

FIGURE 3: OPTIMAL PRICE when $\frac{\sqrt{3}+3}{6}<\beta<1$

All that remains to show is that the allocation function q^* in Proposition 3 is such that for every $\beta \geq \frac{5+\sqrt{5}}{10}$ by $q^*(t) = 1$ if and only if $t - \beta t^2 \geq p^*(\beta)$. I skip this straightforward step and the proof is complete. ■

Appendix 2: Extension to multiple agents with interdependent values

Consider the following standard multiagent model with interdependent values. Let A be a set of allocations and N be a set of agents, both finite. Endow each $i \in N$ with the type space $T_i = [\underline{t}_i, \bar{t}_i]$ and valuation function $v_i : A \times T \rightarrow \Re$ where $T = \times_{j \in N} T_j$. Assume that each v_i is continuously differentiable in t_i and denote the derivative by $v_i'(a, \cdot, t_{-i})$. An allocation function is a map $q : T \rightarrow A$. A mechanism is a list $(q, (x_i)_{i \in N})$ where $x_i : T \rightarrow \Re$ for every i.

A mechanism $(q, (x_i)_{i \in N})$ is *ex post Nash incentive compatible* if for every i, for every pair (s_i, t_i) of i's types, and for every type profile t_{-i} of the rest of the agents,

$$v_i(q(s_i, t_{-i}), t_i, t_{-i}) - x_i(s_i, t_{-i}) \leq v_i(q(t_i, t_{-i}), t_i, t_{-i}) - x_i(t_i, t_{-i}).$$

An allocation function q is *ex post Nash implementable* if for some $(x_i)_{i \in N}$, the mechanism $(q, (x_i)_{i \in N})$ is ex post Nash incentive compatible.

Proposition 5 (Extension of Proposition 1 to a multiagent environment) *An allocation function $q(\cdot)$ is ex post Nash implementable if for every i and t_{-i}, there exists a binary relation $\preceq_{t_{-i}}$ on A such that*

1. *$s_i \leq t_i$ implies $q(s_i, t_{-i}) \preceq_{t_{-i}} q(t_i, t_{-i})$ and*

2. *$a' \preceq_{t_{-i}} a$ implies $v_i'(a', t_i, t_{-i}) \leq v_i'(a, t_i, t_{-i})$ for every t_i.*

The proof uses Proposition 6 below and is identical to the proof of Proposition 1 except for the necessary changes in notation.

If we make the additional assumption that q is efficient, then Proposition 5 is an exact analog of Theorem 5.1 in Jehiel and Moldovanu (2001) and Proposition 3 in Bergemann and Välimäki (2002).

In the linear model of Jehiel and Moldovanu, $v_i(a,t) = \sum_{j \in N} \alpha_{ij}(a) t_j$ and monotone differences property takes the following form: for every i and t_{-i}, $a' \preceq_{t_{-i}} a \Rightarrow \alpha_{ii}(a') \leq \alpha_{ii}(a)$. Jehiel and Moldovanu start with an arbitrary unordered allocation set A and order it for every i and t_{-i} such that monotone differences property is satisfied. They are interested in implementing the efficient allocation rule $q^E(\cdot)$, which is obtained by choosing

$$q^E(t) \in \arg\max_{a \in A} \sum_{i \in N} \sum_{j \in N} \alpha_{ij}(a) t_j$$

for every t. The efficient allocation, of course, need not satisfy the monotonicity property. Jehiel and Moldovanu further impose the condition

$$\text{for all } i \text{ and } t_{-i},\ a' \preceq_{t_{-i}} a \Rightarrow \sum_{j \in N} \alpha_{ji}(a') < \sum_{j \in N} \alpha_{ji}(a).$$

If this condition is satisfied then for every i and t_{-i}, the map

$$(a, t_i) \mapsto \sum_{i \in N} \sum_{j \in N} \alpha_{ij}(a) t_j$$

satisfies the single crossing property. Consequently q^E is monotone and implementation follows.

The multiagent version of Proposition 2 is as follows.

Proposition 6 (Extension of Proposition 2 to a multiagent environment) *The following statements are equivalent:*

1. *The allocation function $q(\cdot)$ satisfies*

$$\int_{s_i}^{t_i} v_i(q(y, t_{-i}), y, t_{-i}) dy \leq v_i(q(t_i, t_{-i}), t_i, t_{-i}) - v_i(q(t_i, t_{-i}), s_i, t_{-i})$$

for every (i, t_{-i}, s_i, t_i).

2. *The mechanism* $(q, (\hat{x}_{iq})_{i \in N})$ *is ex post Nash incentive compatible where*

$$\hat{x}_{iq}(t_i, t_{-i}) = v_i(q(t_i, t_{-i}), t_i, t_{-i}) - \int_{\underline{t}_i}^{t_i} v_i(q(y, t_{-i}), y, t_{-i}) dy$$

for every (i, t_i, t_{-i}).

3. *The allocation function* $q(\cdot)$ *is ex post Nash implementable.*

The proof is identical to the proof of Proposition 2 except for the necessary changes in notation.