On Aggregate Precautionary Saving: Infinitely-Lived Agents*

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Abstract

When does idiosyncratic earnings uncertainty increase aggregate saving? This question is answered in the context of a general equilibrium model where infinitely-lived agents face idiosyncratic labor endowment shocks and can hold a single, risk-free asset to smooth consumption. It is shown that the capital stock always increases (i.e. there is aggregate precautionary saving) when idiosyncratic uncertainty is added and utility functions are strictly concave. It is also shown that aggregate precautionary saving occurs in infinitely-lived agent models if and only if borrowing constraints bind in equilbrium for some agents.

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1 Introduction

When households are asked their reasons for saving the three most common responses are saving for emergencies, for old age and for the education of children.[1] The theoretical work on consumption and savings behavior has largely analyzed these three distinct motivations separately with precautionary savings theory capturing the first, life-cycle theory capturing the second and the third being related to models of the family.[2] Within the theory of precautionary saving, one important type of emergency is that resulting from uninsured, random variation in earnings. For the case of earnings uncertainty two natural questions that theory should answer are the following:

1. When does idiosyncratic earnings uncertainty increase individual saving?

2. When does idiosyncratic earnings uncertainty increase *aggregate* saving?

Leland (1968) and Sandmo (1970) address the first question. They address this question within a two-period, partial-equilibrium model in which there is earnings uncertainty in the second period and in which there is only a single, risk-free asset that can be accumulated. They prove that a risk-averse individual will increase savings when earnings uncertainty is added (holding mean earnings constant) as long as the third derivative of the period utility function is positive.[3]

[1]Projector and Weiss (1966, Table A30) document these results for US households. They also find that saving for a bequest is a common response only among very high income or high wealth households. Saving to buy a home is a common response among young households.

[2]These motives could be analyzed jointly. This is precisely the approach taken in the recent applied work on consumption and saving.

[3]This version of their result assumes that utility is additively separable. Within this context the result follows from Jensen's Inequality.

This paper focuses on the second question. The Leland-Sandmo result provides an answer to this question at least for the case of a two-period model of an open economy. However, there seems to be relatively little theoretical work that answers this question in either of the two widely-used frameworks of modern macroeconomics: the overlapping generations and the infinitely-lived agent framework.[4]

This paper answers the second question in the context of a general equilibrium model with a continuum of infinitely-lived agents. In this model each agent faces idiosyncratic shocks to labor endowment and can hold a single, risk-free asset to smooth consumption. In the aggregate, agents can accumulate physical capital. We will say that aggregate saving increases when idiosyncratic uncertainty is added (holding aggregate labor endowment constant) provided that the capital stock is higher in steady state in the presence than in the absence of these shocks. This is our notion of aggregate precautionary saving.[5]

The main results on aggregate precautionary saving proved in this paper apply to the case in which agents face a borrowing constraint. The first result of the paper shows that when idiosyncratic uncertainty is present there is aggregate precautionary saving in any steady-state equilibrium. This result holds regardless of the third derivative properties of the period utility function and regardless of the precise level of the borrowing constraint. The second result of the paper shows that aggregate precautionary saving occurs if and only if the borrowing constraint binds for a positive mass of agents. These results imply that in infinitely-lived agent models aggregate

[4] Aiyagari (1995) and Huggett (1997) analyze precautionary savings in the infinitely-lived agent framework for the special case in which no borrowing is allowed. Laitner (1979), Bewley (1984) and Clarida (1990) prove the existence of steady-state equilibria in the infinitely-lived agent framework. Abel (1985) analyzes precautionary saving in the overlapping generations framework when there is uncertainty over the length of a lifetime.

[5] We stress that this is a comparative steady-state notion of precautionary saving. Precautionary saving could potentially occur under this definition even when, in a national income accounting sense, there is no gross or net saving or investment in the economy.

precautionary saving occurs for different reasons than it occurs in the two-period, partial equilibrium model of Leland and Sandmo. In two-period models precautionary saving is related to third derivative properties of the utility function, whereas in infinitely-lived agent models precautionary saving is independent of these curvature assumptions and is instead associated with endogenously binding borrowing constraints.

The paper is organized in four sections. Section 2 describes the model economy. Section 3 presents the main results. Section 4 concludes.

2 The Economy

We describe our economies in three steps. First, we state our assumptions on pereferences, endowments and the technology. Second, we describe the decision problem of agents. Third, we define our steady-state equilibrium concept.

2.1 Assumptions

There are a continuum of infinitely-lived agents in the economy. The total mass of agents is equal to 1. All agents have identical preferences over consumption that are given by a calculation of expected utility:

$$E[\sum_{j=0}^{\infty} \beta^j u(c_j)]$$

Assumptions A1-2 list the restrictions on the period utility function u and the discount factor β. Assumption A1 ensures that expected utility will be bounded. Assumption A2 assumes that the period utility function u is strictly increasing and strictly concave.

A1 $u(c)$ is bounded and $\beta \in (0, 1)$.

A2 u is strictly increasing, strictly concave and continuously differentiable.

Each agent receives a labor endowment $e \in E$ in efficiency units each period. Assumption A3 lists the restrictions on endowments. Assumption A3 says that in any time period the labor endowment of any agent is potentially uncertain. This uncertainty is represented mathematically by a probabililty measure P defined on $\mathcal{E}$, the Borel sets of E. We will say that there is *idiosyncratic uncertainty* when $P(e) \neq 1$ for any $e \in E$, whereas there is *no idiosyncratic uncertainty* when $P(e) = 1$ for some $e \in E$.

A3 $(E, \mathcal{E}, P)$ is a probability space.

$E = [\underline{e}, \infty) \subseteq R_{+}$, $\mathcal{E}$ are the Borel sets of E and $\int_E edP = 1$

Assumption A3 further restricts labor endowments in two ways. First, the expected labor endowment of any agent is finite and normalized to equal 1. Second, labor endowments are independent and identically distributed for a given agent over time as well as independent across agents. Thus, there is potentially uncertainty for an individual agent but no uncertainty over the aggregate labor endowment which always equals 1.

There is a single firm in the economy that operates a total output function f. Assumption A4 lists the restrictions on f. These assumptions imply that the marginal product of capital is positive and diminishing. Assumption A4 allows for the case of a constant returns to scale production function $F(K, L)$ in capital K and labor L inputs, where capital depreciates at a constant rate δ. In this case the total output function f is given as follows: $f(K) = F(K, 1) + K(1 - \delta)$.

A4 $f(0) = 0$, $f' > 0$ and $f'' < 0$

2.2 Decision Problem

The decision problem that an individual agent faces is a version of the problem studied in the literature which attempts to formalize Friedman's

(1957) permanent-income hypothesis.[6] In this problem, an agent experiences stochastic variation over time in the efficiency of his/her labor endowment. An agent faces a fixed real wage w per efficiency unit of labor and a fixed real interest rate r at which it can borrow and lend. An agent maximizes expected utility. Formally, individual agents are assumed to solve the following dynamic programming problem. In this problem the individual state variable $x = (k, e)$ of a particular agent contains the current period values of asset holding k and the labor endowment shock e. The individual state x lies in the individual state space $X = [\underline{k}, \infty) \times E$. Expectations in the dynamic programming problem are taken with respect to the probability measure P defined on endowment shocks. A solution to this problem is given by optimal decision rules for consumption $c(x)$ and next period asset holding $k(x)$ as well as an optimal value function $V(x)$.

$$V(x) = \max u(c) + \beta E[V(k', e')]$$
$$(c, k') \in \Gamma(x; w, r, \underline{k})$$

We now describe the period budget set $\Gamma(x; w, r, \underline{k})$. Each period an agent chooses how to divide current period resources between consumption c and next period asset holdings k'. Current period resources equal the value of the current period labor endowment ew and the value of asset holdings $k(1 + r)$. In addition, an agent is subject to a borrowing constraint in that next period asset holdings must lie above a level $\underline{k}$. The budget set is given below.

$$\Gamma(x; w, r, \underline{k}) = \{(c, k') : c + k' \leq ew + k(1 + r), c \geq 0, k' \geq \underline{k}\}$$

We put two restrictions on the borrowing constraint $\underline{k}$, given that $w, (1 + r) > 0$. First, the borrowing constaint must allow for strictly positive consumption even when asset holding equals $\underline{k}$ and the endowment shock equals

[6]See Miller (1975), Schechtman and Escudero (1977), Sotomayor (1984) and Chamberlain and Wilson (1984) for an analysis of this decision problem.

$\underline{e}$. Thus, budget sets have nonempty interiors in all states. Second, $\underline{k}$ must lie below the level of the capital stock per person, K, in the economy. These restrictions are summarized as follows:

$$K > \underline{k} \text{ and } \underline{k} < \underline{e}w + \underline{k}(1+r)$$

Lemma 1 below lists some properties that optimal value functions $V(x)$ and optimal decision rules $c(x)$ and $k(x)$ must satisfy. We use these properties in proving the main results of the paper. To prove Lemma 1 we use the following standard results for this type of problem. First, a unique, continuous and bounded function $V(x)$ exists that solves the decision problem stated above. Second, V is strictly increasing and strictly concave in asset holdings k, given strictly positive factor prices w and $(1+r)$. In Lemma 1, $V_1^+(x)$ denotes the right-hand derivative of V with respect to its first argument. In Lemma 1 we will say that borrowing constraints bind when $V_1^+(x) > \beta(1+r)E[V_1^+(k(x), e')]$.

Lemma 1: Assume A1-3. Assume that factor prices are strictly positive.

(1) *$c(x)$ and $k(x)$ are continuous.*

(2) *(i) $V_1^+(x)$ exists and is strictly decreasing in both arguments.*

(ii) $V_1^+(x)$ is bounded

(iii) $V_1^+(x) = max\{u'(c(x))(1+r), \beta(1+r)E[V_1^+(k(x), e')]\}$

(3) *If there is idiosyncratic uncertainty and $\beta(1+r) \leq 1$, then there exists a set $B = [\underline{k}, a) \times A$ where $a > \underline{k}$ and where $A \in \mathcal{E}$ and $P(A) > 0$ such that the borrowing constraint strictly binds for all $x \in B$.*

(4) *If there is idiosyncratic uncertainty and $\beta(1+r) \leq 1$, then for any $k_1 > \underline{k}$ there exists $\epsilon, \delta > 0$ and $A \in \mathcal{E}$ where $P(A) > 0$ such that $k(k,e) \leq k_1 - \epsilon$ for all $k \in [k_1, k_1+\delta)$ and for all $e \in A$.*

Proof: See the Appendix.

In the analysis that follows we make use of the necessary condition in Lemma 1 (2)(iii) rather than the more familiar necessary condition $u'(c(x)) \geq \beta(1+r)E[u'(c(k(x),e'))]$. This latter condition follows from the former when $V_1^+(x) = u'(c(x))(1+r)$. This would follow from Mirman and Zilcha (1975, Lemma 1) at least when both asset holding and consumption are off corners. From the argument in the proof of Lemma 1 (2)(ii) it is clear that $u'(0) = \infty$ is sufficient for $V_1^+(x) = u'(c(x))(1+r)$. As this need not hold when $u'(0) < \infty$ we use the more general necessary condition.

2.3 Equilibrium

In this section we provide a definition of steady-state equilibrium.[7] This equilibrium concept requires that the distribution of individual states x across the population does not change over time. This implies, of course, that factor supplies and factor prices do not change over time either. The distribution is described mathematically by a probability measure y defined on $(X, \mathcal{X})$, where $X = [\underline{k}, \infty) \times E$ is the individual state space and $\mathcal{X}$ are the Borel sets of X. To state the equilibrium concept used in this paper we define a transition function $P(x,B) \equiv P(\{e' : (k(x), e') \in B\})$ on the sets $B \in \mathcal{X}$. The transition function $P(x,B)$ gives the probability that an agent with state x this period will have a state in the set B next period.

Definition: A steady-state equilibrium is $(c(x), k(x), K, w, r, \underline{k}, y)$ *such that*

(1) $c(x)$ *and* $k(x)$ *are optimal decision rules, given* $(w, r, \underline{k})$.

(2) Factor Prices are Competitively Determined:

[7] The treatment of agent heterogeneity is similar to that in Lucas and Prescott (1974), Foley and Hellwig (1975) and Lucas (1980).

$1 + r = f'(K)$ *and* $w = f(K) - f'(K)K$

(3) Markets Clear:

(i) $K = \int_X k dy$ *(ii)* $\int_X e dy = 1$ *and (iii)* $\int_X (c(x) + k(x))dy = f(K)$

(4) Distributions Are Time Invariant:

$y(B) = \int_X P(x, B)dy$ *for all sets* $B \in \mathcal{X}$

(5) Borrowing Constraint Restrictions:

$K > \underline{k}$ *and* $\underline{k} < \underline{e}w + \underline{k}(1 + r)$

3 Results

The main results of the paper are contained in two theorems. Theorem 1 states that the real interest rate r is always lower in economies with idiosyncratic uncertainty than in the same economy without idiosyncratic uncertainty. As the steady-state capital stock increases as the interest rate decreases, Theorem 1 implies that precautionary saving always occurs when idiosyncratic uncertainty is added to the model. This is true regardless of the third derivative properties of the period utility function.

The basic logic used to prove Theorem 1 is as follows. Any statistic of the aggregate state of the economy (y) cannot change over time in a steady-state equilibrium. However, in the economy with idiosyncratic uncertainty the marginal gain in expected utility to an increase in asset holdings when averaged over the population must decrease unless $\beta(1 + r) < 1$. This is shown by integrating the necessary condition from Lemma 1 (2)(iii) using the steady-state distribution y.

It should be noted that in this paper the trivial steady state with zero capital is ignored. Thus, all results implicitly concern steady states with positive capital.

Theorem 1: Assume A1-4.

(i) Consider the model economy without idiosyncratic uncertainty. In a steady-state equilibrium $\beta(1+r) = 1$.

(ii) Consider the model economy with idiosyncratic uncertainty. In a steady -state equilibrium $\beta(1+r) < 1$.

Proof:

It will first be shown that in steady-state equilibrium $\beta(1+r) \leq 1$ for both (i) and (ii). This follows from the three equations below:

$$V_1^+(x) \geq \beta(1+r)E[V_1^+(k(x), e')] \equiv \beta(1+r)E[V_1^+(x')|x]$$

$$\int_X V_1^+(x)dy \geq \beta(1+r)\int_X E[V_1^+(x')|x]dy$$

$$1 \geq \beta(1+r)$$

The first equation above holds by Lemma 1 (2) and by taking expectations with respect to x'. The second equation integrates the first using the steady-state distribution y. By Lemma 1 (2) these integrals are finite. The third equation follows from the second after noting two facts. First, Stokey and Lucas (1989, Thm. 8.3) implies that $\int_X E[V_1^+(x')|x]dy = \int_X V_1^+(x')dy^*$, where $y^*(B) \equiv \int_X P(x, B)dy$ for $B \in \mathcal{X}$. Second, the definition of steady-state equilibrium implies that $y^* = y$.

It will now be shown that when there is idiosyncratic uncertainty $\beta(1+r) \neq 1$. The key step is to show that $\beta(1+r) = 1$ implies that there is a set B of positive measure ($y(B) > 0$) on which the borrowing constraint binds. From this fact and from the arguments given above the following four inequalities hold:

$$\int_B V_1^+(x)dy > \beta(1+r)\int_B E[V_1^+(x')|x]dy$$

$$\int_{X/B} V_1^+(x)dy \geq \beta(1+r)\int_{X/B} E[V_1^+(x')|x]dy$$

$$\int_X V_1^+(x)dy > \beta(1+r)\int_X E[V_1^+(x')|x]dy$$

$$\beta(1+r) < 1$$

From Lemma 1 (3) there is a set $B \in \mathcal{X}$ such that $x \in B$ implies that $V_1^+(x) > \beta(1+r)E[V_1^+(x')|x]$. We now show that $y(B) > 0$. Define $k_1 \equiv sup\{k : y([k, \infty) \times E) = 1\}$. Clearly, $y([k_1, a) \times E) > 0$ for any $a > k_1$. As $y([k_1, a) \times E) = \int_X P(x, [k_1, a) \times E)dy$, we know that $P(x, [k_1, a) \times E) > 0$ on a set of x of positive measure. Since $P(x, [k_1, a) \times A) = P(x, [k_1, a) \times E)P(A)$, it is also true that $y([k_1, a) \times A) > 0$ for any set $A \in \mathcal{E}$ where $P(A) > 0$.

Suppose by way of contradiction that $k_1 > \underline{k}$. Define the set $D = [\underline{k}, k_1) \times E$. Lemma 1 (4) implies that $P(x, D) = 1$ for all $x \in [k_1, a) \times A$, where $P(A) > 0$. Since $P(x, D) = 1$ on a set of positive measure, we have that $y(D) = \int_X P(x, D)dy > 0$ which implies that $k_1 > \underline{k}$ cannot hold. Thus, $y([\underline{k}, a) \times E) > 0$ for any $a > \underline{k}$ and, hence, $y([\underline{k}, a) \times A) \equiv y(B) > 0$ for the set B constructed in Lemma 1(3).

It will now be shown that when there is no idiosyncratic uncertainty $\beta(1+r) \geq 1$. Suppose that $\beta(1+r) < 1$. Define B as the set of $x \in X$ such that $V_1^+(x) > \beta(1+r)V_1^+(k(x), 1)$. Define X/B as the set of $x \in X$ such that $V_1^+(x) = \beta(1+r)V_1^+(k(x), 1)$. As $V_1(x)$ is decreasing in its first argument, $k(x) < k$ within X/B except on a set of measure zero. Within the set B, $k(x) = \underline{k}$ and thus $k(x) \leq k$. These two results and the fact that $\underline{k} < K$ imply that the inequality below holds. The equality is, once again, due to Stokey and Lucas (1989, Thm. 8.3) and to the fact that y is time invariant in steady state. This establishes the contradiction.

$\int_X kdy > \int_X k(x)dy \equiv \int_X E[k'|x]dy = \int_X k'dy$ □

Comment 1. Theorem 1 (ii) can easily be extended to economies with heterogeneity in preferences and endowments. The reason is that the logic used to prove Theorem 1 can be applied separately to each of a finite number of agent types. Here an agent type is characterized by their preferences and their endowment processes. When discount factors differ across types, then the result is that with idiosyncratic uncertainty $\beta(1+r) < 1$ using the highest discount factor β associated with any agent type.

Comment 2. Becker (1980) characterizes steady states in models where agents differ in their discount factors but do not face idiosyncratic uncertainty. He proves that in steady state $\beta(1+r) = 1$, where β is the highest discount factor associated with any agent type. Hernandez (1991) characterizes steady states in models where agents discount at the same rate but differ in preferences and endowments and do not face idiosyncratic uncertainty. He also proves that in steady state $\beta(1+r) = 1$. The argument used in Theorem 1 (i) could easily be modified to handle economies with preference and endowment heterogeneity and thus to offer a different proof of these results.

The second result of the paper characterizes why aggregate precautionary saving occurs in the infinitely-lived agent models considered in Theorem 1. Theorem 2 below states that precautionary saving occurs if and only if borrowing constraints bind for a positive mass of agents in equilibrium. Recall that borrowing constraints are binding when $V_1^+(x) > \beta(1+r)E[V_1^+(k(x), e')]$.

> *Theorem 2: Assume A1-4. In a steady-state equilibrium $\beta(1+r) < 1$ if and only if the borrowing constraint binds for a positive mass of agents.*

Proof:

We first prove the necessity of binding borrowing constraints for precautionary savings by contraposition. Thus, suppose that borrowing constraints do not bind for a positive mass of agents in a steady-state equilibrium. Then the first equation below holds except on a set of measure zero. The second equation below follows by integration as the value of the integral does not depend on the value of the function on a set of measure zero. Lemma 1 (2) guarantees that these integrals are finite. The third equation follows from two facts. First, apply Stokey and Lucas (1989, Thm. 8.3) to get that $\int_X E[V_1^+(x')|x]dy = \int_X V_1^+(x)dy^*$, where $y^*(B) \equiv \int_X P(x,B)dy$. Second, note that in steady-state equilibrium $y^* = y$. The third equation completes the proof by contraposition as the negation of $\beta(1+r) < 1$ is $\beta(1+r) \geq 1$.

$$V_1^+(x) = \beta(1+r)E[V_1^+(x')|x]$$

$$\int_X V_1^+(x)dy = \beta(1+r)\int_X E[V_1^+(x')|x]dy$$

$$1 = \beta(1+r)$$

What is left is to prove the sufficiency of binding borrowing constraints. Thus, suppose that borrowing constraints bind for a positive mass of agents. Let B be the set of states for which the constraint binds and X/B be the set for which the constraint does not bind. We then know that the first two equations hold after integrating the necessary condition from Lemma 1 (2) over the set B and its complement X/B. The third equation below follows by combining the previous two equations.

$$\int_B V_1^+(x)dy > \beta(1+r)\int_B E[V_1^+(x')|x]dy$$

$$\int_{X/B} V_1^+(x)dy = \beta(1+r)\int_{X/B} E[V_1^+(x')|x]dy$$

$$\int_X V_1^+(x)dy > \beta(1+r)\int_X E[V_1^+(x')|x]dy$$

The result follows from this last equation from applying, once again, Stokey and Lucas (1989, Thm. 8.3). □

Comment 3. Theorem 2 can be restated to apply to economies with a positive mass of each of a finite number of agent types, where type refers to both preferences and endowment processes as discussed in Comment 1. The theorem could then be stated as follows: $\beta_i(1+r) < 1$ for each agent type i iff borrowing constraints bind for a positive mass of each agent type. The proof is straight forward.

4 Conclusion

The conventional wisdom among many economists is that the phenomena of precautionary saving is theoretically related to third derivative properties of utility function representations of preferences. This conventional wisdom is mistaken for the question of the theoretical determinants of *aggregate* precautionary saving. This paper focuses on theoretical models populated by infinitely-lived agents with a preference structure that is commonly used to study issues related to physical capital accumulation at both the individual and the aggregate level. Within this context it has been shown that (1) aggregate precautionary saving always occurs when period utility functions are strictly concave and (2) aggregate precautionary saving occurs if and only if borrowing constraints are binding for some agents.

Future work on the question of aggregate precautionary saving could proceed in several directions. One direction is to investigate more general preference and endowment structures. One motivation for investigating more general preference specifications is the recent emphasis on the potential importance of habit persistence and of departures from expected utility preference specifications for addressing asset pricing puzzles. Another direction is to investigate aggregate precautionary saving within the overlapping generations framework.

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5 Appendix

Proof of Lemma 1:

(1) The Theorem of the Maximum together with the strict concavity of u and V generates this result.

(2) (i) As the value function is strictly increasing and strictly concave in its first argument, the right-hand derivative exists (Rockafellar (1970, Thm. 23.1) and is strictly decreasing in k. It is clear that $V(k,e)=V(k',e')$ when $ew+k(1+r)=e'w+k'(1+r)$. Thus, $V_1^+(x)$ is also strictly decreasing in e.

(2) (ii) To prove that $V_1^+(x)$ is bounded we argue as follows. First, Lemma 1(2)(i) implies that $V_1^+(x) \leq V_1^+(\underline{k},\underline{e})$ for all $x \in X$. Second, Rockafellar (1970, Thm. 29.1) implies that any subgradient of V with respect to k is of the form $\lambda_1(1+r)$, where λ_1 is a 'multiplier' associated with the resource constraint in our dynamic programming problem. Thus, we know that $V_1^+(\underline{k},\underline{e}) \leq \lambda_1(1+r)$ if a subgradient exists. Finally, Rockafellar (1970, Cor. 28.3.1) states Kuhn-Tucker conditions that are necessary and sufficient for a solution to concave programming problems. These conditions are stated in terms of subdifferentials so the objective function need be concave but not necessarily differentiable.

Corollary 28.3.1 applies to our dynamic programming problem as the budget set has nonempty interior. Since we know that a solution to the dynamic programming problem exists, Corollary 28.3.1 states that there exist finite, positive values $(\lambda_1,\lambda_2,\lambda_3)$ satisfying the following restriction at the optimal levels of (c,k'). In the equation below $h(c,k') \equiv u(c)+\beta E[V(k',e')]$, ∂h denotes the subdifferential and $(\lambda_1,\lambda_2,\lambda_3)$ are multipliers associated with the resource constraint, the borrowing constraint and the non-negativity constraint on consumption.

$$(0,0) \in \partial h(c,k') + \lambda_1(-1,-1) + \lambda_2(0,1) + \lambda_3(1,0)$$

(2) (iii) First show $V_1^+(x) \geq max\{u'(c(x))(1+r), \beta(1+r)E[V_1^+(k(x),e')]\}$. The opposite inequality will then also be shown to hold. Clearly for $\epsilon > 0$, $V(k+\epsilon,e)-V(k,e) \geq u(c(k,e)+\epsilon(1+r))-u(c(k,e)$. Similarly, $V(k+\epsilon,e)-V(k,e) \geq \beta E[V(k(k,e)+\epsilon(1+r),e')]-\beta E[V(k(k,e),e')]$ Dividing by ϵ and taking limits produces $V_1^+(x) \geq u'(c(x))(1+r)$ and $V_1^+(x) \geq \beta(1+r)E[V_1^+(k(x),e')]$. The second inequality relies on differentiating under the expectation operator. Billingsley (1986, Thm. 16.8) shows that $V_1^+(x)$ bounded is sufficient for this to be valid. Boundedness was established in Lemma 1(2)(ii).

Now show that $V_1^+(x) \leq max\{u'(c(x))(1+r), \beta(1+r)E[V_1^+(k(x),e')]\}$. The following must hold for $\epsilon > 0$:

$$V(k+\epsilon,e)-V(k,e) \leq u'(c(k,e))\Delta c + \beta E[V_1^+(k(k,e),e')]\Delta k'$$

$$V(k+\epsilon,e)-V(k,e) \leq max\{u'(c(k,e), \beta E[V_1^+(k(k,e),e')]\}\epsilon(1+r)$$

The first inequality above follows from the concavity of u and V, after defining $\Delta c \equiv c(k+\epsilon, e) - c(k, e)$ and $\Delta k' \equiv k(k+\epsilon, e) - k(k, e)$. The second inequality follows from the first as the budget constraint implies that $\Delta c + \Delta k' = \epsilon(1+r)$ The result follows from dividing each side of the second inequality by ϵ and taking limits.

(3) Lemma 1 (2)(i) and the fact that there is idiosyncratic uncertainty implies that there is $\hat{e} \in E$ and a set $A = E \cap [0, \hat{e}]$ such that $0 < P(A) < 1$ and such that the strict inequality below holds for all $e \in A$. The two weak inequalities below follow because $\beta(1+r) \leq 1$ and because $V_1^+(x)$ is decreasing in k.

$$V_1^+(\underline{k}, e) > E[V_1^+(\underline{k}, e')] \geq \beta(1+r)E[V_1^+(\underline{k}, e')] \geq \beta(1+r)E[V_1^+(k(\underline{k}, e), e')]$$

Lemma 1 (2)(iii) then implies that the following equation holds for all $e \in A$.

$$V_1^+(\underline{k}, e) = u'(c(\underline{k}, e))(1+r) > \beta(1+r)E[V_1^+(k(\underline{k}, e), e')]$$

The continuity of $u'(c(x))$ then implies that the result holds for $x \in [\underline{k}, a) \times \hat{e}$ for some $a > \underline{k}$. As $V_1^+(x)$ is decreasing in e and as asset holding is at the borrowing constraint, the result must hold for all $x \in B = [\underline{k}, a) \times A$.

(4) We first establish that $k(k, e) < k_1$ on a set of positive probability when $k = k_1$. Suppose by way of contradiction that this does not hold. Thus, $k(k_1, e) \geq k_1$ for all $e \in A$ where $P(A) = 1$. By Lemma 1 (2) and by the fact that asset holdings are off the corner the equality below holds. The inequality holds as $\beta(1+r) \leq 1$ and as $V_1^+(x)$ is decreasing in k.

$$V_1^+(k_1, e) = \beta(1+r)E[V_1^+(k(k_1, e), e')] \leq E[V_1^+(k_1, e')]$$

Given that there is idiosyncratic uncertainty and that $V_1^+(x)$ is strictly decreasing in e, this cannot hold for all $e \in A$. Contradiction.

Define $\epsilon_1 = k_1 - k(k_1, e)$ using any $e \in E$ for which $\epsilon_1 > 0$ and for which any open neighborhood of e lieing in E has positive probability. Set $\epsilon = \epsilon_1/2$. Since $k(x)$ is continuous (Lemma 1 (1)), there is a sufficiently small neighborhood (constructed using the Euclidean metric) around (k_1, e) lieing in X such that $k(x) \leq k_1 - \epsilon$ for all x in that neighborhood. Clearly, this also holds for a neighborhood $[k_1, k_1 + \delta) \times A$ lieing inside the previously constructed neighborhood. The set A can by choosen to have positive probability. $\Box$