INSTITUTO TECONOLÓGICO AUTÓNOMO DE MÉXICO

CENTRO DE INVESTIGACIÓN ECONÓMICA

Working Papers Series

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July 2013
Working Paper 13-05

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Choosing two Finalists *and* the Winner[1]

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July 2013

Abstract: We study a class of boundedly rational choice functions, namely *two-stage choosers,* which operate as follows. The decision maker uses two criteria in two stages to make a choice. First she shortlists the top-2 alternatives, i.e. two finalists, according to one criterion. Next, she chooses the winner in this binary shortlist using the second criterion. The criteria are linear orders ranking the alternatives, and they may or may not be explicitly welfare-relevant. For example, they may reflect the distinct preferences of a short-run- and a long-run-self. Alternatively the first criterion may be a list which the decision maker uses to browse alternatives, while the second gives her preferences. Using the concepts of *choice reversers* in a set, i.e., alternatives whose removal from a set affect choice, and *hidden choice* of a set, i.e., the alternative chosen when the choice is removed, we give four logically independent axioms on choice behavior which jointly characterize two-stage choosers.

Keywords: Boundedly Rational Choice, Choice Reversal, Shortlisting, Multiple Rationales, Limited Attention

JEL code: D0

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[1] We thank Paola Manzini, Marco Mariotti and Fernando Payro for helpful comments.

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1 Introduction

The standard model in choice theory is that of a decision maker who maximizes a given preference in every choice problem. Despite its elegance and simplicity, it has long been recognized, at least since Simon (1955), that this model is not fully satisfactory as a descriptive device. Decision makers often subscribe to different behavioral rules such as satisficing rather than optimizing and their decisions may depend on multiple criteria. Their choices are affected by psychological phenomena and cognitive constraints such as attention and framing, which do not feature in the standard model. Furthermore, experiments show that this model is not a fully satisfactory predictive device either, as choices do not always conform to the maximization of a preference. Kahnemann and Tversky (2000) is a definitive survey of this experimental evidence.

In light of these findings, several recent papers have investigated the consequences of various forms of bounded rationality on choice. A typical exercise in this genre starts by positing a boundedly rational choice function, i.e., a procedure for making choices which departs from the standard model. Next comes a characterization of the procedure through axioms on choice behavior, which highlights (i) the types of irrational behavior (in the standard sense) that the procedure gives rise to, and (ii) how the procedure reveals via choices the economic and, if they exist, psychological ingredients on which it is built. This paper aims to contribute to this literature.

We are interested in the following choice function, which we call a *two-stage chooser*. The decision maker is endowed with two possibly conflicting criteria that rank alternatives. She first discards all feasible alternatives except the top-2 according to the first criterion. Then she chooses among these two alternatives using the second criterion. The criteria are linear orders, i.e., complete and transitive binary relations which contain no nontrivial indifferences.

Two-stage choosers assume two distinct interpretations. In the first interpretation both criteria are welfare-relevant as in the choice procedures axiomatized in Manzini and Mariotti (2007) and de Clippel and Eliaz (2012). Multiple criteria appear naturally in many examples. A decision maker could order restaurants in terms of their proximity to her current location as well as the quality of the food they serve. She could order meals in terms of their calorie content and their taste. In evaluating classes in which to enroll, she could care about how easy it is to get a good grade, as well as how beneficial a class could be in professional life. Importantly such criteria will typically disagree. Hence any choice procedure relying on multiple criteria needs to identify a rule to resolve potential conflicts. A two-stage chooser employs a clear rule in such situations: the first criterion vetoes any feasible alternative which is not its top-2 and the second criterion makes the choice among the two alternatives that have not been vetoed.

Alternatively, one could view the first criterion as a welfare-irrelevant list used by the decision maker to browse alternatives, as in the choice procedures studied in Rubinstein and Salant (2006) and Yildiz (2012). In this case a two-stage chooser models a decision maker who, perhaps because of limited attention or in order to avoid cognitive costs associated with having to evaluate numberous alternatives, considers only the first two alternatives that she sees. The second criterion gives her preferences. This interpretation is given as an example of how frames may affect choices in Rubinstein and Salant (2008) where the attention constraint and the listing of alternatives are taken to be frames.

Our main result is a characterization of two-stage choosers. The formulation of our axioms relies on the following concepts. *Choice reversers* in a set, *reversers* in short, are those alternatives whose removal affect choice. Clearly, the choice itself is a reverser. In general a set may have multiple reversers, in which case we say that it has *choice reversal.* Finally the *hidden choice* in a set is the alternative chosen in the subset obtained when the choice of the set is removed from the set.

We use four axioms in our characterization. The first axiom requires the following consistency of choice reversals: for every set, the removal of every reverser should lead to the choice of the same alternative, the hidden choice. The second axiom says that if the hidden choice in a set is chosen over the choice in binary comparison, then the set has choice reversal. The third axiom requires each reverser of a set to remain a reverser in any subset which (i) contains it, and (ii) has choice reversal. Finally the fourth axiom says that in any set with choice reversal, there are exactly two reversers. We list the axioms and give the characterization of two-stage choosers in Section 4. In Section 5, we discuss the kind of irrationalizable behavior exhibited by two-stage choosers. Section 6 is fully devoted to the proof of the main result. We show via examples in the Appendix that the axioms are logically independent.

An immediate modification of two-stage choosers is a procedure where the top $k > 2$ alternatives of the first criterion are shortlisted in every set. Our conjecture, as outlined in the conclusion, is that the resulting choice function is characterized by a slight modification of our axiomatic system, with the fourth axiom replaced by the following condition: if a set has choice reversal, it must have exactly k reversers. Our case for concentrating on binary shortlists, at least as a challenging starting point, rests on the cognitive appeal associated with choosing from smaller sets, as elaborated in Schwartz (2005).

Relation to literature Manzini and Mariotti (2007) study a choice function defined by two rationales, which are asymmetric (but not necessarily complete, nor transitive) binary relations. In the first stage the decision maker shortlists those feasible alternatives which are not dominated by any feasible alternative in the first rationale. In the second stage she eliminates any shortlisted alternative which is dominated by some shortlisted alternative in the second rationale. The procedure, by definition, is required to end up with a unique alternative that survives, which is the choice.[3] A two-stage chooser is not a special case of this procedure where the rationales are linear orders, as it shortlists the second best alternative in the first criterion. Consequently, as we show in Corollary 1 at the end of Section 4, the irrationalizable behavior exhibited by these two choice procedures are quite different.

De Clippel and Eliaz (2012) study, as we do, a decision maker endowed with two potentially conflicting linear orders. In their *reason-based choice*, these linear orders assign scores to feasible alternatives. The score of an alternative is the number of other feasible alternatives it dominates in a linear order. Each alternative therefore receives two scores, one from each linear order, and is evaluated on the basis of its lower score. The choice in a set is that alternative with the largest lower score. Of course, there may be two such alternatives and the procedure defines a *choice correspondence* rather than a choice function.

A related but distinct thread of the literature looks at choice procedures where a single linear order and a procedural constraint together determine choice. Salant and Rubinstein (2008) give as an example a decision maker with limited attention, who browses through alternatives using a list and who can only pay attention to at most k alternatives. In every set, the decision maker chooses among the top k alternatives of the list the best one according to her preference. When $k = 2$, this is precisely the choice function we study in this paper.[4]

Masatlıoğlu et al. (2012) and Lleras et al. (2011) study choice functions with *limited consideration*. This procedure models a decision maker who *filters out* certain feasible alternatives using a specific rule prior to choice. Two-stage choosers constitute a special case of this particular family of choice

[3] See also Dutta and Horan (2013) for related work on how this procedure reveals the underlying rationales and under what conditions the revealed rationales are unique.

[4] Salant and Rubinstein (2008) study extended choice functions of the form $c(A, f)$ which allow the choice in a set A to depend on the frame f. They characterize limited attention using axioms such extended choice functions. Since we are interested in standard choice functions, our axioms and approach are different.

functions, where any alternative which is not top-2 according to the first criterion is filtered out. However there are choice functions with limited consideration, which are not two-stage choosers. We discuss the exact relationship in detail in Section 5.2.

Eliaz et al. (2011) study a decision maker who identifies not one but two alternatives as "choice," which they interpret to be the *two finalists*. Among others, they characterize the procedure which selects the top-2 alternatives of a given linear order. The choice function we study here could be seen as incorporating in their problem the next natural step, in which a final decision is made between the two finalists. There is a subtle yet crucial difference, however, between the problem of choosing two finalists on the one hand and shortlisting two finalists and choosing a winner, on the other. In the former exercise both finalists are observable, whereas in the latter a unique alternative, *the winner*, is observed alone. This creates certain difficulties with respect to the revelation of the first criterion whose function is to determine the finalists. We address these difficulties in Section 6 which is entirely devoted to the proof of our characterization.

2 Basic Concepts

We consider a standard choice environment. Let X be a finite set of alternatives and Σ be the set of all nonempty subsets of X. Any $A \in \Sigma$ is a *choice problem*. We will denote the cardinality of A by $|A|$. To economize on space, we will omit set brackets and denote subsets of X as strings of alternatives. For example we will write xyz instead of $\{x, y, z\}$ or x instead of $\{x\}$. In the latter case, the context will clarify whether we are referring to the set or the alternative.

A *linear order* $\succ$ is a binary relation on X which is (i) *complete*, (ii) *transitive* and (iii) *asymmetric*. Hence $\succ$ is a linear order if for every $x, y, z \in X$, (i) $x \succ y$ or $y \succ x$, (ii) $x \succ z$ whenever $x \succ y$ and $y \succ z$, (iii) $x = y$

whenever $x \succ y$ and $y \succ x$.

If $A \in \Sigma$ and $\succ$ is a linear order, let $\max_{\succ}(A)$ be the $\succ$-maximal element of A, and $T^k_{\succ}(A)$ be the set of top k elements in A according to $\succ$. If $k > |A|$, then we set $T^k_{\succ}(A) = A$.

A *choice function* is a map $c : \Sigma \to X$ satisfying $c(A) \in A$ for every choice problem A. A choice function c is *rationalizable* if there exists a linear order $\succ$ such that $c(A) = \max_{\succ}(A)$ for every A. If this is the case, $\succ$ is said to rationalize c. It is well-known that a choice function c is rationalizable if and only if it satisfies the following axiom:[5]

Independence of Irrelevant Alternatives (IIA) If $x \in A \subset B$ and $x = c(B)$ then $x = c(A)$.

IIA requires the choice of a set to remain the choice in any subset where it belongs. If c satisfies IIA, it is rationalized by the binary relation $\succ$ defined by $x \succ y \Leftrightarrow x = c(xy)$.[6]

We will heavily rely on the following symbolism in our analysis. For a given choice function c, let

$$\begin{aligned} R^c(A) &= \{x \in A : c(A) \neq c(A \backslash x\},\ \text{and} \\ h^c(A) &= c(A \backslash c(A)) \end{aligned}$$

for every non-singleton choice problem A. Also let

$$\tilde{\Sigma}^c = \{A \in \Sigma : |R(A)| \neq 1\}.$$

We will mostly suppress the dependence of these objects on c and write $R(\cdot)$, $h(\cdot)$ and $\tilde{\Sigma}$. With the exception of the proof of Proposition 2 below, our arguments will involve a given choice function and this omission should not cause confusion.

[5] We will denote the subset relation by $\subseteq$ and its asymmetric part by $\subset$.

[6] See, for example, Moulin (1985) for the proof.

The set $R(A)$ collects those alternatives in A whose removal change the choice. Note that $c(A) \in R(A)$ in a trivial manner. An alternative y is a *nontrivial choice reverser* in A, if $y \in R(A)$ and $y \neq c(A)$. We will refer to the alternative $h(A)$ as the *hidden choice*; it is the alternative that is chosen when $c(A)$ is removed from A. Finally a choice problem $A \in \tilde{\Sigma}$ if and only if it contains nontrivial choice reversers. Note that any set in $\tilde{\Sigma}$ must contain at least three alternatives.

Below we will frequently refer to members of $R(A)$ as *reversers*, members of $A \backslash R(A)$ as *nonreversers* and members of $\tilde{\Sigma}$ as *sets with choice reversal.*

Let us record here the following simple observation, which gives a useful characterization of rationalizability.

Theorem 1 *A choice function c is rationalizable if and only if no set contains a nontrivial choice reverser.*

Proof. We will show that IIA is equivalent to the absence of nontrivial choice reversers. Suppose A has choice reversal. Then there exists an alternative $y \neq c(A)$ such that $c(A \backslash y) \neq c(A)$. However $c(A) \in A \backslash y \subset A$, and this contradicts IIA.

Suppose now that c fails IIA. Hence for some x, A and B, $x \in A \subset B$, $x = c(B)$ but $x \neq c(A)$. There must exist alternatives $b_1, b_2, ..., b_K$ such that $B \backslash A = b_1 ... b_K$. Denote by B_k the set $A \cup b_1 ... b_k$ for every $k = 1, ..., K$ and set $B_0 = A$. If for every $k \geq 1$, $x = c(B_k)$, then, in particular, $c(B_1) = x$, $B_1 \in \tilde{\Sigma}$ and $R(B_1) \supseteq x b_1$. Otherwise $x = c(B_k)$ but $x \neq c(B_{k-1})$ for some k, implying $B_k \in \tilde{\Sigma}$ and $R(B_k) \supseteq x b_k$. ■

Hence any irrationalizable choice function necessarily exhibits the following inconsistency of choices: for some set A and some alternative y, $c(A \backslash y) \neq c(A) \neq y$. In other words, addition of alternative y to the set $A \backslash y$ reverses the choice from $c(A \backslash y)$ to $c(A)$.

3 Two-Stage Choosers

We are interested in characterizing the following class of choice functions.

Definition 1 A choice function c is a *two-stage chooser* (TSC) if there exists a pair of linear orders $(\succ_1, \succ_2)$ such that for every set A,

$$c(A) = \max_{\succ_2}(T^2_{\succ_1}(A)).$$

A decision maker (DM) using a TSC is endowed with two linear orders which she uses to make choices. We will call these linear orders *criteria.* In any choice problem A, she first shortlists the top-2 alternatives in A according to the first criterion. Next she chooses in the binary shortlist the best alternative according to the second criterion. We would like to give two distinct interpretations of this procedure.

Scenario 1: Multiple Rationales The DM is to choose a meal from a menu. She has two criteria in mind. The first criterion $\succ_1$, associated with her long-run-self, orders the the meals in decreasing order of calorie content. (Imagine that no two meals have the exact same calorie content.) Her second criterion $\succ_2$, associated with her short-run-self, dictates that the meals she finds tastier are better. The DM, fully aware of both criteria, adapts the following behavioral rule: choose among two dishes with the lowest calorie content, the one that tastes better.

Scenario 2: List and Limited Attention The DM uses a given list $\succ_1$ to browse alternatives. However she has an attention constraint: in any set she can only consider the first two alternatives she sees. She chooses among them the one that she likes better according to her preference $\succ_2$.

Although both are compatible with the TSC family, Scenarios 1 and 2 are different in a fundamental way. In Scenario 1, both criteria are welfare-relevant and the decision maker needs to resolve potential conflicts in order to operationalize them. This resonates with papers on multicriteria decision-making, for instance Manzini and Mariotti (2007), Cherepanov et al. (2012) and de Clippel and Eliaz (2012). In Scenario 2, the first criterion is a welfare-irrelevant listing of alternatives and all welfare-relevant information is contained in the second criterion. This is in line with recent work on limited consideration in choice-making, for instance Masatlıoğlu et al. (2012) and Lleras et al. (2011). A TSC is a special form of choice with limited consideration and this is specifically mentioned in the aforementioned papers. In Section 5, we will elaborate on the exact relationship between TSCs and choice functions with limited consideration. We should also note that Salant and Rubinstein (2008) mention Scenario 2 as an example of how frames (e.g., attention order and attention capacity) affect choice.

Before proceeding, let us highlight certain features of TSCs that follow directly from the definition. First note that the criteria that define a TSC and the order in which they are applied to a choice problem are fixed. The DM always uses the first criterion to create a binary shortlist and always uses the second criterion to choose from the shortlist. Furthermore the TSC family contains all rationalizable choice functions. Indeed if the two criteria are identical, then the associated TSC will always select the maximal element of that criterion in every set. Since the criteria are linear orders, a TSC is necessarily decisive, in other words, a unique alternative is selected in the shortlist. Finally note that if c is a TSC with the associated criteria $\succ_1$ and $\succ_2$, then $c(A) = c(T^k_{\succ_1}(A))$ for every $k \geq 2$. Of course, $c(A)$ need not be $\max_{\succ_1}(A)$, nor $\max_{\succ_2}(A)$. However if $x \succ_1 y$ and $x \succ_2 y$ for some $x, y \in A$, then $y \neq c(A)$.

Example 1 Let $X = wxyz$, $y \succ_1 z \succ_1 w \succ_1 x$ and $w \succ_2 x \succ_2 y \succ_2 z$. The associated TSC is given by:

A	wx	wy	wz	xy	xz	yz	wxy	wxz	wyz	xyz	$wxyz$
$c(A)$	w	w	w	x	x	y	w	w	y	y	y

Note that even though w is the best alternative according to the second criterion, it fails to make the shortlist in any set containing y and z. Also note that $\tilde{\Sigma} = \{X, wyz, xyz\}$ with $R(A) = yz$ for every $A \in \tilde{\Sigma}$, $h(X) = h(wyz) = w$ and $h(xyz) = x$. Hence in any set with choice reversal, the choice reversers are the top two alternatives and the hidden choice is the third-best alternative in the first criterion. ▲

We would like to make the following four observations about Example 1, which inspire the axioms in the next section.

1. For any set A in $\tilde{\Sigma}$ and any choice reverser a in A, $c(A \backslash a) = h(A)$. Hence hidden choice is the choice following the removal of any choice reverser, not just the choice. For instance $R(X) = yz$ and $c(X \backslash y) = c(X \backslash z) = w$.

2. If a set does not belong to $\tilde{\Sigma}$ then its choice is chosen over its hidden choice when no other alternative is feasible. Take $wxz \notin \tilde{\Sigma}$, for example, where $c(wxz) = w$, $h(wxz) = x$ and $c(wx) = w$. If a set belongs to $\tilde{\Sigma}$, on the other hand, the hidden choice is chosen over the choice when no other alternative is feasible. For example, $c(X) = y$, $h(X) = w$ and $c(wy) = w$.

3. Reversers in X, remain the reversers in every subset which has choice reversal.

4. In any set with choice reversal, there is exactly one nontrivial choice reverser, which is z.

As a side note, note that the alternate first criterion $\succ_1'$ where $z \succ_1' y \succ_1' w \succ_1' x$, coupled with $\succ_2$ would lead to the exact same choice function. A TSC can never reveal the first criterion uniquely as the switching of the top-2 alternatives of X in the first criterion do not affect choices. The second criterion, on the other hand, is uniquely pinned down by choices in doubletons.

Example 1 is indicative of the kind of boundedly rational behavior admitted by TSCs. Imagine adding alternative z to the choice problem xy. According to binary choices z is inferior to both x and y. However its inclusion in xy reverses the choice from x to y. Similarly if z is included in wy, the choice changes w to y. This phenomenon arises naturally in the TSC family. Adding an element to a choice problem changes the choice to an alternative which was previously available but not chosen, precisely because the original choice may now, with the new alternative in the set, fail to be shortlisted. If this happens then we can infer that (i) both the new element and the new choice are superior to the original choice in the first criterion, and (ii) the original choice must have been superior to the new choice in the second criterion.

4 Axiomatization

In this section, we will provide an axiomatic system which characterizes the TSC family and which shed light on choice reversals produced by TSCs. Our axioms are as follows.

A1 For every A and every $x \in R(A)$, $c(A \backslash x) = h(A)$.

A2 If $c(c(A)h(A)) = h(A)$, then $A \in \tilde{\Sigma}$.

A3 If $x \in R(A)$, $x \in B \subset A$ and $B \in \tilde{\Sigma}$, then $x \in R(B)$.

A4 If $A \in \tilde{\Sigma}$, then $|R(A)| = 2$.

Axiom A1 disciplines the choices made when a reverser is removed from a set. Recall that $h(A)$ is by definition $c(A \backslash c(A))$ and this is regardless of whether set A has choice reversal or not. Now if A has any other (nontrivial) reverser, its removal should lead to the choice of $h(A)$ as well. A1 has an immediate consequence: $h(A) \notin R(A)$ for any set A. Axiom A2 gives a sufficient condition for a set to have choice reversal: if the hidden choice is chosen over the choice in binary comparison, then the set has choice reversal. We will shortly prove that in the presence of the rest of the axioms above, this sufficient condition is also necessary for a set to have choice reversal. Axiom A3 says that if x is a reverser in A, B is a subset of A containing x, and B has choice reversal, then x is a reverser in B as well. Note that the superset A may or may not have choice reversal. Axiom A4 says that there should be at most one nontrivial choice reverser in any set. Thus a choice function satisfies A4 iff in any set with choice reversal, there are exactly two reversers.

The appendix presents four choice functions showing that the axiomatic system is tight: no three of these four axioms imply the remaining one. Note that if c is rationalizable, then $\tilde{\Sigma} = \varnothing$ by Theorem 1 and consequently the axiomatic system reduces to the following condition: $c(c(A)h(A)) = c(A)$ for every A. This condition is the *weak path independence* of Yildiz (2012). Since every rationalizable choice function satisfies weak path independence, it also satisfies A1-A4.

Our main result is the characterization of the TSC family via these four axioms.

Theorem 2 A choice function is a two-stage chooser if and only if it satisfies A1-A4.

We prove Theorem 2 in Section 6 below. In the rest of this section, we give certain consequences of the axiomatic system A1-A4, which will be critical in the proof. Consider the following four axioms.

B1 For every x, y and z, $c(xz) = x$ whenever $c(xy) = x$ and $c(yz) = y$.

B2 If $A \in \tilde{\Sigma}$, then $c(c(A)h(A)) = h(A)$.

B3 If $A \in \tilde{\Sigma}$, then $c(A) = c(A')$ for every A' such that $R(A) \subseteq A' \subseteq A$.

B4 If $A \in \tilde{\Sigma}$, then $R(A) = R(A')$ and $h(A) = h(A')$ for every A' such that $R(A) \cup h(A) \subseteq A' \subseteq A$.

Axiom B1 is the familiar no-binary-cycle condition, ruling out binary cycles of choice between any three alternatives. One can easily check that under our full domain assumption, this version of the condition implies, apart from being clearly implied by, the absence of cycles of any length. Axiom B2 is the reverse of axiom A2: if a set has choice reversal, then its hidden choice is chosen over its choice in the binary set. Axioms B3 and B4 impose a certain persistence of choice reversals exhibited by TSCs. In any set with choice reversal, B3 requires the choice to stay the same in every one of its subsets containing its choice reversers. B4 says that if a subset of a set with choice reversal contains the reversers *and* the hidden choice of that set, then the subset exhibits the exact same reversal structure as the original set. In particular, if $R(A) = xy$ and $h(A) = z$, then $c(xy) = c(xyz) = c(A)$, $R(xyz) = xy$ and $h(xyz) = z$.

We will now show that axioms A1-A4 imply axioms B1-B4. The subsequent example proves that the reverse implication does not hold.

Proposition 1 If a choice function satisfies A1-A4, then it satisfies B1-B4.

Proof. Fix a choice function c satisfying axioms A1-A4. We will first show B1. Suppose that $c(xy) = x$, $c(yz) = y$ and $c(xz) = z$ for distinct alternatives x, y and z so that a binary cyle exists and without loss of generality let $c(xyz) = x$. Note $R(xyz) = xy$, however $c(xyz \backslash x) \neq c(xyz \backslash y)$ and A1 fails. Hence c satisfies B1. To see B2, B3 and B4 take $A \in \tilde{\Sigma}$ and let x, y and z be the choice, the nontrivial choice reverser and the hidden choice in A respectively. A3 and A4 imply that x (as the choice in A) and z (as the choice in $A \backslash y$) are the only potential reversers in any subset of $A \backslash y$ which contains them. Hence $c(A \backslash y) = c(xz) = z$, giving B2. Similarly x and y are the only potential reversers in any subset of A where they both belong and therefore if $xy \subseteq A' \subset A$, then $c(A') = x$, giving B3. Now suppose that

$xyz \subseteq A' \subset A$. By B3 $c(A') = x$. Let $A \backslash A' = a_1...a_K$, $A_k = A' \cup a_1,,,a_k$ for every $k = 1, ..., K$ and $A_0 = A'$. We must have $c(A\backslash x) = z = c(A_k\backslash x)$ for every k, since otherwise there exists a set A_k such that $y, z \in R(A_k\backslash x)$ by A3 and $a_k \in R(A_k\backslash x)$ by construction, a contradiction to A4. Hence $h(A') = z$ as well. Finally, since $c(xz) = z$, $A' \in \tilde{\Sigma}$ by A2 and $R(A') = xy$ by A3 and A4. This gives B4 and completes the proof. ■

Example 2 Let $X = xyz$, $c(xy) = c(xz) = x$, $c(yz) = y$ and $c(xyz) = z$ so that $\tilde{\Sigma} = \{xyz\}$ and $R(xyz) = xyz$, leading to the failure of A4. Furthermore $h(xyz) = x \in R(xyz)$ and A1 fails. Note that c satisfies B1-B4. B1 follows as the binary choices do not cycle. B2 follows because $c(c(xyz)h(xyz)) = c(xz) = x = h(xyz)$. B3 and B4 follow because the only set that contains $R(xyz)$ is xyz itself. ▲

Note also that if c satisfies A1-A4 and $A \in \tilde{\Sigma}$, then $h(A)$ is chosen over both reversers of A in binary comparison. That $c(h(A)c(A)) = h(A)$ follows directly from B2. Let $r(A)$ denote the reverser of A which is not $c(A)$. Now B3 gives $c(c(A)r(A)) = c(A)$ and B1 gives $c(h(A)r(A)) = h(A)$. Hence if c satisfies A1-A4, it also satisfies the following strengthening of B2:

B2* If $A \in \tilde{\Sigma}$, then $c(ah(A)) = h(A)$ for every $a \in R(A)$.

To finish this section, we will show TSCs lead to irrationalizable behavior of a specific type. Manzini and Mariotti (2007) show that a choice function satisfies IIA if and only if it satisfies no binary cycles, i.e., our B1, and the following axiom:

Always Chosen (AC) If $x \in A$ and $x = c(xa)$ for every $a \in A\backslash x$, then $x = c(A)$.

Since a two-stage chooser necessarily satisfies B1, any irrationalizable behavior it may exhibit comes in the form of a violation of AC. We record this observation without proof.

Corollary 1 *If a two-stage chooser is not rationalizable, then it fails the Always Chosen axiom.*

The irrationalizable two-stage chooser in Example 1, for instance, fails AC: w is chosen over every other alternative in binary comparison, but not in the grand set.

5 Discussion

Before proceeding to prove Theorem 2, we will discuss two important features of two-stage choosers. First we will show that two-stage choosers are uniquely identified by their behavior in two- and three-alternative choice problems. Next we will explore the exact relationship between TSCs on the one hand and the choice functions axiomatized in Masatlıoğlu et al. (2012) and Lleras et al. (2011) featuring limited attention and consideration on the other.

5.1 TSCs are identifiable by choices in small sets

For any $k = 1, ..., |X|$, let Σ_k be the set of choice problems with at most k alternatives. For any choice function c and any set $\Sigma^0 \subset \Sigma$, let $c|_{\Sigma^0}$ be the restriction of c to the domain Σ^0.

A useful property of (standard) rational behavior is that choices in larger sets are completely determined by choices in doubletons. To be precise, if c satisfies IIA, then $c(S)$ is that alternative in S which is chosen over every other alternative in S in binary comparison. Put differently, if c and $\hat{c}$ are two choice functions which satisfy IIA, then $c = \hat{c}$ if and only if $c|_{\Sigma_2} = \hat{c}|_{\Sigma_2}$. Two-stage choosers satisfy an analogous property.

Proposition 2 *If c and $\hat{c}$ are distinct two-stage choosers then $c|_{\Sigma_3} \neq \hat{c}|_{\Sigma_3}$.*

Proof. Suppose c and $\hat{c}$ are two-stage choosers with $c(S) = a \neq b = \hat{c}(S)$ for some S with at least four elements. We will show that for some strict subset S' of S, $c(S') \neq \hat{c}(S')$ as well. Suppose, towards a contradiction, that $c(S') = \hat{c}(S')$ for every $S' \subset S$. Without loss of generality let $c(ab) = \hat{c}(ab) = a$. Then a is not a top-2 alternative in the second criterion defining $\hat{c}$, and consequently $\hat{c}(S \backslash a) = \hat{c}(S) = b$. This implies that $b = c(S \backslash a)$ as well, i.e., $b = h^c(S)$. Since the choice $a = c(S)$ is chosen over the hidden choice b in binary comparison, $S \notin \Sigma^c$, and $c(S \backslash b) = c(S) = a$. Thus $\hat{c}(S \backslash b) = a$ as well and $S \in \Sigma^{\hat{c}}$ with $h^{\hat{c}}(S) = a$. Now let $R^{\hat{c}} = br$ where r is distinct from a and b and note that there exists $s \in S \backslash abr$. It follows that $c(S \backslash s) = a \neq b = \hat{c}(S \backslash s)$, a contradiction. ■

Hence if we know that a decision-maker is using a two-stage chooser and if we observe her choices out of doubletons and tripletons, then there is a unique way to "extend" her choices to larger sets. Clearly her choices in doubletons must satisfy B1, the no binary cycles axiom. Given B1, not all choices in tripletons are admissible by two-stage choosers, as the behavior must comply with the axioms A1-A4. However, Proposition 2 indicates that for every admissible configuration of choices in two- and three-element sets, there is a unique choice function which extends these choices to the whole domain and is a two-stage chooser. We illustrate in an example.

Example 3 Let $X = wxyz$. The decision maker is known to use a two-stage chooser and is observed to make the following choices in two-alternative sets:

A	wx	wy	wz	xy	xz	yz
$c(A)$	w	w	w	x	x	y

This pins down her second criterion to be $w \succ_2 x \succ_2 y \succ_2 z$. Consider what her possible choices could be in three-alternative sets. Example 1 above gives one possibility but, it turns out, there are six more. We list all seven possibilities below:

A	wxy	wxz	wyz	xyz
Case 1	w	w	w	x
Case 2	w	w	w	y
Case 3	w	w	y	y
Case 4	w	x	w	x
Case 5	x	w	w	x
Case 6	x	x	y	x
Case 7	x	x	y	y

To see how we arrive at these 7 cases, first note z can not be the choice in any tripleton since, even if it is shortlisted, it can not beat the other shortlisted alternative in the second criterion. Similarly y can not be the choice in wxy. Hence in each of the four tripletons, there are two possible choices: w or x in wxy and wxz, w or y in wyz, and x or y in xyz. However not all of the resulting $2^4 = 16$ configurations are admissible. Every row in the table contains two distinct choices as A1 dictates that there can not be three different choices across these four sets. For if $c(wxy) = w$, $c(wyz) = y$ and $c(xyz) = x$, for example, the removal of the reversers in the grand set $wxyz$, regardless of what they may be, would lead to the choice of different alternatives. This observation brings down the possible configurations of choices in tripetons to 8. There is one further elimination. Suppose that the choices are $c(wxy) = c(wxz) = c(xyz) = x$ and $c(wyz) = w$. By A1, $c(wxyz) \in wx$. If $c(wxyz) = w$, then $wxyz$ has three reversers, contradicting A4. Hence $c(wxyz) = x$, $wxyz \notin \tilde{\Sigma}$, and $h(wxyz) = c(wyz) = w$. But since $c(wx) = w$, $wxyz$ must have choice reversal, a contradiction.

Now for each one of these 7 cases, there is a unique possible choice in the grand set $wxyz$. In Cases 1, 2, and 6 this choice is necessarily the alternative chosen in three distinct tripletons. In the remaining cases, $wxyz \in \tilde{\Sigma}$, and this observation determines $c(wxyz)$. In Case 3, for example, if $c(wxyz) = w$, then $h(wxyz) = c(xyz) = y$. However $c(wy) = w$ and $wxyz$ can not have choice reversal, a contradiction. It follows that $c(wxyz) = y$. Hence in Cases 3, 4, 5, and 7, the choice in $wxyz$ is the alternative which is inferior in $\succ_2$ among the two alternatives that are the choices in tripletons. ▲

5.2 Limited Attention and Limited Consideration

We mentioned in the introduction that TSCs belong to a larger family of choice functions that have been axiomatized by Masatlıoğlu et al. (2012) and Lleras et al. (2011). In this section we will elaborate this connection.

Consider a decision maker who, in any set A, maximizes a linear order $\succ$ on a nonempty subset $\Gamma(A)$ of A: $c(A) = \max_{\succ} \Gamma(A)$ where $\succ$ is a linear order and $\Gamma : \Sigma \to \Sigma$ is a *consideration map* which satisfies $\varnothing \neq \Gamma(A) \subseteq A$ for all A. The interpretation is as follows. Before making choices, the decision maker filters out some feasible alternatives from the choice problem perhaps due to some psychological or cognitive constraints. Clearly, for this model to have empirical content, one would like to place restrictions on Γ. When Γ is free, the model captures every single choice function c, simply by taking $\Gamma(A) = \{c(A)\}$ for every A.

Masatlıoğlu et al. (2012) study limited attention, i.e., the case where Γ satisfies the following condition: if $x \notin \Gamma(A)$ then $\Gamma(A) = \Gamma(A \backslash x)$. They show that the associated choice function is characterized by the following axiom.

WARP(LA) For every A, there exists some $x_A^* \in A$ such that for every B containing x_A^*,

$$\text{if } c(B) \in A \text{ and } c(B) \neq c(B \backslash x_A^*) \text{ then } c(B) = x_A^*.$$

In related work, Lleras et al. (2011) explore the consequences of limited consideration, i.e., the case where Γ satisfies the following condition: if $A \subset B$

then $\Gamma(B) \cap A \subseteq \Gamma(A)$. They show that the associated choice function is characterized by the following axiom.

WARP(LC) For every A, there exists some $x_A^* \in A$ such that for every B containing x_A^*,

$$\text{if } c(B) \in A \text{ and } c(B') = x_A^* \text{ for some } B' \supset B \text{ then } c(B) = x_A^*.$$

TSCs constitute a special case of both of these choice procedures, where the decision maker filters out all alternatives but the top-2 according to $\succ_1$. This is the content of the following result.

Proposition 3 If c satisfies A1-A4 then c satisfies WARP(LA) and WARP(LC).

Proof. Suppose c satisfies A1-A4. By Proposition 1 c satisfies B1 as well. Hence the binary relation defined by $x \succ_2 y \Leftrightarrow x = c(xy)$ is a linear order. Now for every S, let $x_S^* = \max_{\succ_2}(S)$. Fix A and B such that $x_A^* \in B$ and $c(B) \in A$. To check WARP(LA) suppose $c(B) \neq c(B \backslash x_A^*)$. Then $x_A^* \in R(B)$. Towards a contradiction, suppose $y = c(B) \neq x_A^*$ so that $R(B) = x_A^* y$. By B3, $y = c(x_A^* y)$, a contradiction. To check WARP(LC) suppose $c(B') = x_A^*$ for some $B' \supset B$. Hence $x_A^* \in R(B')$. If $y = c(B) \neq x_A^*$, then, by A3 and A4, x_A^* and y are the only potential choice reversers in any subset of B which contains them. This gives $y = c(x_A^* y)$, a contradiction. ■

The following example shows that the implication in Proposition 3 can not be reversed. In other words, TSCs are not characterized by WARP(LA) *and* WARP(LC).

Example 4 Suppose that $X = xyz$ and consider the following choice function:

A	xy	yz	xz	xyz
$c(A)$	x	y	x	z

Set $x^*_{xyz} = x^*_{xz} = x^*_{yz} = z$ and $x^*_{xy} = x$ to check that c satisfies WARP(LA), and $x^*_{xyz} = x^*_{xz} = x^*_{xy} = x$ and $x^*_{yz} = y$ to check WARP(LC). However c is not a TSC as it fails A1 and A4. ▲

6 The proof of the main result

In this section we prove Theorem 2.

6.1 Necessity

Proof of Theorem 2: "only if". Suppose that c is a TSC and let $(\succ_1, \succ_2)$ be the associated pair of criteria. To begin, take any choice problem A and any alternative $z \notin T^2_{\succ_1}(A)$. Then $T^2_{\succ_1}(A \backslash z) = T^2_{\succ_1}(A)$ and consequently $c(A \backslash z) = c(A)$. Hence $R(A) \subseteq T^2_{\succ_1}(A)$. Now A4 follows directly from definitions. If $A \in \tilde{\Sigma}$ then $R(A)$ contains multiple alternatives and therefore we must have $R(A) = T^2_{\succ_1}(A)$.

To show A3, take x, B and A such that $x \in B \subset A$, $x \in R(A)$ and $B \in \tilde{\Sigma}$. We showed above that $R(A) \subseteq T^2_{\succ_1}(A)$, giving $x \in T^2_{\succ_1}(A)$ and therefore $x \in T^2_{\succ_1}(B)$. Since $B \in \tilde{\Sigma}$, $R(B) = T^2_{\succ_1}(B)$, $x \in R(B)$ as well.

To show A2 fix $A \notin \tilde{\Sigma}$. Let $x = c(A)$ and $z = h(A)$. We need to show that $c(xz) = x$. Now $z = c(A \backslash x)$ implying $z \in T^2_{\succ_1}(A \backslash x) \subset T^3_{\succ_1}(A)$. There are two cases. If $z \in T^2_{\succ_1}(A)$, then $T^2_{\succ_1}(A) = xz$ giving $c(A) = c(xz) = x$. If, on the other hand $z \notin T^2_{\succ_1}(A)$, then $T^2_{\succ_1}(A) = xa$ for some $a \in A \backslash xz$, and $T^2_{\succ_1}(A \backslash a) = xz$. Since $A \notin \tilde{\Sigma}$, $a \notin R(A)$ and $x = c(A \backslash a) = c(T^2_{\succ_1}(A \backslash a)) = c(xz)$. We conclude that c satisfies A2.

We will finish this part of the proof by showing that c satisfies A1. Suppose $A \in \tilde{\Sigma}$ so that $R(A) = T^2_{\succ_1}(A)$. Note that A must have at least three elements. Let $xy = T^2_{\succ_1}(A)$ where $x = c(A) \neq y$. Then $x \succ_2 y$ as $c(A) = c(xy) = x$. Let z be the third-best alternative in A according to $\succ_1$. Hence $T^2_{\succ_1}(A \backslash x) = yz$ and $T^2_{\succ_1}(A \backslash y) = xz$. Since y is a choice reverser, $c(A \backslash y) = z$, indicating $z \succ_2 x$. To finish the argument we need to show that $z = c(A\backslash x)$ as well. Clearly $c(A\backslash x) = c(T^2_{\succ_1}(A\backslash x)) = c(yz) \in yz$. If $y = c(A\backslash x)$, then $y \succ_2 z$ and $\succ_2$ contains a binary cycle: $x \succ_2 y \succ_2 z \succ_2 x$. Hence we must have $c(A\backslash x) = z$. ■

6.2 Sufficiency

We will now show that if c satisfies A1-A4, then it is a TSC. The proof consists of three parts. The first part consists of some preparations where we define, for a given choice function c, a binary relation P_0. In particular we establish that if c satisfies A1-A4, then P_0 is asymmetric and transitive (Lemmas 1 and 4). In this part we also give two technical results in Lemmas 2 and 3. Lemmas 1-4 are used in virtually every subsequent argument of the proof. In the second part, we define two binary relations, P_1 and P_2, which serve as the criteria in our representation. In particular, P_2 is defined by choices in doubletons, while P_1 is a particular completion of P_0. We show that P_1 and P_2 are linear orders if c satisfies A1-A4. In the third part, we finish the proof by showing that for every set A, $c(A) = \max_{P_2}(T^2_{P_1}(A))$ if c satisfies A1-A4.

6.2.1 Preparations

For any choice function c define

$$xP_0y \text{ iff } \exists S \in \tilde{\Sigma} : \begin{cases} x, y \in S, \\ x \in R(S), \\ y \notin R(S). \end{cases}$$

Note that, as usual, we are suppressing the dependence of P_0 on the choice function c for notational convenience. The binary relation P_0 will later help us in the definition of P_1. Our primary goal in this initial part of the proof is to show that if c satisfies A1-A4, then P_0 is asymmetric and transitive. We start by establishing asymmetry.

Lemma 1 If c satisfies A1-A4, then P_0 is asymmetric.

Proof. Take a choice function c satisfying A1-A4 and alternatives $x \neq y$. Suppose towards a contradiction that xP_0y and yP_0x. There must then exist sets $S, T \in \tilde{\Sigma}$ such that $x, y \in S \cap T$, $x \in R(S)$, $y \in R(T)$, $x \notin R(T)$ and $y \notin R(S)$. We will analyze two exhaustive cases identified by the role of y in S.

Case 1: Suppose to begin with that $y = h(S)$. In this case $x \neq h(T)$ as otherwise B2 would imply $x = c(xy) = y$, an impossibility. Let $R(S) = x\alpha$, $R(T) = y\beta$ and $h(T) = \gamma$. By B4,

$$\begin{aligned} R(xy\alpha) &= x\alpha,\ y = h(xy\alpha) \text{ and} \\ R(xy\beta\gamma) &= y\beta,\ \gamma = h(xy\beta\gamma). \end{aligned}$$

Clearly $\beta \neq \gamma$, as hidden choice can not be a choice reverser by A1. If $\alpha = \beta$ or $\alpha = \gamma$, then A3 fails: $xy\alpha \subset xy\beta\gamma$, $y \in R(xy\beta\gamma)$, $xy\alpha \in \tilde{\Sigma}$ however $y \notin R(xy\alpha)$. Hence we need only consider the case where α, β and γ are distinct alternatives.

We will begin by showing that $xy\alpha\beta\gamma \notin \tilde{\Sigma}$. By A3, $c(xy\alpha\beta\gamma) \in \alpha\beta$ as each one of the other feasible alternatives does not belong to either $R(xy\alpha)$ or $R(xy\beta\gamma)$. If $xy\alpha\beta\gamma \in \tilde{\Sigma}$ then $R(xy\alpha\beta\gamma) = \alpha\beta$ by A3, giving $c(xy\beta\gamma) = c(xy\alpha\gamma) = y$, where the first equality is by A1 and the second follows since $c(xy\alpha\gamma) \in R(xy\beta\gamma) = y\beta$ however $\beta \notin xy\alpha\gamma$. Now using A3 again, we get $y \in R(xy\alpha)$, a contradiction. We conclude that $xy\alpha\beta\gamma \notin \tilde{\Sigma}$.

Now we will show that $c(xy\alpha\beta\gamma) = \alpha$. If $c(xy\alpha\beta\gamma) = \beta$ then $c(x\alpha\beta\gamma) = \beta$ as well by A3. We also have $c(x\beta\gamma) = \gamma$ since $\gamma = h(xy\beta\gamma)$ and $y \in R(xy\beta\gamma)$. Now consider the set $xy\alpha\gamma$. Note that $c(xy\alpha\gamma) \neq \gamma$ by A2 since $xy\alpha\beta\gamma \notin \tilde{\Sigma}$, $c(xy\alpha\beta\gamma) = \beta$ and $c(\beta\gamma) = \gamma$. Furthermore $c(xy\alpha\gamma) \neq y$ by A3 as $xy\alpha \in \tilde{\Sigma}$ and $y \notin R(xy\alpha)$. Thus $c(xy\alpha\gamma) \in x\alpha$ and $y \notin R(xy\alpha\gamma)$ by A3. It follows that $c(xy\alpha\gamma) = c(x\alpha\gamma) \in x\alpha$. This gives a violation of A1 as $c(x\alpha\beta\gamma) = \beta$, $c(x\beta\gamma) = \gamma$ and $c(x\alpha\gamma) \in x\alpha$. We conclude that $c(xy\alpha\beta\gamma) = \alpha$.

To finish, note that $h(xy\alpha\beta\gamma) = c(xy\beta\gamma) = c(y\beta)$. Applying B2* in $xy\alpha$ gives $c(\alpha y) = y$. Then $h(xy\alpha\beta\gamma) \neq y$ by A2 and $h(xy\alpha\beta\gamma) = c(xy\beta\gamma) = c(y\beta) = \beta$. Applying A2 again in $xy\alpha\beta\gamma$, we get $c(\alpha\beta) = \alpha$. Thus the choices out of αy, $y\beta$ and $\alpha\beta$ constitute a binary cycle and a violation of B1.

Case 2: Now suppose that $y \neq h(S)$. If $x = h(T)$, we are in the first case with the roles of x and y reversed and a contradiction follows analogously. Suppose therefore that $x \neq h(T)$. Let $R(S) = x\alpha$, $h(S) = \beta$, $R(T) = y\gamma$ and $h(T) = \delta$. Using B4, we get

$$\begin{aligned} R(xy\alpha\beta) &= x\alpha \text{ and } h(xy\alpha\beta) = \beta, \\ R(xy\gamma\delta) &= y\gamma \text{ and } h(xy\gamma\delta) = \delta. \end{aligned}$$

We will first show that if any two alternatives among α, β, γ and δ are the same, we arrive at a contradiction. Clearly if $\alpha = \beta$ or $\gamma = \delta$, A1 fails. If $\alpha = \gamma$ then $c(xy\alpha\beta\delta) = \alpha$ and $xy\alpha\beta\delta \notin \tilde{\Sigma}$ by A3. Since $c(xy\beta) = \beta$ and $c(xy\delta) = \delta$, $h(xy\alpha\beta\delta) = c(xy\beta\delta) \in \beta\delta$ by A1. Applying B2* on $xy\alpha\beta$ yields $c(\alpha\beta) = \beta$, and on $xy\gamma\delta$ yields $c(\gamma\delta) = c(\alpha\delta) = \delta$. Hence both candidates for $h(xy\alpha\beta\delta)$ are chosen over $\alpha = c(xy\alpha\beta\delta)$ in binary sets. Now B2 implies $xy\alpha\beta\delta \in \tilde{\Sigma}$, a contradiction. Similar contradictions obtain if $\alpha = \delta$ or $\beta = \gamma$ or $\beta = \delta$ using almost exactly the same argument. We will skip the details and consider in the rest of the proof the case where α, β, γ and δ are distinct alternatives.

We will now show that $xy\alpha\beta\gamma\delta \notin \tilde{\Sigma}$. If $xy\alpha\beta\gamma\delta \in \tilde{\Sigma}$ then $R(xy\alpha\beta\gamma\delta) = \alpha\gamma$ by A3 and $c(xy\beta\gamma\delta) = c(xy\alpha\beta\delta) = h(xy\alpha\beta\gamma\delta)$ by A1. Clearly $h(xy\alpha\beta\gamma\delta) \notin \alpha\gamma$ by A1 and $h(xy\alpha\beta\gamma\delta) \notin xy$ by A3. Now if $h(xy\alpha\beta\gamma\delta) = \beta$ then $c(xy\alpha\beta\delta) = \beta$, and $\delta \in R(xy\alpha\beta\delta)$ since $c(xy\alpha\beta) \in x\alpha$. It follows that $xy\alpha\beta\delta \in \tilde{\Sigma}$ and $h(xy\alpha\beta\delta) = c(xy\alpha\beta) \in x\alpha$ by A1. Now applying B2* on $xy\alpha\beta$, we find $c(\beta x) = c(\beta\alpha) = \beta$. Hence $\beta = c(xy\alpha\beta\delta)$ is chosen over both candidates for $h(xy\alpha\beta\delta)$, giving $xy\alpha\beta\delta \notin \tilde{\Sigma}$ by B2, a contradiction. Similarly if $h(xy\alpha\beta\gamma\delta) = \delta$ then $xy\beta\gamma\delta \in \tilde{\Sigma}$ as $c(xy\gamma\delta) \in y\gamma$. However B2* gives

$c(\delta y) = c(\delta\gamma) = \delta$ and $xy\beta\gamma\delta \in \tilde{\Sigma}$ by B2, an analogous contradiction. We conclude that $xy\alpha\beta\gamma\delta \notin \tilde{\Sigma}$.

Now there are four subcases to consider depending on the roles played by x and α in $xy\alpha\beta$, and by y and β in $xy\gamma\delta$, and we will consider these cases seperately. We have established above that $c(xy\alpha\beta\gamma\delta) \in \alpha\gamma$. In all subcases that follow we will assume that $c(xy\alpha\beta\gamma\delta) = \alpha$. This is without loss of generality and the arguments can easily be adopted to work if $c(xy\alpha\beta\gamma\delta) = \gamma$.

Case 2a: Suppose that $x = c(xy\alpha\beta)$ and $y = c(xy\gamma\delta)$, making α and γ nontrivial choice reversers. We will build up to a contradiction to A1. Now $c(xy\alpha\beta\gamma) = \alpha$ as $xy\alpha\beta\gamma\delta \notin \tilde{\Sigma}$. Since $c(xy\alpha\beta) = x$, it follows that $R(xy\alpha\beta\gamma) = \alpha\gamma$ and $x = c(xy\alpha\beta) = c(xy\beta\gamma)$ by A1. We also have $c(xy\beta) = \beta$ and $c(xy\delta) = \delta$, implying $c(xy\beta\delta) \in \beta\delta$ by A1. On the other hand $c(xy\gamma\delta) = y$ by assumption and therefore $xy\beta\gamma$, $xy\beta\delta$ and $xy\gamma\delta$ are three subsets of $xy\beta\gamma\delta$ where choices are different. This contradicts A1, as we wished to show.

Case 2b: Suppose that $\alpha = c(xy\alpha\beta)$ and $y = c(xy\gamma\delta)$. To begin, note that $c(y\alpha\beta) = \beta$ since $x \in R(xy\alpha\beta)$ and $\beta = h(xy\alpha\beta)$. We will now build up to a contradiction to this finding. We have $c(xy\beta) = \beta$ and $c(xy\gamma) = y$, giving $c(xy\beta\gamma) \in \beta y$ by A1. Now consider the set $xy\beta\gamma\delta$. A3 implies $c(xy\beta\gamma\delta) \notin x\delta$, as neither x nor δ reverse choice in $xy\gamma\delta$. Furthermore A2 implies $c(xy\beta\gamma\delta) \neq \beta$ as $c(\alpha\beta) = \beta$, $\alpha = c(xy\alpha\beta\gamma\delta)$ and $xy\alpha\beta\gamma\delta \notin \tilde{\Sigma}$. If $c(xy\beta\gamma\delta) = \gamma$, then δ and β are two distinct nontrivial reversers in $xy\beta\gamma\delta$ and A4 fails. Hence we must have $c(xy\beta\gamma\delta) = y = h(xy\alpha\beta\gamma\delta)$, $c(\alpha y) = \alpha$ by A2 and $c(y\beta\gamma\delta) = y$ as x can not be a reverser in $xy\beta\gamma\delta$ by A3. Using A3 again, we conclude that δ can not be a reverser in $y\beta\gamma\delta$ as it is not a reverser in $y\gamma\delta$. Hence $c(y\beta\gamma) = y$. On the other hand, $c(xy\alpha\beta\gamma) = \alpha$ as $xy\alpha\beta\gamma\delta \notin \tilde{\Sigma}$ and $c(xy\beta\gamma) = y$ by A3 which dictates that δ can not be a reverser in $xy\beta\gamma\delta$ either. Note that $c(xy\alpha\beta\gamma)$ is chosen over $h(xy\alpha\beta\gamma) = c(xy\beta\gamma)$ in binary comparison as we established $c(\alpha y) = y$ above, giving $xy\alpha\beta\gamma \notin \tilde{\Sigma}$ by B2. This implies $c(y\alpha\beta\gamma) = \alpha$. Now $c(y\alpha\beta\gamma)$ is chosen over $h(y\alpha\beta\gamma) =$

$c(y\beta\gamma) = y$ in binary comparison, giving $y\alpha\beta\gamma \notin \tilde{\Sigma}$ by B2. It follows that $c(y\alpha\beta) = \alpha$, which is exactly the contradiction we were aiming for.

Case 2c: If $x = c(xy\alpha\beta)$ and $\gamma = c(xy\gamma\delta)$, the argument is parallel to the one in Case 2b, with the roles of x and y reversed. We skip the details.

Case 2d: Finally suppose that $\alpha = c(xy\alpha\beta)$ and $\gamma = c(xy\gamma\delta)$ making x and y nontrivial reversers of $xy\alpha\beta$ and $xy\gamma\delta$ respectively. We will build up to a contradiction to A1. Now $c(y\alpha\beta\gamma\delta) = \alpha$ as $xy\alpha\beta\gamma\delta \notin \tilde{\Sigma}$. Since, by B4, δ is not a reverser in $y\gamma\delta$, it can not reverse choice in $y\alpha\beta\gamma\delta$ either by A3. Hence $c(y\alpha\beta\gamma) = \alpha$. Next we will consider the set $xy\beta\gamma\delta$. By A3 $c(xy\beta\gamma\delta) \notin x\delta$. Furthermore by A2, $c(xy\beta\gamma\delta) \neq \beta$ either as $c(\beta\alpha) = \beta$ and $xy\alpha\beta\gamma\delta \notin \tilde{\Sigma}$. Thus $c(xy\beta\gamma\delta) \in y\gamma$. Note that $c(xy\beta\delta) \in \beta\delta$ by A1 as $c(xy\beta) = \beta$ and $c(xy\delta) = \delta$. Furthermore $c(xy\gamma\delta) = \gamma$, and $xy\beta\delta, xy\gamma\delta \subseteq xy\beta\gamma\delta$. Hence $c(xy\beta\gamma\delta) \neq y$ by A1 and A4. We conclude that $c(xy\beta\gamma\delta) = \gamma$. Note also that $x \notin R(xy\beta\gamma\delta)$ and $\delta \notin R(y\beta\gamma\delta)$ by A3, giving $c(y\beta\gamma) = \gamma$. Finally note that $c(y\alpha\beta) = \beta$ as $x \in R(xy\alpha\beta)$ and $\beta = h(xy\alpha\beta)$. Hence we have $c(y\alpha\beta\gamma) = \alpha$, $c(y\beta\gamma) = \gamma$ and $c(y\alpha\beta) = \beta$ contradicting A1. This completes the proof of Lemma 1. ■

Now we will record, in Lemmas 2 and 3, certain technical consequences of our axioms which will be useful in several key points of the proof.

Lemma 2 Suppose that c satisfies A1-A4. For every A, B and x such that $c(A) \in B \subset A$, $c(B) \neq c(A)$ and $x \in B \backslash c(A)$ (1) $c(A) P_0 x$, and (2) there exists some $y \in A \backslash B$ such that $y P_0 x$.

Proof. Take a, b, x, A and B such that $c(A) = a$, $c(B) = b \neq a$, $a \in B \subset A$ and $x \in B \backslash a$. Now there must exist a set A' and an alternative $y \in A' \backslash B \subset A \backslash B$ such that $B \subset A' \subseteq A$, $A' \in \tilde{\Sigma}$ and $y \in R(A')$ since otherwise $c(A) = c(B)$. Note that $y \neq x$ and, by A3 $a \in R(A')$ as well. Now A4 implies $R(A') = ay$. Since $x \in B \backslash a$, $x \in A' \backslash R(A')$ and consequently $a P_0 x$ and $y P_0 x$. ■

Note in particular that alternative $x \in B \backslash c(A)$ in Lemma 2 could be $c(B)$. Hence if $B \subset A$ and $c(B) \neq c(A)$, then $c(A) P_0 c(B)$. Lemmas 1 and 2 will feature in the proof of our second technical result, which follows.

Lemma 3 Suppose that c satisfies A1-A4. For any distinct x, y and z,

$$\text{if } z \not{P}_0 y \text{ and } z P_0 x, \text{ then } x \neq c(xyz).$$

Proof. Fix a choice function c which satisfies A1-A4 and distinct alternatives x, y and z satisfying $z \not{P}_0 y$ and $z P_0 x$. Since $z P_0 x$, there exists a set $S \in \tilde{\Sigma}$ such that $z \in R(S)$ and $x \in S \backslash R(S)$. Suppose, to begin, that $y \in S$. It follows that $y \in R(S)$ since otherwise $z P_0 y$, a contradiction. Hence, by A4, $R(S) = yz$. Now by B3, $c(S) = c(xyz)$ and since $c(S) \in R(S)$, $c(xyz) \neq x$.

Now suppose $y \notin S$ and using A4 let $R(S) = zw$ for some $w \in S \backslash zx$. Note in particular that $z P_0 s$ for every $s \in S \backslash zw$ by definition. Denote $S' = S \cup y$ and $S'' = S' \backslash w$. We *claim* here that $c(S'') \in yz$. Before proving this *claim*, let us show that if it is correct, the proof is complete. Again, there are two possibilities. If $c(S'') = y$ and $c(xyz) = x$, then by Lemma 2 there exists some $s \in S'' \backslash xyz$ such that $s P_0 z$. However such s necessarily belongs to $S \backslash zw$, and therefore $z P_0 s$ as well, contradicting Lemma 1. If $c(S'') = z$ and $c(xyz) = x$, on the other hand, $z P_0 y$ by Lemma 2, an impossibility. Hence we must have $c(xyz) \neq x$

All that remains to be done is to prove the *claim* that $c(S'') \in yz$. By A3 $R(S') \subset wyz$, as no alternative in $S \backslash wz$ could belong to $R(S')$. Hence there are three possibilities regarding the choice in S'.

Suppose first that $c(S') = y$. If $S' \notin \tilde{\Sigma}$ then $c(S'') = y$. If, on the other hand, $S' \in \tilde{\Sigma}$, then either $w \notin R(S')$ and $c(S'') = c(S) = y$, or $w \in R(S')$ and by A1 $c(S) = c(S'')$. In the latter case $c(S) = c(S'') = z$ since $c(S) \in zw$ and $w \notin S''$. Hence if $c(S') = y$, then $c(S'') \in yz$.

Now suppose that $c(S') = z$. If $S' \notin \tilde{\Sigma}$, then $c(S'') = z$. If, on the other hand, $S' \in \tilde{\Sigma}$, then $R(S') = yz$ since otherwise $z P_0 y$, a contradiction. Hence

$c(S'') = c(S') = z$. Consequently, if $c(S') = z$, then $c(S'') = z$.

Finally suppose that $c(S') = w$. We will first consider the case $S' \notin \tilde{\Sigma}$, where we will exploit the fact that $S \in \tilde{\Sigma}$. Let $\alpha = h(S)$ and $\beta = h(S')$. Hence $\alpha = c(S\backslash z) = c(S\backslash w)$ by A1, and $\beta = c(S'')$ by definition. Note that B2* implies $c(\alpha w) = \alpha$ and A2 implies $c(\beta w) = w$. We conclude $\alpha \neq \beta$. If $\beta = y$, then there is nothing to show. If $\beta \neq y$, then $\beta \in S\backslash w$. Since $c(S''\backslash y) = c(S\backslash w) = h(S) = \alpha \neq \beta = c(S'')$, $y \in R(S'')$, and, by A4, $R(S'') = \beta y$. Now if $\beta \neq z$, $z \in S''\backslash R(S'')$ and there exists $s \in S$ such that sP_0z, contradicting Lemma 1. Hence $\beta = z$ and $c(S'') \in yz$, as we wanted to show. Now suppose that $S' \in \tilde{\Sigma}$. Clearly $w = c(S') \in R(S')$ and by A4 there exists a unique nontrivial reverser in S'. It follows that $z \notin R(S')$ since $z\not{P}_0 y$. If there exists some $s \in S'\backslash wyz$ such that $R(S') = sw$, then sP_0z, a contradiction to Lemma 1, as $s \in S'\backslash wyz \subset S\backslash R(S)$. Hence $R(S') = wy$. Now by A1 $c(S'\backslash w) = c(S'\backslash y)$ and since $S'\backslash y = S$, $c(S'\backslash w) = c(S'\backslash y) \in R(S) = wz$, giving $c(S'') = c(S'\backslash w) = z$. We conclude that if $c(S') = w$, then $c(S'') \in yz$, and this establishes the *claim.* ■

We are now ready to prove that P_0 is transitive if c satisfies our axioms. The proof will rely on Lemmas 1-3.

Lemma 4 If c satisfies A1-A4, then P_0 is transitive.

Proof. Fix a choice function c which satisfies A1-A4 and distinct alternatives x, y and z such that xP_0y and yP_0z. We will show that xP_0z. There exists, by definition, $S \in \tilde{\Sigma}$ such that $x \in R(S)$ and $y \in S\backslash R(S)$. Using A4, write $R(S) = xw$ where $w \in S\backslash xy$ and note that wP_0y as well. If $z \in S$, then $z \notin R(S)$, since otherwise zP_0y. It follows that xP_0z, as we needed.

If $z \notin S$, the argument is less trivial. Our strategy in this case is to show that $S \cup z \in \tilde{\Sigma}$ and $R(S \cup z) = xw$. This will establish that xP_0z.

To prove our claim, let $S' = S \cup z$ and note that $c(S') \in xwz$ by A3. We have yP_0z by hypothesis and $y\not{P}_0x$ by asymmetry of P_0 as established in

Lemma 1. Now Lemma 3 implies that $z \neq c(xyz)$. If $c(S') = z$, then zP_0y by Lemma 2, a contradiction. Hence $c(S') \in xw = R(S)$.

Now suppose, towards a contradiction, that $S' \notin \tilde{\Sigma}$. Let $\alpha = h(S')$ and $\beta = h(S)$. A2 implies $c(\alpha c(S')) = c(S')$. Since $c(S') \in R(S)$, B2* gives $c(\beta c(S')) = \beta$. Thus $\alpha \neq \beta$. Now using B1, we conclude $c(\alpha\beta) = \beta$. Note that $c(S' \backslash c(S')) = \alpha \neq \beta = c(S \backslash x) = c(S \backslash w)$ which implies, regardless of whether $c(S') = x$ or $c(S') = w$, $z \in R(S' \backslash c(S'))$. Hence $h(S' \backslash c(S')) = c(S' \backslash c(S')z) = \beta$ and $S' \backslash c(S') \in \tilde{\Sigma}$ by A2, as $c(\alpha\beta) = \beta$. It follows that $y \in R(S' \backslash c(S'))$ as well since $z \not{P}_0 y$ by Lemma 1. Now if $c(S') = x$, then $w \in R(S' \backslash c(S'))$ since otherwise yP_0w, a contradiction. If $c(S') = w$, on the other hand, $x \in R(S' \backslash c(S'))$ so that $y \not{P}_0 x$. Hence $S' \backslash c(S')$ has at least three reversers, y, z and either x or w, contradicting A4. We conclude $S' \in \tilde{\Sigma}$.

Note, using asymmetry of P_0, $z \notin R(S')$ as otherwise $y, x \in R(S')$ as well, violating A4. Similarly if $s \in S \backslash xw$, $s \notin R(S')$ either, as otherwise $x, w \in R(S')$ as well, again violating A4. We conclude that $R(S') = xw$ and xP_0z, as we needed to show. ■

6.2.2 The criteria

Now we are ready to define the binary relations which will help us express any choice function satisfying A1-A4 as a TSC. For any c and any x and y, define

$$xP_1y \text{ iff } \begin{cases} \text{either } xP_0y, \\ \text{or } y\not{P}_0x \text{ and } c(xy) = x, \end{cases}$$

and

$$xP_2y \text{ iff } c(xy) = x.$$

As usual, our suppression of the dependence of P_1 and P_2 on c should not cause confusion.

The binary relation P_2 is defined straightforwardly by choices on doubletons. The binary relation P_1 is a particular completion of P_0 which follows the choices over doubletons whenever x and y are incomparable in P_0. We now show that P_1 and P_2 are linear orders if c satisfies our axioms. This is straightforward for P_2, whereas Lemmas 1-4 play critical roles in establishing the properties of P_1.

Lemma 5 *If c satisfies A1-A4, then P_1 and P_2 are linear orders.*

Proof. We take a choice function c satisfying A1-A4. Now P_2 is asymmetric and complete by definition of a choice function and it is transitive by B1.

We move on to studying P_1. Given any two alternatives x and y, if xP_0y then xP_1y, if yP_0x then yP_1x and if $x\not{P}_0y$ and $y\not{P}_0x$ then $c(xy)$ determines xP_1y or yP_1x. Hence P_1 is complete by definition. Now we will show that it is asymmetric. Suppose that xP_1y. If xP_0y, then $y\not{P}_0x$ by Lemma 1 and $y\not{P}_1x$ by definition. If $x\not{P}_0y$, on the other hand, then $y\not{P}_0x$ and $c(xy) = x$. It follows that $y\not{P}_1x$ as neither yP_0x nor $c(xy) = y$. We will finish the proof by showing that P_1 is transitive. Suppose xP_1y and yP_1z for distinct x, y and z. By definition this could happen in four distinct scenarios.

Case 1: If xP_0y and yP_0z, then xP_0z as we showed in Lemma 4, and therefore xP_1z as well.

Case 2: Suppose xP_0y, $z\not{P}_0y$ and $c(yz) = y$. We will show that xP_0z. There must exist $S \in \tilde{\Sigma}$ such that $R(S) = xw$ for some $w \in S\backslash xy$ and $y \in S\backslash R(S)$. If $z \in S$, then $z \notin R(S)$, since otherwise $z = w$ and zP_0y, a contradiction. This gives xP_0z as we wanted to show. If $z \notin S$, then call $S' = S \cup z$ and note that $c(S') \in wxz$ by A3. We must have $z \neq c(S')$ since otherwise, as $y = c(yz)$, zP_0y by Lemma 2, a contradiction. Hence $c(S') \in xw = R(S)$. The rest of the proof proceeds exactly as the proof of Lemma 4: it follows that $S' \in \tilde{\Sigma}$, $R(S') = xw$ and therefore xP_0z.

Case 3: Suppose $y\not{P}_0x$, $c(xy) = x$ and yP_0z. We will show that xP_1z. First note that since P_0 is transitive by Lemma 4, $z\not{P}_0x$, since otherwise yP_0x, a contradiction. Next note that since yP_0z and $y\not{P}_0x$, $z \neq c(xyz)$ by Lemma 3. Furthermore $c(xyz) \neq y$ either, since otherwise $xyz \in \tilde{\Sigma}$, with $R(xyz) = yz$, and consequently yP_0x, a contradiction. We conclude that

$c(xyz) = x$. Now if $c(xz) = x$, then xP_1z since $z\not P_0x$ as we noted before. If $c(xz) = z$, on the other hand, then $xyz \in \tilde{\Sigma}$, with $R(xyz) = xy$, and xP_0z.

Case 4: Finally suppose that $y\not P_0x$, $c(xy) = x$, $z\not P_0y$ and $c(yz) = y$. We note by B1 that $c(xz) = x$. All that remains to show is that $z\not P_0x$. Suppose not: zP_0x. Since $z\not P_0y$ by assumption, Lemma 3 implies $x \neq c(xyz)$. Since z is not the choice in any doubleton subset of xyz, $z \neq c(xyz)$ either, by A1. It follows $c(xyz) = y$, $R(xyz) = yz$ and yP_0x, a contradiction. This finishes the proof of Lemma 5. ■

6.2.3 Proof of Theorem 2: "if"

We finish the proof of Theorem 2 by showing that every choice function which satisfies A1-A4 is a two-stage chooser with the associated criteria (P_1, P_2). Our technical Lemmas 2 and 3 will, again, play roles in the argument. We will also critically rely on the formulation of P_1.

Proof of Theorem 2: "if". Take a choice function c which satisfies A1-A4 and a set S, and let $x = c(S)$. If $S \in \tilde{\Sigma}$ and then $R(S) = xy$ for some $y \in S\backslash x$ by A4 and, by definition, xP_0z and yP_0z for all $z \in S\backslash xy$. Since P_1 follows from P_0, $T^2_{P_1}(S) = xy$. Now B3 implies $c(xy) = x$ as well and consequently xP_2y. Hence $x = \max_{P_2} T^2_{P_1}(S)$.

Suppose now that $S \notin \tilde{\Sigma}$. Towards a contradiction, further suppose that $x \notin T^2_{P_1}(S)$. Then, since P_1 is a linear order, there exist distinct alternatives $y, z \in S\backslash x$ such that yP_1z and zP_1x. Thus $z\not P_0y$ since otherwise zP_1y, contradicting the asymmetry of P_1. Furthermore one of two things is true by definition of P_1. Either zP_0x or $x\not P_0z$ and $c(xz) = z$. In the former case, Lemma 3 implies $x \neq c(xyz)$. However since $c(S) = x$ and $xyz \subseteq S$, xP_0y and xP_0z by Lemma 2, a contradiction. In the latter case we must have $c(xyz) = x$, since otherwise Lemma 2 implies xP_0y or xP_0z, a contradiction. It follows that $R(xyz) = xy$ and xP_0z, a contradiction. We conclude that $x \in T^2_{P_1}(S)$.

Now take any $y \in S$ such that yP_2x, i.e., $c(xy) = y$. Let $z = h(S)$. By A2, $c(xz) = x$. It follows, by A1, that $c(xyz) \in xy$. If $c(xyz) = x$, then $R(xyz) = xz$ and hence xP_0y and zP_0y. If $c(xyz) = y$, on the other hand, S must contain at least four alternatives, since $x = c(S)$ and $xyz \subseteq S$. Now Lemma 2 implies xP_0y and aP_0y for some $a \in S \backslash xyz$. Hence $y \notin T^2_{P_1}(S)$ and the proof is complete. ■

7 Conclusion

In this paper we axiomatize a family of boundedly rational choice functions, which we call two-stage choosers. Two-stage choosers could arise when the decision maker has two possibly conflicting preferences and uses the second preference to make a choice out of the top two alternatives of the first. An alternative interpretation is that the decision maker only considers the top two alternatives on a list to be feasible and makes a choice among them using her preference. We leave two questions for future work.

First, consider the natural extension where the two-stage chooser shortlists top-k rather than top-2 alternatives of the first criterion, for $k > 2$. We conjecture that an analog of Theorem 2 will apply and the procedure will be characterized by A1-A3 and the following modification of A4:

A4' For every $A \in \tilde{\Sigma}$, $|R(A)| = k$..

A related open question is the analysis of the generalization where the decision-maker shortlists the top-$k(A)$ elements of the first criterion in every set A. Now the map $k(\cdot)$ could measure the changing attention capacity of the decision-maker, or the changing bargaining power of the first criterion. We leave for future work a systematic study of this generalization for various $k(\cdot)$ functions.

Appendix

The following four examples show that the axiomatic system A1-A4 is tight, in other words, no three among A1-A4 imply the fourth. We will use the following notation: $C_c(x) = \{A : |A| > 1 \text{ and } c(A) = x\}$.

Example 5 A2, A3 and A4 do not imply A1. Let $X = xyzw$. Consider the following choice function:

$$\begin{aligned} C_c(x) &= \{xy, xz, wx, xyz\} \\ C_c(y) &= \{yz, wy, wxy\} \\ C_c(z) &= \{wz, wxz, wyz, X\} \\ C_c(w) &= \varnothing \end{aligned}$$

Note that $\tilde{\Sigma} = \{wxy, wxz, wyz, X\}$, $R(X) = R(wyz) = R(wxz) = wz$ and $R(wxy) = wy$. Hence c satisfies A3 and A4. Moreover the unique set with three or more elements and which does not belong to $\tilde{\Sigma}$ is xyz, where x is the choice and z is the hidden choice. Since $c(xz) = x$, c also satisfies A2. A1 fails because $R(X) = wz$ but $c(X \backslash z) = y \neq x = c(X \backslash w)$. ▲

Example 6 A1, A3 and A4 do not imply A2. Let $X = xyzw$. Consider the following choice function:

$$\begin{aligned} C_c(x) &= \{xy, xz, wx, wxz\} \\ C_c(y) &= \{yz, wy, xyz, wxy, wyz, X\} \\ C_c(z) &= \{wz\} \\ C_c(w) &= \varnothing \end{aligned}$$

Note that $\tilde{\Sigma} = \{xyz, wxy\}$, $R(xyz) = yz$ and $R(wxy) = wy$. Hence c satisfies A3 and A4. Furthermore A1 follows since $c(xyz \backslash y) = c(xyz \backslash z) = x$

and $c(wxy \backslash w) = c(wxy \backslash y) = x$. However A2 fails because $X \notin \tilde{\Sigma}$ but $c(c(X)h(X)) = c(yx) = x = h(X)$. ▲

Example 7 A1, A2 and A4 do not imply A3. Let $X = vwxyz$ and consider the following choice function*:*

$$\begin{aligned}
C_x(v) &= \{vz\} \\
C_x(w) &= \{vw, wx, wy, wz, vwy, vwz, wxy, wxz\} \\
C_x(x) &= \{vx, xy, xz, vwx, vxy, vxz, vwxy, vwxz\} \\
C_x(y) &= \{vy, yz, vyz, wyz, xyz, vwyz, vxyz, wxyz, X\} \\
C_x(z) &= \varnothing
\end{aligned}$$

Note that $\tilde{\Sigma}$ contains X, all subsets of X with four elements, and the following three-element sets: vwx, wyz, xyz. One can check that there is a unique nontrivial reverser in all sets in $\tilde{\Sigma}$, which is v in vwx, $vwxy$ and $vwxz$, and z otherwise. Thus c satisfies A4. One can also check that the removal of the choice and the removal of the nontrivial reverser leads to the choice of the hidden choice in all sets with choice reversal, which is x in X and in all sets that do not contain w, and w otherwise. This gives A1. To check A2, consider the following table listing all sets with three or more elements outside $\tilde{\Sigma}$, their choices and hidden choices.

A	vwy	vwz	wxy	wxz	vxy	vxz	vyz
$c(A)$	w	w	w	w	x	x	y
$h(A)$	y	v	x	x	y	v	v

Now note that the choice is always chosen over the hidden choice in binary comparison. Hence c satisfies A2 as well. A3 fails because $z \in R(X)$, $vwxz \in \tilde{\Sigma}$ but $R(vwxz) = vx$.

Example 8 A1, A2 and A3 do not imply A4. Let $X = vwxyz$ and $\succ_1$ *and* $\succ_2$ *be linear orders given by* $x \succ_1 y \succ_1 z \succ_1 w \succ_1 v$ *and* $v \succ_2 w \succ_2 x \succ_2$

$y \succ_2 z$. Consider the following extension of two-stage choosers: following choice function: $c(A) = \max_{\succ_2}(T^3_{\succ_1}(A))$ for every A. Hence c shortlists the top-3 alternatives in $\succ_1$ chooses from the shortlist using $\succ_1$. The resulting choices are as follows:

$$
\begin{aligned}
C_x(v) &= \{vw, vx, vy, vz, vwx, vwy, vwz, vxy, vxz, vyz\} \\
C_x(w) &= \{wx, wy, wz, wxy, wxz, wyz, vwxy, vwxz, vwyz\} \\
C_x(x) &= \{xy, xz, xyz, vxyz, wxyz, X\} \\
C_x(y) &= \{yz\} \\
C_x(z) &= \varnothing
\end{aligned}
$$

Now X and all its subsets with four alternatives have choice reversal. Clearly A4 fails as all sets with choice reversal have three choice reversers: $R(X) = R(wxyz) = R(vxyz) = xyz$, $R(vwyz) = wyz$, $R(vwxz) = wxz$ and $R(vwxy) = wxy$. It is straightforward to check that c satisfies A1, A2 and A3. ▲

References

[1] G. de Clippel and K. Eliaz (2012), "Reason-Based Choice", Theoretical Economics, 7: 125-162

[2] R. Dutta and S. Horan (2013), "Inferring Rationales from Choice: Identification for Rational Shortlist Methods", UQAM working paper.

[3] K. Eliaz, M. Richter and A. Rubinstein (2011), "Choosing the Two Finalists", Economic Theory, 46: 211-219.

[4] D. Kahnemann and A. Tversky (2000), Choice, Values, and Frames. Cambridge U.K.: Cambridge University Press.

[5] J. Lleras, Y. Masatlıoğlu, D. Nakajima and E. Özbay (2011), "When More is Less: Choice by Limited Consideration", working paper

[6] P. Manzini and M. Mariotti (2007). "Sequentially Rationalizable Choice", American Economic Review, 97(5): 1824-1839

[7] Y. Masatlıoğlu, D. Nakajima and E. Özbay (2012), "Revealed Attention", American Economic Review, 102(5): 2183-2205

[8] H. Moulin (1985), "Choice Functions over a Finite Set: a Summary," Social Choice and Welfare, 2: 147-160.

[9] A. Rubinstein and Y. Salant (2006), "A Model of Choice from Lists", Theoretical Economics, 1: 3-17

[10] A. Rubinstein and Y. Salant (2008), "(A,f), Choice with Frames", Review of Economic Studies, 75: 1287-1296

[11] B. Schwartz (2005), The Paradox of Choice: Why more is Less. Harper Perennial.

[12] H. Simon, (1955). "A Behavioral Model of Rational Choice", Quarterly Journal of Economics, 69: 99-118

[13] K. Yildiz (2012), "List Rationalizable Choice", Working Paper