Omnidirectional Control

Raul Rojas

Freie Universität Berlin
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Abstract

This paper shows how to control a robot with omnidirectional wheels, using as example robots with four motors, and generalizing to n motors. More than three wheels provide redundancy: many combinations of motors speeds can provide the same Euclidean movement. Since the system is over-determined, we show how to compute a set of consistent and optimal motor forces and speeds using the pseudoinverse of coupling matrices. This approach allows us also to perform a consistency check to determine whether a wheel is slipping on the floor or not. We show that it is possible to avoid wheel slippage by driving the robot with a motor torque under a certain threshold.

1 Omnidirectional Wheels

Omnidirectional wheels have become popular for mobile robots, because they allow them to drive on a straight path from a given location on the floor to another without having to rotate first. Moreover, translational movement along any desired path can be combined with a rotation, so that the robot arrives to its destination at the correct angle.

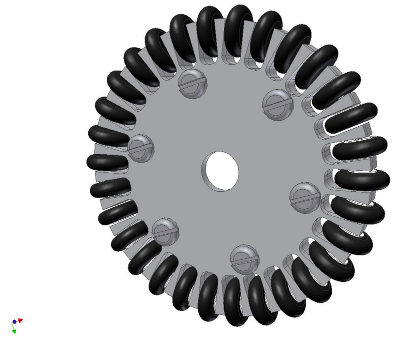

Figure 1: Our omnidirectional wheel design

Omnidirectional wheels are all based on the same general principle: while the wheel proper provides traction in the direction normal to the motor axis, the wheel can slide frictionless in the motor axis direction. In order to achieve this, the wheel is built using smaller wheels attached along the periphery of the main wheel. Fig. 1 shows an example of the kind of wheels that we have been using for our omnidirectional robots since 2002. Our wheel is a variation of the so-called Swedish wheels, which use rollers with a rotation direction which is not parallel nor perpendicular to the motor axis.

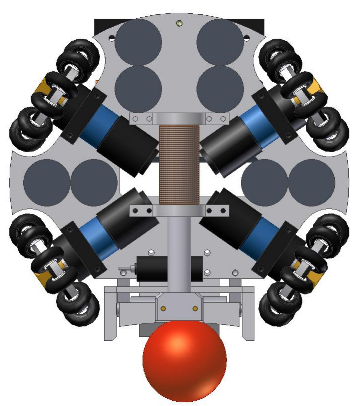

Figure 2: Our omnidirectional four-wheeled robot

Two or more omnidirectional wheels are used to drive a robot. Each wheel provides traction in the direction normal to the motor axis and parallel to the floor. The forces add up and provide a translational and a rotational motion for the robot. If it were possible to mount two orthogonally oriented omnidirectional wheels right under the center of a robot with a circular base, then driving the robot in any desired direction (without rotation) would be trivial. To give the robot a speed (v_x, v_y), with respect to a cartesian coordinate system attached to the robot, each wheel would just have to provide one of the two speed components.

However, since the wheels and motors need some space, this simple arrangement is not possible (the robot would be also very unstable!). The wheels are usually mounted on the periphery of the chassis. More than two wheels can be used, which makes it also easier to cancel any rotational torque which could make difficult to drive the robot on a straight path. Popular configurations are three and four-wheeled omnidirectional robots. Fig. 2 shows the CAD design of the omnidirectional robot which we used at RoboCup 2004 in Lisbon.

Each wheel can move the robot forward, but since they are located on the periphery of the robot, they can also rotate the robot's frame. In order to derive the relationship between the motors' torques and the movement of the robot, we need to analyze the geometry of the problem.

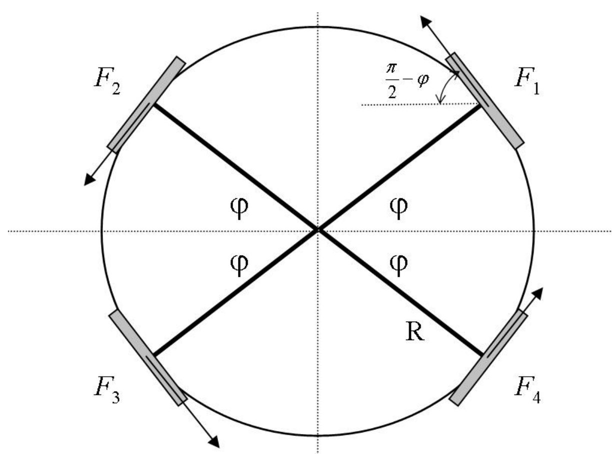

Figure 3: Arrangement of the wheels and distribution of forces

Let us use a motor with four wheels as our first example. For simplicity, the robot has two symmetry axes, as shown in the diagram (Fig. 3). Let us call φ the angle of the wheels with respect to the horizontal axis (the

x-direction), as shown on the diagram. When the four motors are activated, we obtain four traction forces F_1, F_2, F_3, F_4 from the motors, which add up to a translational force and a rotational torque. Each traction force F_i is the torque of the motor multiplied by the radius of the wheel. The sum of the forces depends on the exact wheel arrangement.

2 Force Coupling matrix

We are interested in the movement of the robot along the x and y direction. In order to simplify the expressions we will derive, we consider the instantaneous acceleration and velocity of the robot with respect to its own reference frame. For example, a robot moving forward will have a certain positive velocity in the y direction and zero in the x direction. We call the translational velocity and the angular velocity of the robot the "Euclidean magnitudes", different from the individual motor speeds and accelerations.

The translational acceleration of the center of mass of the robot (which we assume is located at the geometrical center of our circular robot), is given by

$$a = \frac{1}{M}(F_1 + F_2 + F_3 + F_4)$$

where M is the mass of the robot. The angular acceleration is given by

$$\dot{\omega} = \frac{R}{I}(f_1 + f_2 + f_3 + f_4)$$

where R is the radius of the robot, f_i denotes the magnitude of the force F_i, for $i = 1, \ldots, 4$, and I is the moment of inertia. The computation is possible using this expression, because the forces are tangent to the circular frame of the robot and point in the same rotational direction, so that we can work just with the magnitudes of the force vectors. The magnitudes f_1, f_2, f_3, f_4 can be positive or negative, according to the direction of rotation of the motor (counterclockwise or clockwise). The positive rotation directions are as shown in Fig. 3.

We can compute the x and y components of the robot's acceleration, by considering the respective components of each force. Then from the geometry of the problem, shown in Fig. 3,

$$Ma_x = -f_1 \sin\varphi - f_2 \sin\varphi + f_3 \sin\varphi + f_4 \sin\varphi$$

and

$$Ma_y = f_1 \cos\varphi - f_2 \cos\varphi - f_3 \cos\varphi + f_4 \cos\varphi$$

For a homogeneous cylinder $I = \frac{1}{2}MR^2$, for a ring $I = MR^2$. For any mass distribution strictly between a concentration of mass in the middle and concentration in the periphery, $I = \alpha MR^2$, with $0 \leq alpha \leq 1$. We can express the above acceleration equations as a matrix vector multiplication

$$(a_x, a_y, \dot{\omega})^{\mathrm{T}} = \frac{1}{M} \begin{pmatrix} -\sin\varphi & -\sin\varphi & \sin\varphi & \sin\varphi \\ \cos\varphi & -\cos\varphi & -\cos\varphi & \cos\varphi \\ \frac{MR}{I} & \frac{MR}{I} & \frac{MR}{I} & \frac{MR}{I} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}$$

Using $I = \alpha MR^2$, the expresion can be simplified to

$$(a_x, a_y, \dot{\omega})^{\mathrm{T}} = \frac{1}{M} \begin{pmatrix} -\sin\varphi & -\sin\varphi & \sin\varphi & \sin\varphi \\ \cos\varphi & -\cos\varphi & -\cos\varphi & \cos\varphi \\ \frac{1}{\alpha R} & \frac{1}{\alpha R} & \frac{1}{\alpha R} & \frac{1}{\alpha R} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}$$

We can further simplify this matrix by using the same units (meters per second) for the planar and angular acceleration. Instead of operating with $\dot{\omega}$ we can work with $R\dot{\omega}$. The new expression is then

$$(a_x, a_y, R\dot{\omega})^{\mathrm{T}} = \frac{1}{M} \begin{pmatrix} -\sin\varphi & -\sin\varphi & \sin\varphi & \sin\varphi \\ \cos\varphi & -\cos\varphi & -\cos\varphi & \cos\varphi \\ \frac{1}{\alpha} & \frac{1}{\alpha} & \frac{1}{\alpha} & \frac{1}{\alpha} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}$$

We call the 3×4 matrix in the expression above the force coupling matrix C_α.

Given any four motor states (and the associated torques) it is then straightforward to compute the acceleration in the x and y directions, as well as the tangential acceleration of the robot's frame periphery.

Note that the forces can cancel. If, for example, $f_1 = f_3 = 1$, and $f_2 = f_4 = -1$, then the robot stands still while the wheels work against each other. Much energy is wasted, but the robot does not move.

We are assuming here that the wheels cannot slip, that is, all the torque from the motors is transmitted to the robot via the floor. This is an unrealistic assumption which we discuss further down.

It is interesting to note from the above expression, that the rotational acceleration depends on the mass distribution of the robot. A point-mass robot can be accelerated infinitely fast around its center ($\alpha = 0$). A robot where the mass is distributed on a ring with very large radius (larger than the robot itself) will be accelerated around its center very slowly (because $\alpha \gg 1$).

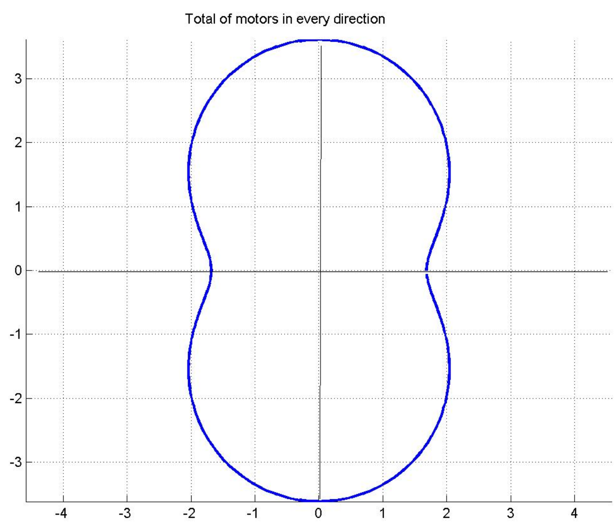

Figure 4: Total number of effective motor force in all directions around the origin for a four-wheeled asymmetrical omnidirectional robot

It is also interesting to visualize how many motors cooperate effectively when driving forwards or sideways. For a symmetrical robot with four motor axis at $\varphi = 30$ degrees, driving forward can be done faster than driving sideways (s9nce the wheels point more towards the front). Fig. **??** shows how many "effective motors" we have in each direction. When accelerating forward the four motors are equivalent to $4\cos\varphi = 3.7$ motors. When driving sideways, is is only equivalent to around 1.8 motors. When going diagonally at 45 degrees, the effective acceleration corresponds to the force of two motors.

3 Euclidean magnitudes

We can compute the final velocities of the wheels, and the velocity of the robot on the plane, as well as its angular velocity, by integrating the movement equations with respect to time. However, we have to think of the robot in Euclidean space, compute its trajectory there, and derive from this the velocity of each individual wheel. First let us look at the geometry of the problem.

Let us group the individual speeds of the four motors in the vector $(v_1, v_2, v_3, v_4)^{\mathrm{T}}$ and the Euclidean velocity and tangential rotational speed of the robot in the vector $(v_x, v_y, R\omega)^{\mathrm{T}}$. If the robot is moving as determined by the vector $(1, 0, 0)$, this means that it is moving sideways without rotating. When the robot moves with speed 1 to the right, the wheels rotate with speed $\sin\varphi$ (with the appropriate sign). This is easy to see from the diagram in Fig. 5. The large wheel provides one of the components of the horizontal movement (that is, $\sin\varphi$), while the small peripheral wheels provide the other orthogonal component (i.e. $\cos\varphi$).

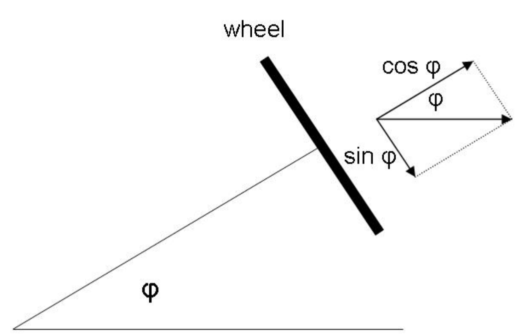

Figure 5: Rotation of large and small wheels, when the robot moves sideways.

The same kind of computation can be done when the robot is moving forward without rotating.The wheel movement is then the component $\cos\varphi$. Using the convention that the positive rotation direction is the direction of the right-hand thumb when we hold the motor axis in the hand, we obtain the following expression for the correspondences between the Euclidean and motor speeds:

$$(v_1, v_2, v_3, v_4)^{\mathrm{T}} = \begin{pmatrix} -\sin\varphi & \cos\varphi & 1 \\ -\sin\varphi & -\cos\varphi & 1 \\ \sin\varphi & -\cos\varphi & 1 \\ \sin\varphi & \cos\varphi & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ R\omega \end{pmatrix}$$

The matrix in this expression is very similar to the transpose of the coupling matrix C_α. This matrix, which we denote by D, is the velocity coupling matrix.

Let us denote the vector $(a_x, a_y, R\dot{\omega})^{\mathrm{T}}$ by a, the vector $(f_1, f_2, f_3, f_4)^{\mathrm{T}}$ by f, and the vector of motors speeds $(v_1, v_2, v_3, v_4)^{\mathrm{T}}$ by m. Then the following identities hold

$$\begin{aligned} a &= C_\alpha f \\ m &= Dv \end{aligned}$$

Integrating over a time interval Δt, we have $\Delta v = \Delta t \times a$, that is, $\Delta m = \Delta t \times DC_\alpha f$.

The motors have tick counters which allow to measure their speed in real time with the on-board electronics. For the purpose of controlling the robot, we want to know how measured wheel speeds $m = (v_1, v_2, v_3, v_4)^{\mathrm{T}}$ map to the Euclidean magnitudes $v = (v_x, v_y, R\omega)^{\mathrm{T}}$, that is, we would like to invert the expression $m = Dv$. This is not possible in general because the matrix D is not a square one, and therefore, is not invertible. However, we can look for a matrix D^+ such that $D^+D = I_3$, where I_3 is the 3×3 identity matrix. If such a matrix exists, then given Euclidean magnitudes v it is possible to find the corresponding wheel speeds m, which, in turn, reproduce the original Euclidean magnitudes. This is so because if

$$m = Dv$$

Then

$$D^+m = (D^+D)v = I_3v = v$$

The matrix D^+ exists, it is the so-called pseudoinverse of the matrix D. In the special case of a robot with two symmetry axes and the same angle φ

for each motor axis, we can even write down a simple expression for D^+. Assuming that $\sin\varphi$ and $\cos\varphi$ are both different from zero, then

$$D^+ = \frac{1}{4}\begin{pmatrix} -\frac{1}{\sin\varphi} & -\frac{1}{\sin\varphi} & \frac{1}{\sin\varphi} & \frac{1}{\sin\varphi} \\ \frac{1}{\cos\varphi} & -\frac{1}{\cos\varphi} & -\frac{1}{\cos\varphi} & \frac{1}{\cos\varphi} \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

It is straightforward to check that $D^+D = I_3$. It can also be checked that D^+ has all the properties of the pseudoinverse of D.[1]

Summarizing: the matrix C_α transforms the traction forces of the motors into Euclidean accelerations. The matrix D transforms Euclidean speeds into motor speeds. If we accelerate the motors for a time interval Δt, then, as we saw above, the additional Euclidean speeds are proportional to the motor forces according to the expression

$$(\Delta v_1, \Delta v_2, \Delta v_3, \Delta v_4)^{\mathrm{T}} = \Delta t \times DC_\alpha\,(f_1, f_2, f_3, f_4)^{\mathrm{T}}$$

The product DC_α can be computed and is equal to

$$DC_\alpha = \frac{1}{M\alpha}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} + \frac{1}{M}\begin{pmatrix} 1 & a & -1 & -a \\ a & 1 & -a & -1 \\ -1 & -a & 1 & a \\ -a & -1 & a & 1 \end{pmatrix}$$

where $a = \sin^2\varphi - \cos^2\varphi$. Therefore

$$\begin{pmatrix} \Delta v_1 \\ \Delta v_2 \\ \Delta v_3 \\ \Delta v_4 \end{pmatrix} = \Delta t\left(\frac{1}{M\alpha}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} + \frac{1}{M}\begin{pmatrix} 1 & a & -1 & -a \\ a & 1 & -a & -1 \\ -1 & -a & 1 & a \\ -a & -1 & a & 1 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}\right)$$

What this expression shows is that if we keep the motor torques constant and we integrate over time, there is a portion of the motor wheel velocities which is due to the rotation of the robot (produced by the left matrix in the expression), and another due to the movement on the plane. If α is small, the motors can be accelerated fast when the motor torques cooperate (for

[1] Both D^+D and DD^+ are symmetric, $D^+DD^+ = D^+$, and $DD^+D = D$.

example, when $(f_1, f_2, f_3, f_4)^{\mathrm{T}} = (1, 1, 1, 1)^{\mathrm{T}}$). If α is large, it takes longer to achieve the desired angular velocity than to achieve the desired translational speed.

What the above expression implies is that we must decouple the control for the Euclidean velocity from the control for the rotation. We then apply the necessary wheel torques over a time Δt in order to reach the velocity (v_x, v_y), and on top of that, we overlay the necessary wheel accelerations for reaching the angular velocity ω, but we apply these torques for a period $\Delta t'$, different in general from Δt. The only robot for which we can have $\Delta t = \Delta t'$ is one with $\alpha = 1$. Such a robot corresponds to a ring of mass M and radius R, that is, a robot with all its mass at the periphery.

4 Generalizing to n wheels

We can now generalize the points discussed above to an omnidirectional robot with $n \geq 3$ wheels. For this generalization assume that all the angles of the motor axis are measured relative to the x direction in the coordinate system of the robot. Call the angles of the motor axis for the n wheels $\theta_1, \theta_2, \ldots, \theta_n$. The driving direction of the i-th wheel is therefore $\theta_i + \pi/2$.

Arguing in the same way as in the previous section, and referring to Fig. 5, we find that the velocities coupling matrix D for the Euclidean velocity and rotational velocity of the robot is given by

$$(v_1, v_2, v_3, v_4)^{\mathrm{T}} = \begin{pmatrix} -\sin\theta_1 & \cos\theta_1 & 1 \\ -\sin\theta_2 & \cos\theta_2 & 1 \\ \vdots & \ldots & 1 \\ -\sin\theta_n & \cos\theta_n & 1 \end{pmatrix} (v_x, v_y, R\omega)^{\mathrm{T}}$$

This is an $n \times 3$ matrix. It maps Euclidean and tangential rotational speeds to motor speeds. D has the same function as in the case of the four motor robot discussed previously.

If the rank of the coupling matrix is at least three, then for any given Euclidean combination $(v_x, v_y, \omega R)^{\mathrm{T}}$, there is a combination of motor speeds

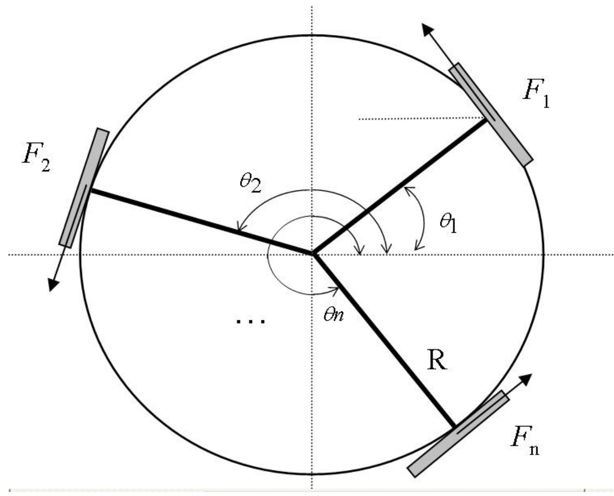

Figure 6: Arrangement of n wheels and distribution of forces

which can produce such movement.

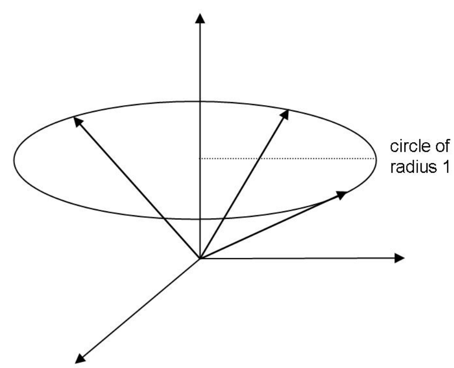

Figure 7: Three different axis directions are independent

It is easy to see that the rank of the coupling matrix is at least three, when at least three different motor axis directions have been chosen. Note that the rows of the matrix D are of the form $(-\sin\theta_i, \cos\theta_i, 1)$. These are vectors in three dimensional space, going from the origin to the periphery of a circle of radius 1 and positioned at $z = 1$ (see Fig. 7). Clearly, no three such different vectors can lay on the same plane, that is, they are linearly independent, and therefore the rank of the coupling matrix is at least three. It cannot be greater than three, because the matrix has only three rows, so it is exactly three.

When the rank of an $n \times 3$ matrix D is three, the product D^+D is the 3×3 identity matrix I_3, where D^+ denotes the pseudoinverse of D. We prove this below, first let us examine an important consequence.

Given the vector of Euclidean and rotational velocities $(v_x, v_y, \omega)^T$, we compute the needed motors speeds using the matrix D, obtaining

$$(v_1, v_2, \ldots, v_n)^T = D(v_x, v_y, R\omega)^T$$

Applying these motors speeds we obtain the Euclidean and rotational velocity back, because

$$D^+(D(v_x, v_y, R\omega)^T) = I_3(v_x, v_y, R\omega)^T = (v_x, v_y, \omega)^T$$

The matrix D^+, therefore, maps motor speeds to Euclidean velocities correctly. This is a formal proof of what we have been arguing in this paper, that three omnidirectional wheels, at three different angles, are sufficient for omnidirectional driving. The rest of the wheels $(n-3)$ provide redundancy to the system.

It rests to prove that for the $n \times 3$ matrix D, $D^+D = I_3$. We make use of singular value decomposition. We know that any $n \times m$ matrix D can be written as USV^T, where U and V are orthogonal square matrices, and S is an $m \times n$ diagonal matrix, with as many entries different from zero in the diagonal, as the rank of the matrix. In our case S is a $3 \times n$ matrix, with three nonzero elements in the diagonal. The pseudoinverse for C is given by the expression

$$D^+ = V(S^TS)^{-1}S^TU^T$$

where S^TS is an $n \times n$ diagonal matrix with three nonzero diagonal elements $\sigma_1, \sigma_2, \sigma_3$, and the rest $n-3$ diagonal elements are equal to zero. $(S^TS)^{-1}$ is the $n \times n$ matrix with three diagonal non-zero elements $1/\sigma_1, 1/\sigma_2, 1/\sigma_3$, and the rest of the elements equal to zero. The product D^+D is then

$$D^+D = V(S^TS)^{-1}S^T(U^TU)SV^T = V(S^TS)^{-1}(S^TS)V^T = VIV^T = I_3$$

This completes our proof.

It is easy to see that the force coupling matrix C_α for a robot with n wheels at the angles $\theta_1, \ldots, \theta_n$ is the matrix

$$C_\alpha = \frac{1}{M}\begin{pmatrix} -\sin\theta_1 & -\sin\theta_2 & \ldots & -\sin\theta_n \\ \cos\theta_1 & \cos\theta_2 & \ldots & \cos\theta_n \\ \frac{1}{\alpha} & \frac{1}{\alpha} & \ldots & \frac{1}{\alpha} \end{pmatrix}$$

where M is the mass of the robot, and α the constant in the moment of inertia $I = \alpha M R^2$. This matrix is the generalization of the matrix for the four wheeled robot discussed in the previous section.

Having the force coupling matrix C_α for the motor tangential forces and Euclidean accelerations, and also the velocities coupling matrix D for the transformation of Euclidean velocities into motor velocities, it is possible to

- transform motor forces into Euclidean accelerations:

$$a = C_\alpha f$$

- transform Euclidean accelerations into motor forces:

$$f = C_\alpha^+ a$$

- transform Euclidean speeds into motor speeds:

$$m = Dv$$

- transform motor speeds into Euclidean speeds:

$$v = D^+ m$$

5 Wheel control of a four wheeled symmetrical robot

Let us assume that we drive the motors in the form described above, for a robot with four symmetrically mounted motor axis, starting from rest, and with forces f_1, f_2, f_3, f_4 in each motor for a time t, and that the wheels do not slip on the ground. It is clear from the equations in the previous sections, that if $f_1 + f_2 + f_3 + f_4 = 0$, the robot will not rotate around its center. Can we find settings for the wheels, under this constraint, which allow us to accelerate the robot in any desired Euclidean direction, without letting it rotate? (That is, a_x and a_y can be arbitrary, within the velocity constraints of the motors, while $\dot{\omega}$ is zero).

This is indeed the case, because in the equations

$$a_x = \frac{1}{M}(-f_1 \sin\varphi - f_2 \sin\varphi + f_3 \sin\varphi + f_4 \sin\varphi)$$

and

$$a_y = \frac{1}{M}(f_1 \cos\varphi - f_2 \cos\varphi - f_3 \cos\varphi + f_4 \cos\varphi)$$

the combination of forces $p = (1, -1, -1, 1)$ drives the robot forward with acceleration $\frac{4}{M}\cos\varphi$. The combination $q = (1, -1, -1, 1)$ moves the robot sideways to the right with acceleration $\frac{4}{M}\sin\varphi$. Any linear combination of the two vectors, of the form

$$\frac{a_x}{\sin\varphi} q + \frac{a_y}{\cos\varphi} p$$

accelerates the robot in the direction (a_x, a_y). Here we are assuming that neither $\cos\varphi$ nor $\sin\varphi$ are equal to zero, otherwise we would have a very simple robot with four parallel wheels and this whole analysis would be unnecessary.

Also, since the sum of components of p and q is zero, any linear combination of both vectors does not let the robot rotate. Therefore we have found a vector of motor forces, for the desired direction, and which does not rotate the robot.

Now let us compute the forces necessary for the desired angular velocity. The vector $r = (1, 1, 1, 1)$ does not produce any forward or sideways acceleration (as can be tested by substituting in the acceleration equations). The angular acceleration with r is given by

$$\dot{\omega} = \frac{(f_1 + f_2 + f_3 + f_4)}{\alpha R} = \frac{4}{\alpha M R}$$

.

Therefore, we see that we can decouple the control problem for the robot into two parts:

- We can find motor forces which accelerate the robot in the desired Euclidean direction, without making it rotate.
- We can also find forces which provide angular acceleration for the robot, without making it displace on the ground.

The sum of both set of forces is the force to be applied to the robot. However, as explained before, the forces for the linear displacement must be applied for a different total time as the forces for the rotation.

In our system, we send the desired final linear velocity (v_x, v_y) to the robot and accelerate the wheels using a PID controller. Another PID controller receives the desired final angular velocity ω and controls the wheels. Both PID controllers are interleaved. The wheel accelerations overlap and we obtain the desired robot behavior.

6 Control of a four wheeled asymmetrical robot

When the robot does not have two axes of symmetry, we have to be more careful with the control of the wheels, as we show in this section.

Assume that the robot has four wheels, that the two motor axes in front are positioned at angles φ with respect to the horizontal, and the two rear motors at angle θ also with respect to the horizontal. The relationship between Euclidean accelerations and motor forces is given by

$$(a_x, a_y, R\dot{\omega})^{\mathrm{T}} = \frac{1}{M} \begin{pmatrix} -\sin\varphi & -\sin\varphi & \sin\theta & \sin\theta \\ \cos\varphi & -\cos\varphi & -\cos\theta & \cos\theta \\ \frac{1}{\alpha} & \frac{1}{\alpha} & \frac{1}{\alpha} & \frac{1}{\alpha} \end{pmatrix} (f_1, f_2, f_3, f_4)^{\mathrm{T}}$$

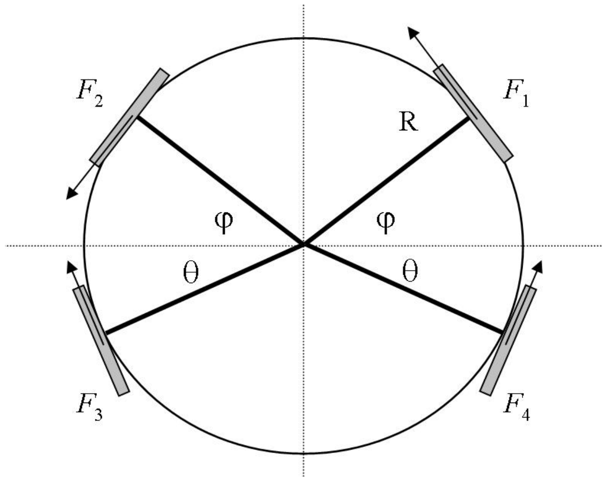

Figure 8: Arrangement of the wheels and distribution of forces for an asymmetrical robot

The velocity coupling matrix D for motor and Euclidean speeds is now given by

$$D = \begin{pmatrix} -\sin\varphi & \cos\varphi & 1 \\ -\sin\varphi & -\cos\varphi & 1 \\ \sin\theta & -\cos\theta & 1 \\ \sin\theta & \cos\theta & 1 \end{pmatrix}$$

From this expressions we can observe that motor forces proportional to $(-1, -1, 1, 1)$ produce an acceleration vector $(a_x, a_y, \dot{\omega})^{\mathrm{T}} = 2(\sin\varphi + \sin\theta, 0, 0)^{\mathrm{T}}$. That is, for driving sideways we need the same absolute force from each motor, so that the robot does not rotate.

However, notice that now the increment of the velocities of the individual wheels is different than in the case of the symmetrical robot. The increment to wheel velocities is given (as shown before) by

$$(\Delta v_1, \Delta v_2, \Delta v_3, \Delta v_4)^{\mathrm{T}} = \Delta t \times DC_\alpha \left(f_1, f_2, f_3, f_4\right)^{\mathrm{T}}$$

The matrix DC_α is now equal to

$$DC_\alpha = \frac{1}{M\alpha}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} + \frac{1}{M}\begin{pmatrix} 1 & a & b & c \\ a & 1 & c & b \\ b & c & 1 & d \\ c & b & d & 1 \end{pmatrix}$$

where $a = \sin^2\varphi - \cos^2\varphi$, $d = \sin^2\theta - \cos^2\theta$, $b = -\sin\varphi\sin\theta - \cos\varphi\cos\theta$, and $c = -\sin\varphi\sin\theta + \cos\varphi\cos\theta$. A combination of forces $(-1, -1, 1, 1)$ leads to motor speed increments equal to

$$\begin{pmatrix} \Delta v_1 \\ \Delta v_2 \\ \Delta v_3 \\ \Delta v_4 \end{pmatrix} = \begin{pmatrix} -1 + b + c - a \\ -1 + b + c - a \\ 1 - b - c + d \\ 1 - b - c + d \end{pmatrix}$$

. As can be seen, the front wheels have the same increment in speed, which is different from the increment in speed of the rear wheels. Curiously, although the torques in all wheels are of the same magnitude (plus or minus one), the final motor speeds are different in the front and rear motors. This clearly shows that all forces are transmitted through the robot chassis and that wheels accelerate according to the resultant of all forces involved.

The pseudoinverse of D can be computed with some algebraic effort. It is given by

$$D^+ = \begin{pmatrix} -i & -i & i & i \\ j & -j & -(1-j) & (1-j) \\ k & k & (1-k) & (1-k) \end{pmatrix}$$

where $i = 1/(2\sin\varphi + 2\sin\theta)$, $j = \cos\varphi/(2\cos^2\varphi + 2\cos^2\theta)$, and $k = \sin\theta/(2\sin\varphi + 2\sin\theta)$. It is easy to check that $D^+D = I_3$. Having C_α and D we can control the robot in the usual way.

7 Critical force and a limit example

When we start all four motors used in our robots, the respective x and y components of the torques work against each other. When the robot is rolling forward (in the positive y direction), for example, the force components sideways (in the x direction) cancel. However, if the forces being cancelled are excessive, the wheels will lose their grip and will start to slip on the ground. The result is a robot which cannot drive accurately enough.

Given a direction, let us call the maximum torque that a motor can put on the floor in that direction and without slipping, the critical torque for that direction. An extreme example can help to understand this concept.

Assume that in our four-wheeled robot the angle φ is very small and near to 90 degrees. In that case, the two frontal and the two rear wheels are almost parallel. If all motors are started with maximum torque $(1, -1, -1, 1)$, the robot should move forward, because the resultant force on the robot has only a component in the positive y direction. However, the opposing torques are so large that the wheels will probably slip on the wheel and will start spinning. The robot will move wildly on the floor.

Fig. 9 shows what percentage of the velocity of the robot is provided by the small wheels when the robot is moving from the origin towards any angle between 0 and 2π . The figure shows several curves for different motor axis angles φ. At $\varphi = 45$ degrees, for example, the small wheels provide around 30% of the total speed, in any direction in the graph. When the motors are positioned at $\varphi = 30$ degrees, then less than 15% of the forward velocity is provided by the small wheels. However, when driving sideways, around 50% of the velocity comes from them. The extreme example are wheels with motors at an angle of 9 degrees. When driving sideways, more than 80% of the total speed is provided by the small wheels.

Different values of φ mean that different projections of the motor torques have to be considered. If φ is almost zero, the wheels are almost parallel, are aligned in the forward direction, and the robot can accelerate forward at the limit imposed by the grip of the wheels on the carpet. The critical torque limit is therefore related to the friction between the wheels and the carpet, and the angle in which the wheels oppose each other.

In the limit case mentioned above (φ almost 90 degrees), if we are careful enough to start the motors slowly, then the robot will be accelerated forward. The forward movement is provided by the passive rolling wheels and the

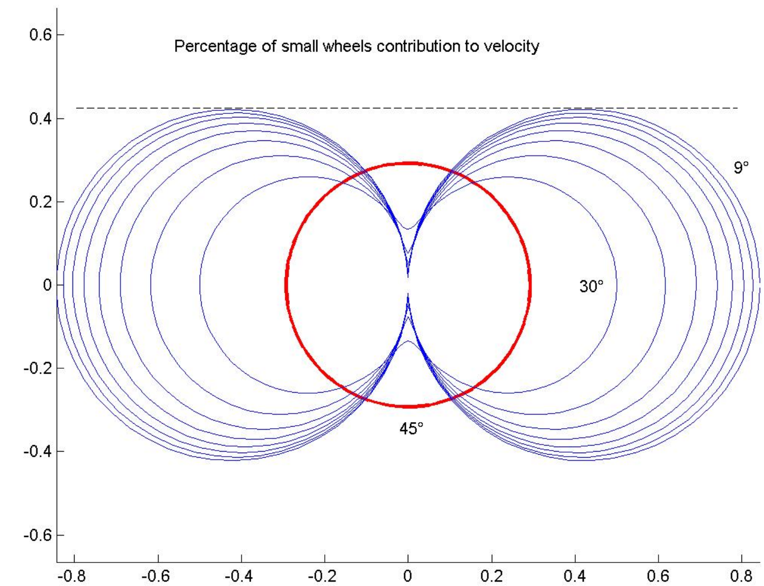

Figure 9: Contribution of the small wheels rotation to the total movement as percentage, for all directions between zero and 2π. For the robot with motors at 45 degrees, the percentage is constant in all directions.

wheels themselves have to roll very slowly. If the forward velocity of the robot is v_y, the needed tangential velocity of the wheel is $v_y \cos(\varphi)$. In other words, if the motor can drive the wheel at a maximum speed of v, the robot can reach a maximum forward speed of $v/\cos(\varphi)$, that is, the robot can be, in the limit of $\varphi = \pi/2$, infinitely fast! This is so because the acceleration on the robot is always present and the robot can roll forward passively. In reality, the passive wheels have more and more friction with their axis the faster they roll. Eventually this friction limits the forward acceleration of the robot.

8 How to drive without wheel slippage

Assume that you want to drive our robot forward, trying to avoid any significant wheel slippage. Assume that it has been determined experimentally that when the voltage for the DC motors is a maximum of 2 Volts the wheels will not slip (we can do this by holding the robot, and increasing the voltage until the wheels start to slip). We would like to drive as fast as possible (that is, with the maximum possible voltage for the motors) but without slipping. What we have to do then is to start the motors with $V_0 = 2$ and let the robot roll forward. After a few milliseconds, the induced current E in the rotor decreases the effective voltage on the rotor's solenoid to $V_0 - E$, and the motor torque correspondingly. We can now increase the value of the voltage to $V_1 = V_0 + E$, and now the motor torque corresponds to the effective voltage $V_0 + E - E = V_0$. Repeating this adjustment periodically, allows us to drive the motors with the maximum possible torque which does not let the wheels slip. Fig. 10 shows the result of a computer simulation for a certain DC motor. As can be seen, the motor angular speed increases along a curve. When the adjustment to the voltage is made (at discrete intervals) the curve changes to the corresponding curve for the higher voltage. In the limit, when the adjustment is made very often, the motor torque remains constant, the wheel acceleration too, and the angular velocity of the motor increases linearly. The slope of the curve is the maximum allowable acceleration before the wheels slip. However: this result depends on a perfect mass distribution so that all wheels exert the same pressure on the floor. This assumption cannot be guaranteed when the robot is driving fast, because acceleration can lift sometimes the front or side wheels from the floor, or at least diminish the pressure they exert on the floor. Slipping wheels are a fact of life and they have to be handled in the PID controller.

9 Identifying slipping wheels

Let us consider a symmetrical robot. Let us call m the four-dimensional vector $(v_1, v_2, v_3, v_4)^{\mathrm{T}}$ of tangential motor speeds, D the velocities coupling matrix, and v the three-dimensional vector $(v_x, v_y, R\omega)^{\mathrm{T}}$. The fact that the

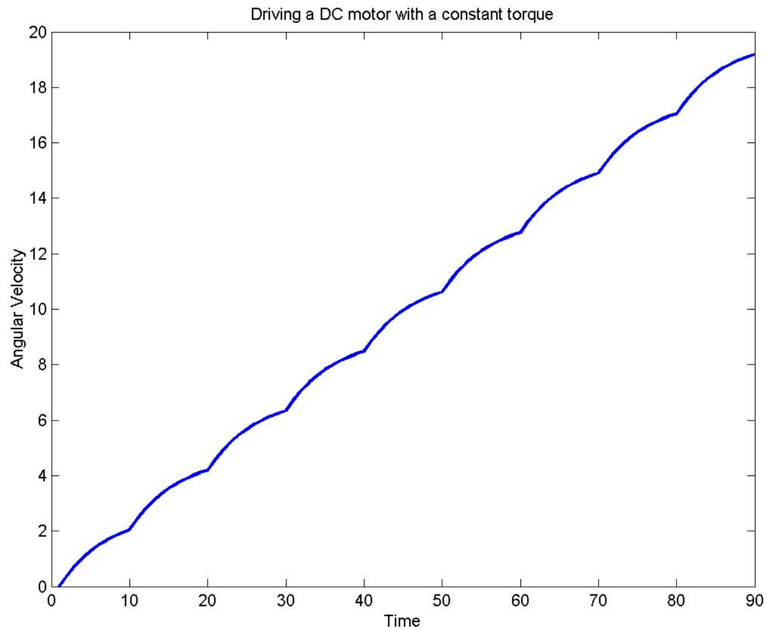

Figure 10: Driving a DC motor with a constant torque

Euclidean velocities of the robot and the tangential velocities of the motors are connected by the expressions

$$m = Dv \quad v = D^{+}m$$

gives us the possibility of testing for inconsistencies in the motors speed and thus detect wheel slippage.

The controller on the robot gets the desired vector v by radio communication and transforms v into the necessary motor speeds s. After some time, the tick counters in the motors provide a vector of current motor speeds m'. We can test if wheels are slipping using the in-built redundancy of our motor values. Since $v' = D^{+}m'$ and $m' = Dv'$, then it must be true that $m' = DD^{+}m$. If not, then one or more wheels are slipping on the ground. It would be extremely implausible that they all slip at a rate which allows the expression to remain valid.

A simple computation shows that for a symmetrical robot with angle φ:

$$DD^+ = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 1 & 3 & 1 \\ 1 & -1 & 1 & 3 \end{pmatrix}$$

Checking spinning wheels can be done very easily by multiplying with this matrix, because we should obtain $(I - DD^+)m' = 0$. The matrix $I - DD^+$ is given by

$$\begin{aligned} I - DD^+ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 1 & 3 & 1 \\ 1 & -1 & 1 & 3 \end{pmatrix} \qquad (1) \\ &= \frac{1}{4}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \qquad (2) \end{aligned}$$

The consistency check, as this matrix shows, reduces to the test

$$v_1 - v_2 + v_3 - v_4 = 0$$

or equivalently

$$v_1 + v_3 = v_2 + v_4$$

that is, the sum of the speeds of motor 1 with its opposite motor 3, is equal to the sum of the speed of motor 2 with its opposite motor 4. The speeds used before, that is $(1, 1, 1, 1)$ for rotation, $(1 - 1, -1, 1)$ for going forward, and $(-1, -1, 1, 1)$ for driving sideways, they all fulfill the above consistency check. Any linear combination of them passes also the consistency check.

It is interesting to note that this slippage check does not depend on the angle φ. It is a universal test for omnidirectional four motor autonomous robots, with wheels placed symmetrically at an angle φ.

Slippage check can be also computed for any other configuration of n motors. We just have to compute the matrix $(I - DD^+)$, and this provides the check we need. Consistent motor speeds m satisfy all the condition $(I-DD^+)m = 0$.

In the case of an asymmetrical robot with motor axis at angles φ and θ, we can also compute algebraically the matrix $(I-DD^+)$.

10 Handling slipping wheels

If we detect that the the wheel rotation is not consistent, that is, that one or more wheels are slipping, can we do something about it? As this section shows, in principle we can try to correct the motor movement without control losing time, that is, by doing a correction on top of the normal PID control. Let us consider a symmetrical robot with the same angle φ for all motor axes with respect to the horizontal.

First, note that there are motor torque combinations which do not accelerate the robot. One such combination, for a symmetrical robot, is $f = (1,-1,1,-1)$. The wheels pull in the forward and backward direction simultaneously. In such a case, the Euclidean magnitudes are zero, i.e.

$$a = C_\alpha f = 0$$

This means that the vector f is an element of the kernel of C_α (the set of vectors mapped to zero by this matrix).

Interestingly, if we have two combinations f and f' of motor forces which produce the same Euclidean accelerations a, then

$$a = C_\alpha f$$

and

$$a = C_\alpha f'$$

Therefore

$$C_\alpha(f - f') = 0$$

This means that $f - f'$ belongs to the kernel of C_α. Also, if the vector g belongs to the kernel of C_α, then the vector of forces f produces the same accelerations as the vector of forces $f + g$ because

$$a = C_\alpha(f + g) = C_\alpha f + C_\alpha g = C_\alpha f$$

For a robot with four wheels the rank of the matrix C_α is exactly three (we showed this in section x). Since the sum of the rank of an $n \times m$ matrix and the dimension of the kernel is equal to $\max(n, m)$, this means that the dimension of the kernel of C_α is one. This in turn implies that any vector in the kernel of C_α is of the form $\lambda(1, -1, 1, -1)$. Curiously enough, this is the first row in the matrix $(I - DD^+)$ computed in the previous section for a four-wheeled symmetrical robot. What this means is that motor speeds (v_1, v_2, v_3, v_4) are consistent if they are orthogonal to the kernel of the matrix C_α, that is, to all vectors of the form $\lambda(1, -1, 1, -1)$. Motor speeds are inconsistent if they include a projection in the direction of the vector $(1, -1, 1, -1)$. The solution? If we have inconsistent motor speeds $m = (v_1, v_2, v_3, v_4)^{\mathrm{T}}$, we compute the projection of m in the direction of the unit vector $\frac{1}{2}(1, -1, 1, -1)^{\mathrm{T}}$ and subtract it from m. The corrected m is thus:

$$m' = m - (m \cdot \frac{1}{2}(1, -1, 1, -1)^{\mathrm{T}})\frac{1}{2}(1, -1, 1, -1)^{\mathrm{T}}$$

The correction can be simplified to

$$m = m - \frac{v_1 - v_2 + v_3 - v_4}{4}(1, -1, 1, -1)^{\mathrm{T}}$$

We can visualize this result as follows: in the four dimensional space of motor values, there is a three dimensional subspace of consistent and non-slipping motor values (a three-dimensional hyperplane). Normal to this hyperspace we have the vector of wheel speeds $(1, -1, 1, -1)$. Anytime wheels are slipping, we are wasting energy because the vector of motor values contains a component in the $(1, -1, 1, -1)$ direction. What we have to do is to orthogonally project the vector of motor values back onto the hyperplane of consistent motor values, by just subtracting the component in the $(1, -1, 1, -1)$ direction.

Incidentally, when we map Euclidean accelerations a to motor forces f, we always obtain consistent results because we use the expression

$$f = C_\alpha^+ a$$

Since C_α and C_α^+ are both of rank three, accelerations a (in three-dimensional space) are mapped one-to-one to forces f (in a three-dimensional subspace of the four-dimensional space). If this three dimensional subspace would include any element g of the kernel of C_α, then there would exist an acceleration a, different from zero, such that $g = C_\alpha^+ a$. Since $C_\alpha g = 0$, then we would not have $a = C_\alpha C_\alpha^+ a$, as we should have.

Also, if g belongs to the kernel of C_α, any force of the form $f + g$ is equivalent to f, because $C_\alpha f = C_\alpha(f + g)$. For an acceleration a, there is a unique force f such that $f = C_\alpha^+ a$. Since f is always smaller in length than $f + g$ (because g is orthogonal to the subspace where f lives), we have the optimal situation where the forces computed require the minimal consumption of energy. Motors do not waste energy, when force combinations in the kernel of C_α are avoided.

11 Driving without one motor

It can happen that a robot loses one of the four motors while driving because the rotor contacts evaporate (this can happen when the motor is driven at very high speeds over its voltage specification). The advantage of a four motor robot is that it can recover from such a loss. It can continue to drive accurately and can still behave as an omnidirectional robot.

Assume, for example, that the first motor in Fig. 3 malfunctions. The wheel can still roll, but the motor does not provide torque. In that case in the set of motor forces (f_1, f_2, f_3, f_4), f_1 is always zero. However, the three remaining motors can still drive the robot forward, sideways, and can let it rotate. Furthermore, the necessary motor torques are independent. The vector $p = (0, 0, -1, 1)$ of motor forces moves the robot forward without letting it rotate, the vector $q = (0, -1, 1, 0)$ moves the robot sideways to the right without letting it rotate, and the vector of forces $r = (0, 1, 0, 1)$ rotates the robot counterclockwise, without displacing its center of mass. It is easy to see that these (and multiples of them) are the only torque combinations with such properties.

As before, we can decouple the displacement of the robot from the rotation. We modify the control strategy programmed in the PID controller in the robot. We can drive the robot in the direction (v_x, v_y) by setting a combination of forces proportional to

$$\frac{v_x}{\sin(\varphi)}q + \frac{v_y}{\cos(\varphi)}p.$$

The robot will accelerate in the desired direction, and the two inactive motors will roll passively, without causing major problems. On top of that movement, we can let the robot rotate using any multiple of the vector of forces r. The PID controller will not try to provide power from the two "disconnected" motors. Of course, the robot will be only half as fast as before, because we will be using only half of the motor power, but the robot will be still manageable.

If two, even three, motors are damaged it is still possible to drive with the robot, but it ceases to be omnidirectional. In that case the high-level control must change the sequence of commands sent to the robot.