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Abstract

We provide a framework to evaluate whether or not a seller can increase his revenue in interacting with a privately informed buyer by using money-back guarantees (MBGs). The buyer's value for the good exhibits *fit risk* and his type is multidimensional giving the probability of fit as well as the value in case of fit. The seller has the option to offer a MBG together with the good. We reformulate the optimal mechanism design problem and show that typically the optimal mechanism contains MBGs. Furthermore choosing the optimal mechanism is tantamaount to choosing two prices: (i) a discount price at which no MBG is offered and (ii) a regular (higher) price which comes with a MBG. We also analyze two limit scenarios where private information is one-dimensional. If the seller knows the probability of fit but not its value, then MBGs are not useful. If, on the other hand, the value of fit is commonly known but its probability is buyer's private information, then MBGs can be used to *extract full surplus* from the buyer.

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1 Introduction

A money-back guarantee (MBG) is a seller's commitment to refund the price back to the buyer in case of a return.[1] For goods without a defect and which are returned on the basis of buyer-specific fit issues, it is in the seller's discretion to offer MBG. In this paper we study optimal mechanism design for a monopolistic seller who has recourse to such voluntary MBGs.[2] We formulate this question in a screening model which generalizes the standard textbook scenario in two ways. First the possibility of offering MBG gives the seller an additional tool to screen buyer types. Hence the allocation space is multidimensional. Relatedly, the buyer's private information is multidimensional as well, giving his value for the good and his value for MBG.

The environment we consider is characterized by *fit risk*: the buyer's value for the good is either $v > 0$, or 0, depending on whether or not it fits his specific needs. Fit occurs with probability p and it can only be determined after purchase. The buyer's type is the pair (p, v). Modeling p as part of buyer's private information falls squarely with our focus on buyer-specific reasons for returns as opposed to, say, defects which should affect all buyer types symmetrically.

In the absence of MBGs, our model reduces to a special case where the relevant private information is the product pv, the buyer's expected value for the good. This is the classical single-dimensional problem. With the possibility of offering MBG, however, the seller could serve two types differently if their valuation for MBG is different, even if they have the same expected value for the good.

In previous literature (e.g., Armstrong, 1996; Manelli and Vincent, 2007), optimal

[1] Heiman et al. (2002) observe that retailers and manufacturers provide MBGs to help resolve fit risk. Product returns are an enormous phenomenon in the US market, exceeding $100 billion annually in the US (Stock et al. 2002). The impact of MBGs on purchase decisions can vary greatly by product category, consumer type and distribution channel. For example, Anderson et al. (2009) found that for one catalog retailer, average product return rates were 23%, 14% and 29% for women's tops, men's tops and women's footwear respectively, and that offering MBGs increased demand by 16%, 9% and 53% respectively in these same categories.

[2] Our concern in this paper is not on returns due to product defects–these are generally covered by warranties and consumes have some legal protection for these occurrences. Rather, our focus is on returns which are subject to voluntary retailer discretion. In our context, MBGs reflect the fact that in offering this guarantee the retailer publicly agrees to fully refund the purchase price to a dissatisfied customer even when the product is not defective.

mechanism design with multidimensional types has proved a challenge, as the methods of Myerson (1981) can not be applied in suitably characterizing incentive constraints, while at the same time tractably separating the choice of the allocation from the choice of payments. We are, however, able to give a recipe for solving our mechanism design problem by exploiting the nature of MBGs. We critically rely on the following restriction: if a MBG is offered, it must be a *full* reimbursement of the price. This restriction may be with loss of generality, as perhaps the seller could benefit from offering a partial MBG. However it has strong support in business practice. Moreover in their guideline for guarantee practices, the Federal Trade Commission defines MBGs as a full reimbursement:

> A seller... should use the term "Money Back Guarantee"... only if the seller refunds the full purchase price... at the purchaser's request. (FTC, Guide for Advertising Warranties and Guarantees)

We exploit in our analysis the nontrivial relationship between the payment and the allocation introduced by MBGs. We show (Corollary 1 to Proposition 1) that optimal mechanism design is tantamount to choosing two prices optimally: a high price which comes with a MBG and a low price at which no MBG is offered. Using such a mechanism the seller is able to separate types with higher p from those for whom fit is less likely. The first group receives a low price with no MBG while the second group is charged a higher price but they are given the MBG. The optimal choice of these two prices depends on the distribution of types. We show (Proposition 2) that if either p and v are statistically independent, or if their joint distribution is smooth, then the optimal mechanism contains some MBGs. As we show in the Appendix, this analysis applies, almost verbatim, to a more general and higher dimensional scenario where the non-fit value is not fixed at zero, and it is also a part of buyer's type.

We then relax the restriction that the MBGs should be full while turning attention to two limiting scenarios regarding buyer's type. Interestingly, the full reimbursement condition is satisfied by the optimal mechanism in both scenarios, although MBGs assume starkly contrasting roles. We show (Proposition 3) that if p is common knowledge and the buyer's type is v, then MBGs play no role in the optimal mechanism. In this scenario, MBG acts as money, in that its marginal impact on the buyer's utility does not depend

on the buyer's private information. Hence any mechanism with MBG can be replicated, in payoffs to both parties as well as in incentive properties, by a mechanism without MBG by appropriately adjusting payments. This is precisely the content of Proposition 3.

In the alternate scenario where v is common knowledge and buyer's type is p, we show (Proposition 4) that the seller *extracts full surplus* from the buyer using a MBG which fully reimburses the price. In other words there is an incentive compatible and individually rational mechanism which is efficient, which gives full MBGs to all types and which leaves all types with zero information rent. To the best of our knowledge, this is the first instance in the literature where full surplus extraction occurs with a single agent in an economically meaningful incomplete information environment.

Relation to literature A common theoretical explanation for MBGs is that they help signal quality. In these models, consumers are uncertain about quality, and the better quality firm gains consumer trust by offering MBGs (e.g., Mann and Wissink, 1990; Moorthy and Srinavasan, 1995). The other traditional explanation for MBGs is that a risk neutral seller can gain from offering the MBG as insurance to a risk averse consumer, who runs the risk of being dissatisfied by the product (e.g., Heal, 1977; Che, 1996). More recently, it has been suggested that MBGs can arise when the seller has a higher salvage value for the returned product than does the dissatisfied customer (Davis et al., 1995 and McWilliams, 2012).

Our model is one of screening. To eliminate salvage value as a possible explanation, we assume zero production and zero return costs, while modeling consumers as risk neutral eliminates the standard insurance motivation. Whereas the signaling models assume that the seller has private information, we assume the reverse, i.e., that the buyer is privately informed.

Others have studied the pure screening role of MBGs and other types of warranties, but in differing frameworks. Matthews and Moore (1987), though related, is more about warranties than MBGs. The risk averse buyer's type is one dimensional. Possibility of providing different quality warranties makes the allocation set multidimensional. Warranties serve as insurance and different kinds of warranties are used to screen the types. Courty and Li (2000) consider a model where a buyer has two possible types, each characterized by an uncertainty about the value of the good. The more informed buyer type gets a refund equal to the sellers fixed cost of production (zero in our case), while the less

informed buyer can get a refund greater than, equal to or less than the cost (depending on the functional form of the distribution of types). In the second part of the paper, like us, they consider a continuum of types. However, to reduce complexity, they only consider a class of examples where the distribution and the realized value are a function of a common parameter. Thus, they essentially reduce their analyses to one dimensional (albeit complex) types. Furthermore, the allocation set does not have refunds. Screening is achieved by providing or not providing the good (cutoffs) and charging different prices. Matthews and Persico (2005) show that in a model with two kinds of buyers, those who know their value and those who are uncertain about it, a monopolist could benefit from MBGs if the proportion of the second kind of buyers to the first is high enough. In comparison, in our model all types are uncertain about their values and consequently, under general conditions, it is always to the benefit of the seller to offer MBGs.

2 Environment and main results

A seller of an indivisible good is interacting with a privately informed buyer. Both parties are risk-neutral. The seller has no cost. The good has a random value to the buyer in the sense of exhibiting the following form of fit risk: with probability p it fits the buyer's needs and its value is $v > 0$. Otherwise, with probability $1 - p$, there is no fit and the value of the good is 0. Whether the good fits or not is only observed by the buyer after purchase. The buyer's type is the pair (p, v), distributed in the type space $T = (0, 1)^2$ with a strictly positive density f.

An *outcome* is a triple (α, γ, π) where $\alpha \in \{0, 1\}$ indicates sale or no sale, $\gamma \geq 0$ is a money-back guarantee (MBG), and $\pi \geq 0$ is the price. If the good does not change hands, i.e., if $\alpha = 0$, then there is no payment and no MBG, i.e., $\gamma = \pi = 0$.[3] Hence the outcome belongs to the set

$$C = \{(\alpha, \gamma, \pi) \in \{0, 1\} \times \Re_+ \times \Re_+ : \alpha = 0 \Rightarrow \gamma = \pi = 0\}.$$

Given $(\alpha, \gamma, \pi) \in C$ and $(p, v) \in T$, the payoffs are determined as follows. Both parties receive zero payoff if $\alpha = 0$. If $\alpha = 1$, the buyer pays the seller π. Next he observes the

[3] These restrictions are without loss of generality and help us keep the analysis simple. Optimality of the sellers revenue and the buyer's participation constraint would imply that there is no loss of payoff in setting $\gamma = \pi = 0$ when $\alpha = 0$. Optimal revenue would imply that $\pi \geq 0$ when $\alpha = 1$.

fit of the good and returns the good if and only if the MBG exceeds its value.[4] Returning the good is costless on both parties. The resulting expected payoff of the buyer is

$$\alpha(p\max\{v,\gamma\}+(1-p)\gamma)-\pi.$$

The seller has no value for the good. He receives the payment π and pays the MBG γ if the good is returned. Hence his expected payoff is given by

$$\begin{aligned}\pi-(1-p)\alpha\gamma \text{ if } \gamma &\leq v,\\ \pi-\alpha\gamma \text{ if } \gamma &> v.\end{aligned}$$

Note that if there is no MBG, i.e., if $\gamma=0$, we specialize to the single dimensional scenario where the buyer's value is pv and the payoffs are $\alpha(pv)-\pi$ and π respectively for the buyer and the seller.

A *mechanism* maps types (p,v) into outcomes (α,γ,π). Abusing notation, we will denote a mechanism by a triple of functions $(\alpha,\gamma,\pi):T\to C$. The following two conditions are the classical incentive constraints in mechanism design.

Incentive compatibility (IC): for any two types (p,v) and (p',v')

$$\begin{aligned}&\alpha(p,v)(p\max\{v,\gamma(p,v)\}+(1-p)\gamma(p,v))-\pi(p,v)\\ &\qquad\geq \alpha(p',v')(p\max\{v,\gamma(p',v')\}+(1-p)\gamma(p',v'))-\pi(p',v').\end{aligned}$$

Individual rationality (IR): for any type (p,v)

$$\alpha(p,v)(p\max\{v,\gamma(p,v)\}+(1-p)\gamma(p,v))-\pi(p,v)\geq 0.$$

These two conditions say that (1) the buyer cannot gain by misreporting to the mechanism, and (2) the truthful report earns the buyer a payoff at least equal to his outside option, which we assume is zero. Note that our incentive compatibility condition is in dominant strategies. We will next introduce a key condition to rule out partial MBGs.

Full reimbursement (FR):. For any type (p,v), $\gamma(p,v)>0$ implies $\gamma(p,v)=\pi(p,v)$.

FR requires any positive MBG to be a full reimbursement of the price of the good. Note, importantly, that FR does not impose that a MBG be offered by the seller.

[4] This restriction may be viewed as an implication of sequential rationality of the buyer.

For any mechanism (α, γ, π) which satisfies the three conditions above and for any type (p, v), let $R^{(\alpha,\gamma,\pi)}(p, v)$ be the seller's ex post payoff from using (α, γ, π) when the buyer type is (p, v), i.e.,

$$R^{(\alpha,\gamma,\pi)}(p, v) = \begin{cases} \pi(p, v) - (1-p)\alpha(p, v)\gamma(p, v) & \text{if } \gamma(p, v) \leq v, \\ \pi(p, v) - \alpha(p, v)\gamma(p, v) & \text{if } \gamma(p, v) > v. \end{cases}$$

We are interested in the following *optimal mechanism design problem*:

$$\max_{(\alpha,\gamma,\pi):T\to C} \int_0^1 \int_0^1 R^{(\alpha,\gamma,\pi)}(p, v) f(p, v) dp dv$$
$$\text{s.t. IC, IR and FR.} \tag{P1}$$

We will call a mechanism *optimal* if it solves P1.

In principle, the seller could offer a high enough MBG which will induce the buyer to return the good regardless of its fit. If this is true for some type (p, v), under FR, the buyer's payoff is $\gamma(p, v) - \pi(p, v) = 0$. Correspondingly, the seller's ex post payoff from interacting with this type is $\pi(p, v) - \gamma(p, v) = 0$ as well. Our first observation is that the seller will never find offering such a high MBG profitable.

Lemma 1 *Suppose (α, γ, π) satisfies IC, IR and FR. If $\gamma(p', v') > v'$ for some (p', v'), then there exists a mechanism $(\tilde{\alpha}, \tilde{\gamma}, \tilde{\pi})$ which satisfies IC, IR, FR such that for every (p, v), $\tilde{\gamma}(p, v) \leq v$ and $R^{(\alpha,\gamma,\pi)}(p, v) = R^{(\tilde{\alpha},\tilde{\gamma},\tilde{\pi})}(p, v)$.*

Proof. Fix (α, γ, π) in satisfaction of IC, IR and FR. Define $(\tilde{\alpha}, \tilde{\gamma}, \tilde{\pi})$ as follows:

$$(\tilde{\alpha}(p, v), \tilde{\gamma}(p, v), \tilde{\pi}(p, v)) = \begin{cases} (\alpha(p, v), \gamma(p, v), \pi(p, v)) & \text{if } \gamma(p, v) \leq v, \\ (0, 0, 0) & \text{if } \gamma(p, v) > v. \end{cases}$$

Clearly $\tilde{\gamma}(p, v) \leq v$ for all (p, v) and $(\tilde{\alpha}, \tilde{\gamma}, \tilde{\pi})$ satisfies FR. Furthermore $R^{(\tilde{\alpha},\tilde{\gamma},\tilde{\pi})}(p, v) = R^{(\alpha,\gamma,\pi)}(p, v)$ for all (p, v) as whenever $\gamma(p, v) > v$, $R^{(\alpha,\gamma,\pi)}(p, v) = 0$ by FR.

Note that for every (p, v)

$$\begin{aligned} &\tilde{\alpha}(p, v)(p \max\{v, \tilde{\gamma}(p, v)\} + (1-p)\tilde{\gamma}(p, v)) - \pi(p, v) \\ &\qquad = \alpha(p, v)(p \max\{v, \gamma(p, v)\} + (1-p)\gamma(p, v)) - \pi(p, v). \end{aligned}$$

This is trivially true if $\gamma(p,v) \leq v$ as $(\tilde{\alpha}(p,v), \tilde{\gamma}(p,v), \tilde{\pi}(p,v)) = (\alpha(p,v), \gamma(p,v), \pi(p,v))$. $\gamma(p,v) > v$, on the other hand both sides are zero. Hence the two mechanisms earn the buyer the same payoff regardless of his type. IR of $(\tilde{\alpha}, \tilde{\gamma}, \tilde{\pi})$ directly follows from this observation.

Also note that if $(p', v') \neq (p, v)$

$$\begin{aligned}\alpha(p',v')(p\max\{v,\gamma(p',v')\} + (1-p)\gamma(p',v')) - \pi(p',v') \\ \geq \tilde{\alpha}(p',v')(p\max\{v,\tilde{\gamma}(p',v')\} + (1-p)\tilde{\gamma}(p',v')) - \pi(p',v').\end{aligned}$$

As before this is trivially true if $\gamma(p',v') \leq v'$. If $\gamma(p',v') > v'$, on the other hand, the left-hand side is $p(\max\{v,\gamma(p',v')\} - \gamma(p',v')) \geq 0$ while the right-hand side is 0. Combining the inequalities in the last two displays and using the hypothesis that (α, γ, π) satisfies IC, we conclude that $(\tilde{\alpha}, \tilde{\gamma}, \tilde{\pi})$ satisfies IC as well. ■

Inspired by Lemma 1, let us formulate the condition that the good be returned to the seller for a MBG only if it is not a fit, as a feasibility condition.

Fit compatibility (FC): For any type (p,v), $\gamma(p,v) \leq v$.

If a mechanism satisfies FC, then the buyer will return the good only if it is not a fit.[5] Restricting attention to mechanisms satisfying FC simplifies the seller's payoff and, by Lemma 1, is without loss of generality in solving for the optimal mechanism. In other words, in order to find an optimal mechanism the seller need only solve

$$\max_{(\alpha,\gamma,\pi):T\to C} \int_0^1 \int_0^1 [\pi(p,v) - (1-p)\alpha(p,v)\gamma(p,v)] f(p,v)dpdv$$
$$\text{s.t. IC, IR, FR and FC.} \tag{P2}$$

Next we introduce a class of mechanisms which are feasible in P2.

Definition 1 *Let $0 \leq k \leq m \leq 1$. A mechanism (α, γ, π) is a (k,m) mechanism if*

$$(\alpha(p,v), \gamma(p,v), \pi(p,v)) = \begin{cases} (1,m,m) & \text{if } v \geq m \text{ and } p \leq \frac{k}{m}, \\ (1,0,k) & \text{if } p > \frac{k}{m} \text{ and } pv \geq k, \\ (0,0,0) & \text{otherwise.} \end{cases}$$

[5] Previously we had assumed, due to sequential rationality, that the good be returned if and only if *MBG exceeds the buyer's valuation.* Under the reformulated mechanism, Lemma 1 states that the good is returned if and only if *the good does not fit.*

The (k, m) class contains three kinds of mechanisms as illustrated below, depending on which types receive MBG. If $0 < k < m = 1$, as in the first diagram, the seller does not offer MBGs to any type. The mechanism allocates the good to the agent if his expected value $pv \geq k$ at the price $\pi = k$. This is exactly the case where multidimensionality of the consumer's type (p, v) is inconsequential and the mechanism divides different types with respect to the product pv.

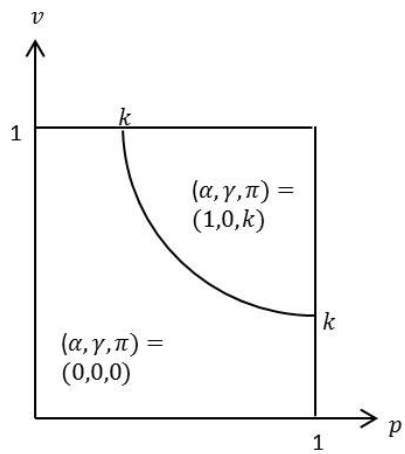

Figure 1
$0 < k < m = 1$

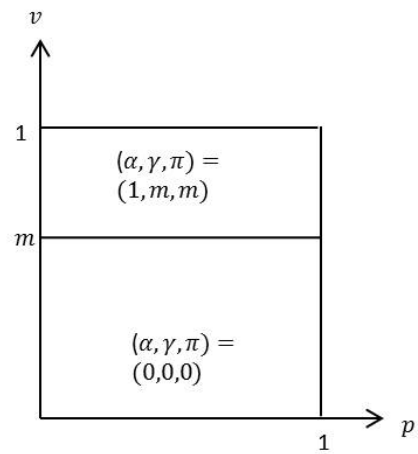

Figure 2
$0 < k = m < 1$

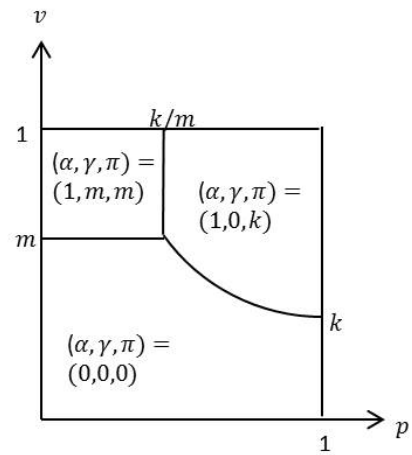

Figure 3
$0 < k < m < 1$

At the other extreme are mechanisms which give MBGs to all types who receive the good. These mechanisms have $m = k$. A typical such mechanism is given in the second diagram. Mixing of these two policies is also feasible, as in the third diagram, by choosing $0 < k < m < 1$. In this case, the mechanism allocates the good to some types at a discount price k with no MBG and to other types at a higher price m with a full MBG.

It is straightforward to check that any (k, m) mechanism satisfies conditions IC, IR, FR and FC and is therefore feasible in problem P2. Next we will show that if a mechanism is feasible in problem P2 then it is "almost" a (k, m) mechanism. First we record a useful consequence of conditions IC and FR.

Lemma 2 *If (α, γ, π) satisfies IC and FR, then there exists $m \in (0, 1)$ such that if $\gamma(p, v) > 0$, then $\gamma(p, v) = m$.*

Proof. Suppose, towards a contradiction, that for two distinct types (p, v) and (p', v'), $0 < \gamma(p', v') < \gamma(p, v)$. Then $\alpha(p', v') = \alpha(p, v) = 1$ and the payoff to the (p, v) type from

a truthful report is

$$pv + (1-p)\gamma(p,v) - \pi(p,v) = p(v - \gamma(p,v))$$

where we use FR in substituting $\gamma(p,v)$ for $\pi(p,v)$. If, instead, the (p,v) type reports (p',v'), then his payoff would be

$$pv + (1-p)\gamma(p',v') - \pi(p',v') = p(v - \gamma(p',v')).$$

IC implies $p(v-\gamma(p,v)) \geq p(v-\gamma(p',v'))$, i.e., $\gamma(p',v') \geq \gamma(p,v)$ since $p > 0$, a contradiction. ■

For any mechanism (α,γ,π) which is feasible in P2, let $(T_1^{(\alpha,\gamma,\pi)}, T_2^{(\alpha,\gamma,\pi)}, T_3^{(\alpha,\gamma,\pi)})$ be the partition of the type space defined by

$$\begin{aligned}
(p,v) &\in T_1^{(\alpha,\gamma,\pi)} \Leftrightarrow \alpha(p,v) = 1 \text{ and } \gamma(p,v) > 0, \\
(p,v) &\in T_2^{(\alpha,\gamma,\pi)} \Leftrightarrow \alpha(p,v) = 1 \text{ and } \gamma(p,v) = 0 \\
(p,v) &\in T_3^{(\alpha,\gamma,\pi)} \Leftrightarrow \alpha(p,v) = 0.
\end{aligned}$$

Note that all types in $T_1^{(\alpha,\gamma,\pi)}$ receive the same MBG by Lemma 2. We will say that two mechanisms are *almost identical* if they generate the same partitions, except perhaps at the boundaries. Given our assumption that types have a strictly positive density, almost identical mechanisms earn the seller the same revenue as they differ only on a set of zero measure.

Proposition 1 *Any mechanism which satisfies IC, IR, FR and FC is almost identical to some (k,m) mechanism.*

Proof. Let (α,γ,π) satisfy IC, IR, FR and FC. The proof relies on Lemma 2 as well as the following three observations.

Claim 1: If $\alpha(p,v) = 1$, then $\alpha(p',v') = 1$ for all (p',v') such that $p' > p$ and $v' > v$.

Proof of Claim 1: Suppose that $\alpha(p,v) = 1$ but $\alpha(p',v') = 0$ for some (p',v') such that $p' > p$ and $v' > v$. If $\gamma(p,v) = 0$, then

$$0 \geq p'v' - \pi(p,v) > pv - \pi(p,v) \geq 0$$

where the weak inequalities follow from incentive compatibility. This is clearly impossible. If $\gamma(p,v) = m > 0$, on the other hand, the impossibility follows similarly from incentive compatibility:

$$0 \geq p'(v' - m) > p(v - m) \geq 0.$$

Claim 2: If $\alpha(p,v) = \alpha(p',v') = 1$ and $p' < p$, then $\gamma(p',v') \geq \gamma(p,v)$.

Proof of Claim 2: Suppose that $\alpha(p,v) = \alpha(p',v') = 1$, $p' < p$ but $\gamma(p',v') < \gamma(p,v)$. Then, by Lemma 2, for some $m > 0$ $\gamma(p,v) = m$ and $\gamma(p',v') = 0$. Incentive compatibility gives

$$p(v - m) \geq pv - \pi(p',v') \text{ and } p'v' - \pi(p',v') \geq p'(v' - m).$$

Rearranging we get $pm \leq \pi(p',v') \leq p'm$, which is an impossibility since $m > 0$.

Claim 3: If $\gamma(p,v) > 0$, then $\gamma(p',v') = \gamma(p,v)$ for all $(p',v') \in (0,p) \times (\gamma(p,v),1)$.

Proof of Claim 3: Suppose $\gamma(p,v) = m > 0$, $p' < p$ and $v' > m$. We will first show that $\alpha(p',v') = 1$. If not, incentive compatibility implies $0 \geq p'(v' - m)$, an impossibility. Now suppose $\gamma(p',v') = 0$. The incentive compatibility conditions are exactly as in the proof of Claim 2 and the same contradiction follows.

Going back to the proof of Proposition 1, if one of the sets in the partition $(T_1^{(\alpha,\gamma,\pi)}, T_2^{(\alpha,\gamma,\pi)}, T_3^{(\alpha,\gamma,\pi)})$ is empty, then the result follows straightforwardly. Suppose all three sets are nonempty. By Lemma 2, there exists $m^* > 0$ such that for any $(p,v) \in T_3^{(\alpha,\gamma,\pi)}$, $\gamma(p,v) = m^*$. By Claim 3 above,

$$\sup\{p : \gamma(p,v) = m^*\} = \sup\{p : \gamma(p,v') = m^*\}$$

for any $v, v' > m^*$. Let p^* be this supremum and $k^* = m^*p^*$. By Claim 1 above, $\alpha(p,v) = 1$ if $p > p^*$ and $v > m$. It follows that $\gamma(p,v) = 0$ for such (p,v) since $p > p^*$. It follows that $k^* = \inf\{p'v' : p > p^* \text{ and } v > m\}$ and if for some (p,v) such that $v \leq m$ and $pv > k^*$, $\alpha(p,v) = 0$, IC fails. Hence $(\alpha(p,v), \gamma(p,v)) = (1,0)$ for all (p,v) such that $p > p^*$ and $pv > k^*$. By IC and IR $\pi(p,v) = k^*$ for any such type. Hence (α,γ,π) is a (k,m) mechanism with $(k,m) = (k^*,m^*)$. ■

Hence, the seller need only find the optimal one among the (k,m) mechanisms, as we record in the following corollary.

Corollary 1 *If the pair* (k^*, m^*) *solves*

$$\max_{k,m} \int_m^1 \int_0^{k/m} mpf(p,v)dpdv + \int_m^1 \int_{k/m}^1 kf(p,v)dpdv + \int_k^m \int_{k/v}^1 kf(p,v)dpdv$$
$$s.t.\ 0 \leq k \leq m \leq 1 \tag{P3}$$

then the (k^*, m^*) *mechanism solves the optimal mechanism design problem P1.*

Problem P3 is our reformulation of the optimal mechanism design problem P1. The objective of P3 is precisely the expected payoff of the seller at a (k, m) mechanism. The first double-integral is over all types which receive the good at the price m and together with the option of returning the good for the MBG m. The seller's revenue at any such type is $m - (1-p)m = pm$. The second and third double-integrals give the seller's expected payoff over all types which receive the good at a discount k but without the MBG. Note that if $m = 1$, the objective becomes $\int_k^1 \int_{k/v}^1 kf(p,v)dpdv$, the revenue in the mechanism which involves no MBGs (Figure 1 above), and if $m = k$, the objective becomes $\int_m^1 \int_0^1 mpf(p,v)dpdv$, the revenue in the mechanism which gives MBGs to all types who receive the good (Figure 2 above).

Next we exhibit the use of this result in computing an optimal mechanism in a specific example.

Example 1 Suppose that p and v are independently distributed and p *is* uniformly distributed. Then it can be easily shown that the optimal mechanism has $k^* = m^*$. That is, the good is always sold with MBG. If both p and v are independent and uniformly distributed. Then the expected payoff of the seller in a (k, m) mechanism is

$$\int_m^1 \int_0^{k/m} mpdpdv + \int_m^1 \int_{k/m}^1 kdpdv + \int_k^m \int_{k/v}^1 kdpdv.$$

Maximizing the expression with respect to (k, m) we find that the optimal mechanism is the (k, m) mechanism with $k^* = m^* = \frac{1}{2}$. Hence the optimal mechanism offers MBGs to all types who receive the good at the price of $\frac{1}{2}$. The corresponding expected revenue of the seller is $\frac{1}{8}$.

Next we will show that under general conditions, the optimal mechanism contains MBGs. In other words, the solution to the reformulated problem P3 has $m < 1$.

Proposition 2 *The seller optimally offers money-back guarantee to some types if one of the following two conditions holds:*

1. *p and v are independently distributed.*

2. *The density f is continuously differentiable.*

Proof. For any $k \in (0,1)$, consider the $(k,1)$ mechanism. This mechanism offers no MBGs. We will show that if $m \in (k,1)$ is sufficiently close to 1, then the seller's expected payoff is larger in the (k,m) mechanism than it is in the $(k,1)$ mechanism.

Switching from the $(k,1)$ mechanism to a (k,m) mechanism entails a loss of expected revenue for all types (p,v) such that $p \in (\frac{v}{m}, \frac{k}{m})$ and $v \in (m,1)$. At any such type the $(k,1)$ mechanism earns the seller k, the price of the good, whereas the $(k,1)$ mechanism brings the expected revenue mp, difference between price m and the expected MBG payment $(1-p)m$ back to the buyer. Note $mp < k$ for this range of p. The benefit from said switch occurs at types (p,v) such that $p \in (0, \frac{v}{m})$, $v \in (m,1)$. The $(k,1)$ mechanism does not serve these types. The (k,m) mechanism serves these types with the MBG m and earns the seller mp in expectation. Hence the switch is profitable if

$$E[mp|p \in (0, \frac{k}{m}),\, v \in (m,1)] > E[k|p \in (\frac{v}{m}, \frac{k}{m}),\, v \in (m,1)].$$

To establish that this is the case, we will show that for some $m \in (k,1)$

$$\frac{\Pr\{p \in (\frac{v}{m}, \frac{k}{m}) \text{ and } v \in (m,1)\}}{E[p|p \in (0, \frac{k}{m}),\, v \in (m,1)]} < \frac{m}{k}.$$

Since the numerator in the left-hand side converges to 0 as m goes to 1, and since the right-hand side is larger than 1, and it suffices if the denominator of the left-hand side has positive limit, i.e.,

$$\lim_{m \to 1} E[p|p \in (0, \frac{k}{m}),\, v \in (m,1)] > 0.$$

Taking the conditional expectation

$$E[p|p \in (0, \frac{k}{m}),\, v \in (m,1)] = \frac{\int_m^1 \int_0^{k/m} pf(p,v)dpdv}{\int_m^1 \int_0^{k/m} f(p,v)dpdv}.$$

Suppose that p and v are independently distributed with strictly positive densities f_p and f_v respectively. The conditional expectation becomes

$$\frac{\int_m^1 \int_0^{k/m} pf(p,v)dpdv}{\int_m^1 \int_0^{k/m} f(p,v)dpdv} = \frac{\int_m^1 f_v(v)dv \int_0^{k/m} pf_p(p)dp}{\int_m^1 f_v(v)dv \int_0^{k/m} f_p(p)dp} = \frac{\int_0^{k/m} pf_p(p)dp}{\int_0^{k/m} f_p(p)dp}$$

and

$$\lim_{m\to 1} E[p|p \in (0, \frac{k}{m}),\ v \in (m,1)] = \frac{\int_0^k pf_p(p)dp}{\int_0^k f_p(p)dp} > 0$$

by the positivity of the densities and the number k, as we wanted to show.

If f and p are not independent, we apply L'Hopital's rule:

$$\lim_{m\to 1} \frac{\int_m^1 \int_0^{k/m} pf(p,v)dpdv}{\int_m^1 \int_0^{k/m} f(p,v)dpdv} = \lim_{m\to 1} \frac{\frac{d}{dm}\left[\int_m^1 \int_0^{k/m} pf(p,v)dpdv\right]}{\frac{d}{dm}\left[\int_m^1 \int_0^{k/m} f(p,v)dpdv\right]}.$$

Define

$$\begin{aligned} H(m,v) &= \int_0^{k/m} pf(p,v)dp, \text{ and} \\ G(m,v) &= \int_0^{k/m} f(p,v)dp. \end{aligned}$$

Since f is continuously differentiable, so are the integrands in these expressions and we can use Leibnitz Theorem as follows:

$$\begin{aligned} \lim_{m\to 1} \frac{\frac{d}{dm}\left[\int_m^1 H(m,v)dv\right]}{\frac{d}{dm}\left[\int_m^1 G(m,v)dv\right]} &= \lim_{m\to 1} \frac{-H(m,m) + \int_m^1 \frac{\partial}{\partial m} H(m,v)dv}{-G(m,m) + \int_m^1 \frac{\partial}{\partial m} G(m,v)dv} \\ &= \lim_{m\to 1} \frac{-H(m,m) + \int_m^1 \frac{k}{m} f(\frac{k}{m},v)dv}{-G(m,m) + \int_m^1 f(\frac{k}{m},v)dv} \\ &= \lim_{m\to 1} \frac{H(m,m)}{G(m,m)}. \end{aligned}$$

Now $m \mapsto H(m,m)$ and $m \mapsto G(m,m)$ are continuous because f is so. Hence

$$\lim_{m\to 1} \frac{H(m,m)}{G(m,m)} = \frac{H(1,1)}{G(1,1)} = \frac{\int_0^k xf(x,1)dx}{\int_0^k f(x,1)dx} = \Pr\{p|p < k \text{ and } v = 1\} > 0,$$

which is what we needed to show. ■

3 Limit cases: one-dimensional private information

In this section, we will study two special cases in which the buyer's private information, v or p. In these cases we can more directly analyze the standard optimal mechanism design problem where the feasibility constraints are IC and IR. We will show that the full reimbursement condition FR as well as the fit compatibility condition FC are satisfied by the optimal mechanism in both cases. However the nature of optimal mechanisms are quite different.

Scenario 1: v is private information and p is common knowledge. Suppose that $v \in (0,1)$ is buyer's type, while $p \in (0,1)$ is common knowledge. The following definitions are direct adaptations of those given for the multidimensional model earlier. A *mechanism* is a map $(\alpha,\gamma,\pi) : (0,1) \to C$ associating an outcome $(\alpha(v),\gamma(v),\pi(v))$ with every type v. A mechanism (α,γ,π) is *incentive compatible* if for every $v, v' \in (0,1)$, $\alpha(v)[pv + (1-p)\gamma(v)] - \pi(v) \geq \alpha(v')[pv + (1-p)\gamma(v')] - \pi(v')$. A mechanism (α,γ,π) is *individually rational* if for every $v \in (0,1)$, $\alpha(v)[pv + (1-p)\gamma(v)] - \pi(v) \geq 0$. If a mechanism (α,γ,π) is incentive compatible and individually rational, it earns the seller the expected profit $\int_0^1 [\pi(v) - (1-p)\alpha(v)\gamma(v)]f(v)dv$ where f is the distribution of v. The revelation principle (Myerson, 1981) tells us that the profit maximizing selling strategy is given by the mechanism which maximizes this expected payoff within the class of incentive compatible and individually rational mechanisms. Hence the optimal mechanism design in this scenario is :

$$\max_{(\alpha,\gamma,\pi)} \int_0^1 [\pi(v) - (1-p)\alpha(v)\gamma(v)]f(v)dv$$
$$\text{s.t. IC and IR.} \tag{P4}$$

The following result indicates that in this scenario, the seller has no incentive to use MBGs.

Proposition 3 *There is a solution to problem P4 which involves no money-back guarantees.*

Proof. Let (α, γ, π) be incentive compatible and individually rational with $\gamma \neq 0$, i.e., some type receives MBG. Consider the alternate mechanism (α, γ', π') where

$$\begin{aligned} \gamma'(v) &= 0, \text{ and} \\ \pi'(v) &= \pi(v) - (1-p)\alpha(v)\gamma(v) \end{aligned}$$

for all v. Suppose that the buyer's type is v and he reports v' to the mechanism (α, γ', π'). His payoff is

$$\begin{aligned} \alpha(v')pv - \pi'(v') &= \alpha(v')pv - \pi(v') + (1-p)\alpha(v')\gamma(v') \\ &= \alpha(v')[pv + (1-p)\gamma(v')] - \pi(v') \end{aligned}$$

which is exactly his payoff from reporting v' to the mechanism (α, γ, π) when his type is v. Hence (α, γ', π') is incentive compatible and individually rational because (α, γ, π) is so. Furthermore the two mechanisms generate the same revenue for the seller ex post at every type v. Hence in order to solve P4 it suffices to maximize seller's expected profit by selecting an incentive compatible and individually rational mechanism from among those that involve no MBGs. ■

The intuition behind Proposition 3 is that the part of buyer's payoff from a nonzero MBG, $\alpha(1-p)\gamma$, is type-independent, just like his utility for money in this quasilinear framework. Hence the seller can substitute a strictly positive γ with a lower π, without changing incentive properties of the mechanism, while keeping his ex post revenue constant.

Scenario 2: p is private information and v is common knowledge Suppose now that the buyer's type is $p \in (0,1)$, while $v \in (0,1)$ is common knowledge between the buyer and the seller. The definition of a mechanism remains the same as above, except

that the argument of (α, γ, π) is p rather than v. The optimal mechanism design problem is

$$\max_{(\alpha,\gamma,\pi)} \int_0^1 [\pi(p) - (1-p)\alpha(p)\gamma(p)]dF(p)$$
$$\text{s.t. IC and IR.} \tag{P5}$$

Note now, as opposed to Scenario 1, that the buyer's payoff from a MBG depends on his type p, which opens the possibility that MBGs can be used to increase seller revenue.

Proposition 4 *There is a solution to problem P5 which offers full money back guarantees and leaves the buyer with zero payoff regardless of his type.*

Proof. Consider the mechanism $(\alpha^*(p), \gamma^*(p), \pi^*(p)) = (1, v, v)$ for all p. First note that this mechanism is incentive compatible, as it is constant in the type p. Next note that it is individually rational as the payoff to truthful reporting is $\alpha^*(p)(pv+(1-p)\gamma^*(p))-\pi^*(p) = 0$ for all p. To show that it is optimal for the seller, take any other incentive compatible and individually rational mechanism (α, γ, π). For any type p, the payoff of the seller from $(\alpha^*, \gamma^*, \pi^*)$ is

$$\begin{aligned} \pi^*(p) - (1-p)\alpha^*(p)\gamma^*(p) &= pv \\ &\geq \alpha(p)pv \\ &\geq \pi(p) - (1-p)\alpha(p)\gamma(p) \end{aligned}$$

where the first inequality is by $\alpha(p) \in \{0, 1\}$ and the second is by the individual rationality of (α, γ, π). Note that the last expression is the payoff of the seller from the mechanism (α, γ, π) when the buyer's type is p. Hence $(\alpha^*, \gamma^*, \pi^*)$ earns a weakly larger payoff to the seller compared to (α, γ, π) at every type. This completes the proof. ■

We would like to point out that if MBGs are not admissible, then Scenarios 1 and 2 are identical. With the possibility of MBGs, however, they lead to two very distinct outcomes. Whereas in Scenario 1 MBGs do not improve the seller's profit, in Scenario 2 it becomes feasible through MBGs to fully extract the buyer's surplus. This is noteworthy, especially because full surplus extraction occurs in a single-agent framework. In contrast, Cremer and McLean (1988) show that the seller can extract full surplus in a multiagent problem with interdependent values, using a mechanism which is not ex post individually rational. We would also like to emphasize that the optimal mechanism of Proposition 4 does not rely on any distributional assumptions regarding the buyer's type.

4 Conclusion

We conclude by highlighting two possible extensions. We consider in this paper *full* MBGs in the general multidimensional-type model. This restriction significantly simplifies our analysis since, by Proposition 1, we can identify the class of mechanisms that offer only full MBGs fairly easily. If partial MBGs are possible, then the class of feasible mechanisms for the seller enlarges. It would be interesting to investigate a tractable description of this class, and analyze conditions under which full MBGs dominate partial MBGs, and vice versa.

A second possible extension pertains to the nature of private information. We assume that if a good does not fit then the buyer's value for it is zero, and this is common knowledge. It is tempting to ask how our analysis applies to a scenario where the non-fit value is also a part of the buyer's private information. In the appendix section, we show that our analysis applies almost verbatim to this three-dimensional type scenario, the only slight difference being in the formulation of the fit compatibility condition. In order to ensure that the good is only returned in case it does not fit, the money back guarantee should be between the fit value and the non-fit value even when the latter is not commonly known to be fixed at zero.

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5 Appendix

Here, we extend our analysis to the more general scenario where the buyer's non-fit value is a part of his private information. As before, with probability p, the good fits the buyer's needs and its value is $v_H > 0$, otherwise, with probability $1-p$, there is no fit and its value is v_L, where $v_H > v_L \geq 0$. Whether the good fits or not is only observed by the buyer after purchase. The triple (p, v_H, v_L) is buyer's type. The type takes values in $T = (0,1) \cup V$ with a strictly positive density f, where

$$V = \left\{(v_H, v_L) : (v_H, v_L) \in (0,1) \times (0,1]^2,\, v_H > v_L\right\}.$$

The following definitions and notation are exact analogs of the two-dimensional scenario studied in the main text. An *outcome* is a triple (α, γ, π) where $\alpha \in \{0,1\}$ indicates sale or no sale, $\gamma \geq 0$ is a money-back guarantee (MBG), and $\pi \geq 0$ is the price. If the good does not change hands, i.e., if $\alpha = 0$, then there is no payment and no MBG, i.e., $\gamma = \pi = 0$. Hence the outcome belongs to the set

$$C = \{(\alpha, \gamma, \pi) \in \{0,1\} \times \Re_+ \times \Re_+ : \alpha = 0 \Rightarrow \gamma = \pi = 0\}.$$

Given $(\alpha, \gamma, \pi) \in C$, the payoffs are determined as follows. Both parties receive zero payoff if $\alpha = 0$. If $\alpha = 1$, the buyer pays the seller π. Next he observes the fit of the good and returns the good if and only if the MBG exceeds his value. Returning the good is costless. The resulting expected payoff of the buyer, depending on the outcome (α, γ, π) and type (p, v), is

$$\alpha(p \max\{v_H, \gamma\} + (1-p)\max\{v_L, \gamma\}) - \pi.$$

The seller receives the payment π and pays the MBG γ if the good is returned. Hence his expected payoff is given by

$$\begin{array}{cl} \pi & \text{if } \gamma \leq v_L \\ \pi - (1-p)\alpha\gamma & \text{if } v_L < \gamma \leq v_H \\ \pi - \alpha\gamma & \text{if } \gamma > v_H \end{array}$$

A mechanism is a triple of functions $(\alpha, \gamma, \pi) : T \to C$. The conditions on mechanisms which are of interest to us are as follows.

Incentive compatibility (IC): for any two types (p, v_H, v_L) and (p, v'_H, v'_L)

$$\begin{aligned} &\alpha(p, v_H, v_L)(p \max\{v_H, \gamma(p, v_H, v_L)\} + (1-p)\max\{v_L, (p, v_H, v_L)\}) - \pi(p, v_H, v_L) \\ &\geq \alpha(p', v'_H, v'_L)(p \max\{v_H, \gamma(p', v'_H, v'_L)\} + (1-p)\max\{v_L, (p', v'_H, v'_L)\}) - \pi(p', v'_H, v'_L). \end{aligned}$$

Individual rationality (IR): for any type (p, v_H, v_L)

$$\alpha(p, v_H, v_L)(p \max\{v_H, \gamma(p, v_H, v_L)\} + (1-p)\max\{v_L, (p, v_H, v_L)\}) - \pi(p, v_H, v_L) \geq 0.$$

Full Reimbursement (FR): for any type (p, v_H, v_L), $\gamma(p, v_H, v_L) > 0$ implies $\gamma(p, v_H, v_L) = \pi(p, v_H, v_L)$.

For any mechanism (α, γ, π) which satisfies IC, IR and FR, and any type (p, v_H, v_L) let $R^{(\alpha,\gamma,\pi)}(p, v_H, v_L)$ be the seller's ex post payoff when the buyer type is (p, v_H, v_L), i.e.,

$$\begin{aligned} &R^{(\alpha,\gamma,\pi)}(p, v_H, v_L) \\ &\quad = \begin{cases} \pi(p, v_H, v_L) & \text{if } \gamma(p, v_H, v_L) \leq v_L, \\ \pi(p, v_H, v_L) - (1-p)\alpha(p, v_H, v_L)\gamma(p, v_H, v_L) & \text{if } v_L < \gamma(p, v_H, v_L) \leq v_H, \\ \pi(p, v_H, v_L) - \alpha(p, v_H, v_L)\gamma(p, v_H, v_L) & \text{if } \gamma(p, v_H, v_L) > v_H. \end{cases} \end{aligned}$$

The *optimal mechanism design problem* is:

$$\max_{(\alpha,\gamma,\pi):T\to C} \int_0^1 \int_0^1 \int_0^{v_H} R^{(\alpha,\gamma,\pi)}(p, v_H, v_L) f(p, v_H, v_L) dp dv_H dv_L$$
$$\text{s.t. IC, IR and FR.} \tag{P6}$$

Problem P6 is that of maximizing the expected payoff of the seller by choosing a mechanism which satisfies the three feasibility constraints above. We will call a mechanism *optimal* if it solves P6.

Lemmas 3 and 4 are the counterparts of Lemma 1 in the main text.

Lemma 3 *Suppose (α, γ, π) satisfies IC, IR and FR. If $\gamma(p', v'_H, v'_L) > v'_H$ for some (p', v'_H, v'_L), then there exists a mechanism $(\tilde{\alpha}, \tilde{\gamma}, \tilde{\pi})$ which satisfies IC, IR, FR such that for every (p, v_H, v_L), $\tilde{\gamma}(p, v_H, v_L) \leq v_H$ and $R^{(\alpha,\gamma,\pi)}(p, v_H, v_L) = R^{(\tilde{\alpha},\tilde{\gamma},\tilde{\pi})}(p, v_H, v_L)$.*

Proof. Fix (α, γ, π) in satisfaction of IC, IR and FR. Define $(\tilde{\alpha}, \tilde{\gamma}, \tilde{\pi})$ as follows:

$$(\tilde{\alpha}(p, v_H, v_L), \tilde{\gamma}(p, v_H, v_L), \tilde{\pi}(p, v_H, v_L)) = \begin{cases} (\alpha(p, v_H, v_L), \gamma(p, v_H, v_L), \pi(p, v_H, v_L)) & \text{if } \gamma(p, v_H, v_L) \leq v_H, \\ (0, 0, 0) & \text{if } \gamma(p, v_H, v_L) > v_H. \end{cases}$$

Clearly $\tilde{\gamma}(p, v_H, v_L) \leq v_H$ for all (p, v_H, v_L) and $(\tilde{\alpha}, \tilde{\gamma}, \tilde{\pi})$ satisfies FR. Furthermore $R^{(\tilde{\alpha},\tilde{\gamma},\tilde{\pi})}(p, v_H, v_L) = R^{(\alpha,\gamma,\pi)}(p, v_H, v_L)$ for all (p, v_H, v_L) as whenever $\gamma(p, v_H, v_L) > v_H$, $R^{(\alpha,\gamma,\pi)}(p, v_H, v_L) = 0$ by FR.

Note that for every (p, v_H, v_L)

$$\begin{aligned} &\tilde{\alpha}(p, v_H, v_L)(p \max\{v_H, \tilde{\gamma}(p, v_H, v_L)\} + (1-p) \max\{v_L, \tilde{\gamma}(p, v_H, v_L)\}) - \pi(p, v_H, v_L) \\ &= \alpha(p, v_H, v_L)(p \max\{v_H, \gamma(p, v_H, v_L)\} + (1-p) \max\{v_L, \gamma(p, v_H, v_L)) - \pi(p, v_H, v_L). \end{aligned}$$

This is trivially true if $\gamma(p, v_H, v_L) \leq v_H$ as

$$(\tilde{\alpha}(p, v_H, v_L), \tilde{\gamma}(p, v_H, v_L), \tilde{\pi}(p, v_H, v_L)) = (\alpha(p, v_H, v_L), \gamma(p, v_H, v_L), \pi(p, v_H, v_L)).$$

When $\gamma(p, v_H, v_L) > v_H$, on the other hand, both sides are zero. IR directly follows from this observation.

Also note that if $(p', v'_H, v'_L) \neq (p, v_H, v_L)$

$$\begin{aligned} &\alpha(p', v'_H, v'_L)(p \max\{v_H, \gamma(p', v'_H, v'_L)\} + (1-p) \max\{v_L, \gamma(p', v'_H, v'_L)\}) - \pi(p', v'_H, v'_L) \\ &\geq \tilde{\alpha}(p', v'_H, v'_L)(p \max\{v_H, \tilde{\gamma}(p', v'_H, v'_L)\} + (1-p) \max\{v_L, \tilde{\gamma}(p', v'_H, v'_L)\}) - \pi(p', v'_H, v'_L). \end{aligned}$$

As before this is trivially true if $\gamma(p', v'_H, v'_L) \leq v'_H$. If $\gamma(p', v'_H, v'_L) > v'_H$, on the other hand, the left-hand side is

$$p \max\{v_H, \gamma(p', v'_H, v'_L)\} + (1-p) \max\{v_L, \gamma(p', v'_H, v'_L)\} \geq 0$$

while the right-hand side is 0. Combining the inequalities in the last two displays and using the hypothesis that (α, γ, π) satisfies IC, we conclude that $(\tilde{\alpha}, \tilde{\gamma}, \tilde{\pi})$ satisfies IC as well. ■

Lemma 4 *Suppose* (α, γ, π) *satisfies IC, IR and FR. If* $\gamma(p', v'_H, v'_L) < v'_L$ *for some* (p', v'_H, v'_L)*, then there exists a mechanism* $(\tilde{\alpha}, \tilde{\gamma}, \tilde{\pi})$ *which satisfies IC, IR, FR such that for every* (p, v_H, v_L)*,* $\tilde{\gamma}(p, v_H, v_L) \geq v_H$ *and* $R^{(\alpha,\gamma,\pi)}(p, v_H, v_L) = R^{(\tilde{\alpha},\tilde{\gamma},\tilde{\pi})}(p, v_H, v_L)$*.*

Proof. Fix (α, γ, π) in satisfaction of IC, IR and FR. Define $(\tilde{\alpha}, \tilde{\gamma}, \tilde{\pi})$ as follows:

$$(\tilde{\alpha}(p, v_H, v_L), \tilde{\gamma}(p, v_H, v_L), \tilde{\pi}(p, v_H, v_L)) = \begin{cases} (\alpha(p, v_H, v_L), \gamma(p, v_H, v_L), \pi(p, v_H, v_L)) & \text{if } \gamma(p, v_H, v_L) \geq v_L, \\ (\alpha(p, v_H, v_L), 0, \pi(p, v_H, v_L)) & \text{if } \gamma(p, v_H, v_L) < v_L. \end{cases}$$

Clearly $\tilde{\gamma}(p, v_H, v_L) \geq v_L$ for all (p, v_H, v_L) and $(\tilde{\alpha}, \tilde{\gamma}, \tilde{\pi})$ satisfies FR. Furthermore $R^{(\tilde{\alpha},\tilde{\gamma},\tilde{\pi})}(p, v_H, v_L) = R^{(\alpha,\gamma,\pi)}(p, v_H, v_L)$ for all (p, v_H, v_L) as whenever $\gamma(p, v_H, v_L) < v_L$, $R^{(\alpha,\gamma,\pi)}(p, v_H, v_L) = \pi(p, v_H, v_L)$.

Note that for every (p, v_H, v_L)

$$\begin{aligned} &\tilde{\alpha}(p, v_H, v_L)(p \max\{v_H, \tilde{\gamma}(p, v_H, v_L)\} + (1-p) \max\{v_L, \tilde{\gamma}(p, v_H, v_L)\}) - \pi(p, v_H, v_L) \\ &= \alpha(p, v_H, v_L)(p \max\{v_H, \gamma(p, v_H, v_L)\} + (1-p) \max\{v_L, \gamma(p, v_H, v_L)\}) - \pi(p, v_H, v_L). \end{aligned}$$

This is true as $\max\{v_L, \tilde{\gamma}(p, v_H, v_L)\} = \max\{v_L, \gamma(p, v_H, v_L)\}$ and $\max\{v_H, \tilde{\gamma}(p, v_H, v_L) = \max\{v_L, \gamma(p, v_H, v_L)\}$ and as $(\tilde{\alpha}(p, v_H, v_L), \tilde{\pi}(p, v_H, v_L)) = (\alpha(p, v_H, v_L), \pi(p, v_H, v_L))$. IR directly follows from this observation.

Also note that if $(p', v'_H, v'_L) \neq (p, v_H, v_L)$

$$\begin{aligned} &\alpha(p', v'_H, v'_L)(p \max\{v_H, \gamma(p', v'_H, v'_L)\} + (1-p) \max\{v_L, \gamma(p', v'_H, v'_L)\}) - \pi(p', v'_H, v'_L) \\ &\geq \tilde{\alpha}(p', v'_H, v'_L)(p \max\{v_H, \tilde{\gamma}(p', v'_H, v'_L)\} + (1-p) \max\{v_L, \tilde{\gamma}(p', v'_H, v'_L)\}) - \pi(p', v'_H, v'_L). \end{aligned}$$

This is true because $\tilde{\gamma}(p', v'_H, v'_L) \leq \gamma(p', v'_H, v'_L)$ for all (p', v'_H, v'_L). Combining the inequalities in the last two displays and using the hypothesis that (α, γ, π) satisfies IC, we conclude that $(\tilde{\alpha}, \tilde{\gamma}, \tilde{\pi})$ satisfies IC as well. ■

Lemmas 3 and 4 will now be stated as a feasibility condition.

Fit Compatibility (FC): for any type (p, v_H, v_L), $v_L \leq \gamma(p, v_H, v_L) \leq v_H$.

Restricting attention to mechanisms satisfying this condition simplifies the seller's payoff and, by Lemma 1, is without loss of generality in solving for the optimal mechanism.

In other words, in order to find the optimal mechanism which solves P6 the seller need only solve

$$\max_{(\alpha,\gamma,\pi):T\to C} \int_0^1 \int_0^1 [\pi(p,v_H,v_L) - (1-p)\int_0^{v_H} \alpha(p,v_H,v_L)\gamma(p,v_H,v_L)] f(p,v_H,v_L) dp dv_H dv_L$$
$$\text{s.t. IC, IR, FR and FC.} \tag{P7}$$

Next we introduce a class of mechanisms which are feasible in P7.

Definition 2 *Let $0 \leq k \leq m \leq 1$. A mechanism (α,γ,π) is a (k,m) mechanism if*

$$(\alpha(p,v_H,v_L),\gamma(p,v_H,v_L),\pi(p,v_H,v_L))$$
$$= \begin{cases} (1,m,m) & \text{if } v_H \geq m,\ p \leq \frac{k-v_L}{m-v_L} \text{ and } v_L \leq k \\ (1,0,k) & \text{if } v_L \leq k,\ p > \max\{0, \frac{k-v_L}{m-v_L}\}, \text{ and } pv_H + (1-p)v_L \geq k; \text{or if } v_L > k \\ (0,0,0) & \text{otherwise.} \end{cases}$$

It is straightforward to check that any (k,m) mechanism satisfies conditions IC, IR, FR and FC and is therefore feasible in problem P7. Next we will show that if a mechanism is feasible in problem P7 then it is "almost" a (k,m) mechanism. First we record a useful consequence of conditions IC,IR, FR and FC.

Lemma 5 *If (α,γ,π) satisfies IC, IR, FR and FC, then there exists $m \in (0,1)$ such that if $\gamma(p,v_H,v_L) > 0$, then $\gamma(p,v_H,v_L) = m$.*

Proof. Suppose, towards a contradiction, that for two distinct types (p,v_H,v_L) and (p',v'_H,v'_L), $0 < \gamma(p',v'_H,v'_L) < \gamma(p,v_H,v_L)$. Then $\alpha(p',v'_H,v'_L) = \alpha(p,v_H,v_L) = 1$ and the payoff to the (p,v_H,v_L) type from a truthful report is

$$pv_H + (1-p)\gamma(p,v_H,v_L) - \pi(p,v_H,v_L) = p(v_H - \gamma(p,v_H,v_L))$$

where we use FR in substituting $\gamma(p,v_H,v_L)$ for $\pi(p,v_H,v_L)$. If, instead, the (p,v_H,v_L) type reports (p',v'_H,v'_L), then his payoff would be

$$pv_H + (1-p)\gamma(p',v'_H,v'_L) - \pi(p',v'_H,v'_L) = p(v_H - \gamma(p',v'_H,v'_L)).$$

IC requires $p(v_H - \gamma(p,v_H,v_L)) \geq p(v_H - \gamma(p',v'_H,v'_L))$ which implies, since $p > 0$, $\gamma(p',v'_H,v'_L) \geq \gamma(p,v_H,v_L)$, a contradiction. ■

For any mechanism (α, γ, π) satisfying these four conditions, let $(T_1^{(\alpha,\gamma,\pi)}, T_2^{(\alpha,\gamma,\pi)}, T_3^{(\alpha,\gamma,\pi)})$ be the partition of the type space defined by

$$\begin{aligned}
(p, v_H, v_L) \in T_1^{(\alpha,\gamma,\pi)} &\Leftrightarrow \alpha(p, v_H, v_L) = 1 \text{ and } \gamma(p, v_H, v_L) > 0, \\
(p, v_H, v_L) \in T_2^{(\alpha,\gamma,\pi)} &\Leftrightarrow \alpha(p, v_H, v_L) = 1 \text{ and } \gamma(p, v_H, v_L) = 0 \\
(p, v_H, v_L) \in T_3^{(\alpha,\gamma,\pi)} &\Leftrightarrow \alpha(p, v_H, v_L) = 0.
\end{aligned}$$

Note that all types in $T_1^{(\alpha,\gamma,\pi)}$ receive the same MBG by Lemma 3. We will say that two mechanisms are *almost identical* if they generate the same partitions, except perhaps at the boundaries. Given our assumption that types have a strictly positive density, almost identical mechanisms earn the seller the same revenue as they differ only on a set of zero measure.

Proposition 5 *Any mechanism which satisfies IC, IR, FR and FC is almost identical to some (k, m) mechanism.*

Proof. Let (α, γ, π) satisfy the conditions. The proof relies on Lemmas 3 and 4 as well as the following three observations.

Claim 1: If $\alpha(p, v_H, v_L) = 1$, then $\alpha(p', v'_H, v'_L) = 1$ for all (p', v'_H, v'_L) such that $p' > p$, $v'_H > v_H$ and $v'_L \geq v_L$.

Proof of Claim 1: Suppose that $\alpha(p, v_H, v_L) = 1$ but $\alpha(p', v'_H, v'_L) = 0$ for some (p', v'_H, v'_L) such that $p' > p$, $v'_H > v_H$ and $v'_L \geq v_L$. If $\gamma(p, v_H, v_L) = 0$, then

$$\begin{aligned}
0 &\geq p'v'_H + (1-p')v'_L - \pi(p, v_H, v_L) \\
&> pv_H + (1-p)v_L - \pi(p, v_H, v_L) \\
&\geq 0
\end{aligned}$$

where the weak inequalities follow from incentive compatibility. This is clearly impossible. If $\gamma(p, v_H, v_L) = m > 0$, on the other hand, the impossibility follows similarly from incentive compatibility:

$$0 \geq p'(v'_H - m) > p(v_H - m) \geq 0.$$

Claim 2: If $\alpha(p, v_H, v_L) = \alpha(p', v'_H, v_L) = 1$ and $p' < p$, then $\gamma(p', v'_H, v_L) \geq \gamma(p, v_H, v_L)$.

Proof of Claim 2: Suppose that $\alpha(p, v_H, v_L) = \alpha(p', v'_H, v_L) = 1$, $p' < p$ but $\gamma(p', v'_H, v_L) < \gamma(p, v_H, v_L)$. Then, by Lemma 3, for some $m > 0$, $\gamma(p, v_H, v_L) = m$ and $\gamma(p', v'_H, v_L) = 0$. Incentive compatibility gives

$$p(v_H - m) \geq pv_H + (1-p)v_L - \pi(p', v'_H, v_L) \text{ and}$$
$$p'v'_H + (1-p')v_L - \pi(p', v'_H, v_L) \geq p'(v'_H - m).$$

Rearranging we get $pm + (1-p)v_L \leq \pi(p', v'_H, v'_L) \leq p'm + (1-p')v_L$, which is an impossibility since $m > 0$, $p' < p$ and $\gamma(p, v_H, v_L) \geq v_L$ (by Lemma 4) implies $m \geq v_L$.

Claim 3: If $\gamma(p, v_H, v_L) > 0$, then $\gamma(p', v'_H, v'_L) = \gamma(p, v_H, v_L)$ for all $(p', v'_H, v'_L) \in (0, p) \times (\gamma(p, v), 1) \times [0, v_L]$.

Proof of Claim 3: Suppose $\gamma(p, v_H, v_L) = m > 0$, $p' < p$, $v'_H > m$ and $v'_L \leq v_L$. We will first show that $\alpha(p', v'_H, v'_L) = 1$. If not, incentive compatibility implies $0 \geq \max\{p'(v'_H - m), p'v'_H + (1-p')v'_L - \pi(p', v'_H, v'_L)\}$. This in turn implies, $0 \geq p'(v'_H - m)$, an impossibility. Now suppose $\gamma(p', v'_H, v'_L) = 0$. The incentive compatibility conditions are:

$$p(v_H - m) \geq pv_H + (1-p)v_L - \pi(p', v'_H, v_L) \text{ and}$$
$$p'v'_H + (1-p')v'_L - \pi(p', v'_H, v_L) \geq p'(v'_H - m).$$

Rearranging we get $pm + (1-p)v_L \leq \pi(p', v'_H, v'_L) \leq p'm + (1-p')v'_L$, which is an impossibility since $m > 0$, $p' < p$, $v'_L \leq v_L$ and $v_L \leq m$ (by Lemma 3).

We can now go back to the proof of Proposition 5. If one of the sets in the partition $(T_1^{(\alpha,\gamma,\pi)}, T_2^{(\alpha,\gamma,\pi)}, T_3^{(\alpha,\gamma,\pi)})$ is empty, then the result follows straightforwardly. We will deal here with the case in which all three sets are nonempty. By Lemma 4, there exists $m^* > 0$ such that for any $(p, v_H, v_L) \in T_3^{(\alpha,\gamma,\pi)}$, $\gamma(p, v_H, v_L) = m^*$. Fix v_L such that $\gamma(p, v_H, v_L) = m^*$. By Claim 3 above, $\sup\{p : \gamma(p, v_H, v_L) = m^*\} = \sup\{p : \gamma(p, v'_H, v_L) = m^*\}$ for any $v_H, v'_H > m^*$. Let $p^*(v_L)$ be this supremum and $k^*(v_L) = v_L + (m^* - v_L)p^*(v_L)$. By Claim 1 above, $\alpha(p, v_H, v_L) = 1$ if $p > p^*$ and $v_H > m^*$. It follows that $\gamma(p, v_H, v_L) = 0$ for such (p, v_H, v_L) since $p > p^*$. Clearly, $k^*(v_L) = \inf\{p'v'_H + (1-p')v_L : p' > p^* \text{ and } v'_H > m\}$ and if for some (p, v_H, v_L) such that $v_H \leq m$ and $pv_H + (1-p)v_L > k^*(v_L)$, $\alpha(p, v_H, v_L) = 0$, incentive compatibility fails. Hence $(\alpha(p, v_H, v_L), \gamma(p, v_H, v_L)) = (1, 0)$ for all (p, v_H, v_L)

such that $p > p^*(v_L)$ and $pv_H + (1-p)v_L > k^*(v_L)$. By incentive compatibility and individual rationality $\pi(p, v_H, v_L) = k^*(v_L)$ for any such type.

Since $\gamma(p, v_H, v_L) = m^*$, by Claim 3 we have $\gamma(p, v_H, 0) = m^*$. Then, from above we have $k^*(0) = m^* p^*(0)$ and $(\alpha(p, v_H, 0), \gamma(p, v_H, 0)) = (1, 0)$ for all $(p, v_H, 0)$ such that $p > p^*(0)$ and $pv_H > k^*(0)$. By incentive compatibility and individual rationality $\pi(p, v_H, 0) = k^*(0)$ for any such type. Now consider two types, (p, v_H, v_L) and $(p, v_H, 0)$ such that $(\alpha(p, v_H, v_L), \gamma(p, v_H, v_L)) = (\alpha(p, v_H, v_L), \gamma(p, v_H, v_L)) = (1, 0)$. Incentive compatibility implies that $\pi(p, v_H, v_L) = \pi(p, v_H, 0) = k^*(0)$. Hence, for all v_L such that $\gamma(p, v_H, v_L) = 0$. We have $k^*(v_L) = k^*(0) = k^*$.

Thus for all (p, v_H, v_L) such that $\gamma(p, v_H, v_L) = m^*$ we have $p^*(v_L) = \frac{k^* - v_L}{m^* - v_L}$. Furthermore, $p^*(v_L) > 0$ implies that $k^* > v_L$.

For all types (p, v_H, v_L) such that $k^* < v_L$ and $m^* < v_H$, incentive compatibility implies $\alpha(p, v_H, v_L) = 1$, $\gamma(p, v_H, v_L) = 0$ and $\pi(p, v_H, v_L) = k^*$.

Hence (α, γ, π) is a (k, m) mechanism with $(k, m) = (k^*, m^*)$. ■

Hence, the seller need only find the optimal one among the (k, m) mechanisms, as we record in the following corollary.

Corollary 2 *If the pair (k^*, m^*) solves*

$$\max_{k,m} \int_0^k \int_m^1 \int_0^{\frac{k-v_L}{m-v_L}} mpf(p, v_H, v_L) dp dv_H dv_L + \int_0^k \int_m^1 \int_{\frac{k-v_L}{m-v_L}}^1 kf(p, v_H, v_L) dp dv_H dv_L$$
$$+ \int_0^k \int_k^m \int_{\frac{k-v_L}{v_H-v_L}}^1 kf(p, v_H, v_L) dp dv_H dv_L + \int_k^1 \int_{v_L}^1 \int_0^1 kf(p, v_H, v_L) dp dv_H dv_L$$
$$s.t.\ 0 \le k \le m \le 1$$

then the (k^*, m^*) mechanism solves the optimal mechanism design problem P6.

Next we will show that under general conditions, the optimal mechanism contains MBGs.

Proposition 6 *If the density f is continuously differentiable then the seller optimally offers money-back guarantee to some types.*

Proof. For any $k \in (0,1)$, consider the $(k,1)$ mechanism. This mechanism offers no MBGs. We will show that if $m \in (k,1)$ is sufficiently close to 1, then the seller's expected payoff is larger in the (k,m) mechanism than it is in the $(k,1)$ mechanism.

Switching from the $(k,1)$ mechanism to a (k,m) mechanism entails a loss of expected revenue for all types (p, v_H, v_L) such that $p \in (\frac{k-v_L}{v_H-v_L}, \frac{k-v_L}{m-v_L})$, $v_L \in (0,k)$ and $v_H \in (m,1)$. At any such type the $(k,1)$ mechanism earns the seller k, the price of the good, whereas the (k,m) mechanism brings the expected revenue mp, difference between price m and the expected MBG payment $(1-p)m$ back to the buyer. Note $mp < k$ for this range of p. The benefit from said switch occurs at types (p,v) such that $p \in (0, \frac{k-v_L}{v_H-v_L})$, $v_L \in (0,k)$ and $v_H \in (m,1)$. The $(k,1)$ mechanism does not serve these types. The (k,m) mechanism serves these types with the MBG m and earns the seller mp in expectation. Hence the switch is profitable if

$$\begin{aligned} E[mp|p &\in (0, \frac{k-v_L}{m-v_L}),\ v_L \in (0,k),\ v_H \in (m,1)] \\ &> E[k|p \in (\frac{k-v_L}{v_H-v_L}, \frac{k-v_L}{m-v_L}),\ v_L \in (0,k),\ v_H \in (m,1)]. \end{aligned}$$

To establish that this is the case, we will show that for some $m \in (k,1)$

$$\frac{\Pr\{p \in (\frac{k-v_L}{v_H-v_L}, \frac{k-v_L}{m-v_L})\ v_L \in (0,k),\ v_H \in (m,1)\}}{E[p|p \in (0, \frac{k-v_L}{m-v_L}),\ v_L \in (0,k),\ v_H \in (m,1)]} < \frac{m}{k}.$$

Since the numerator in the left-hand side converges to 0 as m goes to 1, and since the right-hand side is larger than 1, it suffices to show that the denominator of the left-hand side has positive limit, i.e.,

$$\lim_{m \to 1} E[p|p \in (0, \frac{k-v_L}{m-v_L}),\ v_L \in (0,k),\ v_H \in (m,1)] > 0.$$

Taking the conditional expectation

$$E[p|p \in (0, \frac{k}{m}),\ v_L \in (0,k),\ v_H \in (m,1)] = \frac{\int_0^k \int_m^1 \int_0^{(k-v_L)/(m-v_L)} p f(p, v_H, v_L) dp dv_H dv_L}{\int_0^k \int_m^1 \int_0^{(k-v_L)/(m-v_L)} f(p, v_H, v_L) dp dv_H dv_L}.$$

We apply L'Hopital's rule:

$$\lim_{m\to 1} \frac{\int_0^k \int_m^1 \int_0^{(k-v_L)/(m-v_L)} pf(p, v_H, v_L)dpdv_H dv_L}{\int_0^k \int_m^1 \int_0^{(k-v_L)/(m-v_L)} f(p, v_H, v_L)dpdv_H dv_L}$$

$$= \lim_{m\to 1} \frac{\frac{d}{dm}\left[\int_0^k \int_m^1 \int_0^{(k-v_L)/(m-v_L)} pf(p, v_H, v_L)dpdv_H dv_L\right]}{\frac{d}{dm}\left[\int_0^k \int_m^1 \int_0^{(k-v_L)/(m-v_L)} f(p, v_H, v_L)dpdv_H dv_L\right]}.$$

Define

$$\begin{aligned} H(m, v_H, v_L) &= \int_0^{(k-v_L)/(m-v_L)} pf(p, v_H, v_L)dp, \text{ and} \\ G(m, v_H, v_L) &= \int_0^{(k-v_L)/(m-v_L)} f(p, v_H, v_L)dp. \end{aligned}$$

Since f is continuously differentiable, so are the integrands in these expressions and we can use Leibnitz Theorem as follows:

$$\lim_{m\to 1} \frac{\frac{d}{dm}\left[\int_0^k \int_m^1 H(m, v_H, v_L)dv_H dv_L\right]}{\frac{d}{dm}\left[\int_0^k \int_m^1 G(m, v_H, v_L)dv_H dv_L\right]}$$

$$= \lim_{m\to 1} \frac{\int_0^k [-H(m, m, v_L) + \int_m^1 \frac{\partial}{\partial m} H(m, v_H, v_L)dv_H]dv_L}{\int_0^k [-G(m, m, v_L) + \int_m^1 \frac{\partial}{\partial m} G(m, v_H, v_L)dv_H]dv_L}$$

$$= \lim_{m\to 1} \frac{\int_0^k [-H(m, m, v_L) + \int_m^1 \frac{k}{m} f(\frac{k}{m}, v_H, v_L)dv_H]dv_L}{\int_0^k [-G(m, m, v_L) + \int_m^1 f(\frac{k}{m}, v_H, v_L)dv_H]dv_L}$$

$$= \lim_{m\to 1} \frac{\int_0^k H(m, m, v_L)dv_L}{\int_0^k G(m, m, v_L)dv_L}.$$

Now $m \mapsto H(m,m)$ and $m \mapsto G(m,m)$ are continuous because f is so. Hence

$$\begin{aligned}
\lim_{m\to 1} \frac{\int_0^k H(m,m,v_L)dv_L}{\int_0^k G(m,m,v_L)dv_L} &= \frac{\int_0^k H(1,1,v_L)dv_L}{\int_0^k G(1,1,v_L)dv_L} \\
&= \frac{\int_0^k \int_0^k x f(x,1,v_L)dx}{\int_0^k \int_0^k f(x,1,v_L)dx} \\
&= \int_0^k \Pr\{p|p<k,\, v_H=1, v_L\} > 0,
\end{aligned}$$

which is what we needed to show. ■