On Aggregate Precautionary Saving: When is the Third Derivative Irrelevant?*

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Abstract

When does idiosyncratic earnings uncertainty increase aggregate saving? We address this question in the context of a general equilibrium model where infinitely-lived agents receive idiosyncratic labor endowment shocks, hold a risk-free asset to smooth consumption and face a liquidity constraint. We prove that the steady-state capital stock is always larger in any equilibrium with idiosyncratic shocks and a liquidity constraint than without idiosyncratic shocks (i.e. there is aggregate precautionary saving) as long as utility functions are strictly concave. We also prove that aggregate precautionary saving occurs if and only if the liquidity constraint binds for some agents.

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1 Introduction

When households are asked their reasons for saving the three most common responses are saving for emergencies, for old age and for the education of children.[1] The theoretical work on consumption and saving behavior has largely analyzed these three distinct motivations separately with precautionary saving theory capturing the first, life-cycle theory capturing the second and the third being related to models of the family.[2] Within the theory of precautionary saving, one important type of emergency is that resulting from uninsured, random variation in earnings. For the case of earnings uncertainty two natural questions that theory should answer are the following:

1. When does idiosyncratic earnings uncertainty increase individual saving?

2. When does idiosyncratic earnings uncertainty increase *aggregate* saving?

Leland (1968) and Sandmo (1970) address the first question. They address this question within a two-period, partial-equilibrium model in which there is earnings uncertainty in the second period and in which there is only a single, risk-free asset that can be accumulated. They prove that a risk-averse individual will save more when earnings uncertainty is added (holding mean earnings constant) as long as the third derivative of the period utility function is positive.[3]

[1]Projector and Weiss (1966, Table A30) document these results for US households. They also find that saving for a bequest is a common response only among very high income or high wealth households. Saving to buy a home is a common response among young households. Kennickell and Starr-McCleur (1994) document similar findings.

[2]These motives could be analyzed jointly. This is precisely the approach taken in the recent applied work on consumption and saving.

[3]This version of their result assumes that utility is additively separable. Mirman (1971, p.184) also proves this result.

This paper focuses on the second question. The Leland-Sandmo result provides an answer to this question at least for the case of a two-period, partial-equilibrium model as in this context aggregation is straightforward as factor prices cannot change. However, issues of capital accumulation have traditionally been posed within general equilibrium models with infinitely many time periods. In particular, competitive versions of the infinitely-lived agent growth models of Cass (1965) and Koopmans (1965) as well as the overlapping generations growth model of Diamond (1965) have been the basis for much of the theoretical and applied work on issues related to capital accumulation.[4]

We address the second question within the infinitely-lived agent framework, while leaving for future research an analysis of this question in the overlapping generations framework. The early theoretical work on infinitely-lived agent models where agents face idiosyncratic uncertainty was conducted by Laitner (1979), Bewley (1984) and Clarida (1990). They focused on proving the existence of steady-state equilibria. The same type of framework has also been employed by Aiyagari (1995) and Huggett (1997). Aiyagari uses this framework to examine the optimal tax rate on capital income, whereas Huggett focuses on how the steady state and dynamic properties of models without idiosyncratic uncertainty and liquidity constraints change when these features are added. None of the above papers focus directly on the second question.

In the framework that we consider there are a continuum of infinitely-lived agents. Each agent experiences idiosyncratic shocks to labor endowment. Paralleling the analysis of Leland and Sandmo, we abstract from the labor supply decision and thus assume that labor endowment is inelastically supplied. Again parralleling Leland and Sandmo, we assume that agents can hold only a risk-free asset. While Leland and Sandmo abstract from the pos-

[4]Examples of applied work within these frameworks includes Kotlikoff and Summers (1981), Kydland and Prescott (1982), Auerbach and Kotlikoff (1987) and Abel et al (1989).

sibility that agents might face a borrowing limit that could be endogenously binding, we do not. Thus, in our model an agent faces a fixed limit in the amount that can be borrowed which we refer to as a liquidity constraint.[5] In the aggregate, agents can accumulate physical capital. We will say that there is (positive) aggregate precautionary saving provided that the steady-state capital stock is larger in the presence than in the absence of idiosyncratic labor endowment uncertainty, holding aggregate labor endowment constant. This is our notion of aggregate precautionary saving.[6]

The first result of the paper states that the capital stock is larger in any steady-state equilibrium with idiosyncratic uncertainty than without idiosyncratic uncertainty. Thus, aggregate precautionary saving is a necessary condition for an equilibrium with idiosyncratic uncertainty and a liquidity constraint. This result holds regardless of the third derivative properties of the period utility function, regardless of the amount of idiosyncratic uncertainty and regardless of the precise level of the liquidity constraint. The second result of the paper provides a complete characterization of aggregate precautionary saving in the presence of a liquidity constraint. The result states that there is aggregate precautionary saving if and only if the liquidity constraint binds for a positive mass of agents. Thus, our results emphasize the role of endogenously binding liquidity constraints - a feature that Leland (1968), Sandmo (1970) and much of the literature on precautionary saving has abstracted from.

[5]Section 2 of the paper specifies more precisely what we mean by a liquidity constraint. We note that the presence of borrowing limits in dynamic models is not really the issue. What matters is the exact form of the borrowing limits that agents face. We expand upon this issue in section 4.

[6]This is the standard notion of aggregate precautionary savings (see Skinner (1988), Caballero (1991), Aiyagari (1994) and Huggett (1996)). An alternative but more complicated notion of aggregate precautionary saving would be to compare paths for aggregate capital in the presence and in the absence of idiosyncratic uncertainty, starting from the same initial distribution of capital and holding the path of aggregate labor endowment constant.

We now provide a preview for our results. We find that many economists whose focus is on the precautionary saving literature are surprised that third derivatives are irrelevant for our results, whereas economists whose focus is on capital theory (i.e. models of capital accumulation) are not surprised. Capital theorists are not surprised as steady states in models with infinitely-lived agents can be characterized directly from the Euler equation using only first derivative properties of the utility function. In particular, the fact that in a steady state the distribution of marginal utility cannot change over periods allows one to pin down the steady-state capital stock in one-sector models. It is one of the contributions of this paper to show that the same reasoning, again using only first derivatives, also applies when characterizing steady states in similar models with idiosyncratic uncertainty.

The paper is organized in five sections. Section 2 describes the model economy. Section 3 presents the main results as well as the logic used to prove these results. Section 4 discusses our results. Section 5 concludes.

2 The Economy

We describe our economies in three steps. First, we state our assumptions on preferences, endowments and the technology. Second, we describe the decision problem of agents and state some necessary conditions of optimal decisions. Third, we define our steady-state equilibrium concept.

2.1 Assumptions

There are a continuum of infinitely-lived agents in the economy. The total mass of agents is equal to 1. All agents have identical preferences over consumption that are given by a calculation of expected utility:

$$E[\sum_{j=0}^{\infty} \beta^j u(c_j)]$$

Assumptions A1-2 list the restrictions on the period utility function u and the discount factor β. Assumption A1 ensures that expected utility will be bounded. Assumption A2 lists some standard properties of the period utility function u.

A1 *$u(c)$ is bounded and $\beta \in (0, 1)$.*

A2 *u is strictly increasing, strictly concave and continuously differentiable.*

Each agent receives a labor endowment $e \in E$ in efficiency units each period. Assumption A3 lists the restrictions on endowments. Assumption A3 says that in any time period the labor endowment of any agent is potentially uncertain. This uncertainty is represented mathematically by a probabililty measure P defined on $\mathcal{E}$, the Borel sets of E. We will say that there is idiosyncratic uncertainty when $P(e) \neq 1$ for any $e \in E$, whereas there is no idiosyncratic uncertainty when $P(e) = 1$ for some $e \in E$.

A3 *$(E, \mathcal{E}, P)$ is a probability space.*

$E = [\underline{e}, \infty) \subseteq R_+$, $\mathcal{E}$ are the Borel sets of E and $\int_E e dP = 1$

Assumption A3 further restricts labor endowments in two ways. First, the expected labor endowment of any agent is finite and normalized to equal 1. Second, labor endowments are independent and identically distributed for a given agent over time as well as independent across agents. Thus, there is potentially uncertainty for an individual agent but no uncertainty over the aggregate labor endowment which always equals 1.

There is a single firm in the economy that operates a total output function f. Assumption A4 lists the restrictions on f. These assumptions are the usual ones employed in neoclassical growth theory and imply that the marginal product of capital is positive and diminishing. Assumption A4 allows for the case of a constant returns to scale production function $F(K, L)$ in capital K

and labor L inputs, where capital depreciates at a constant rate δ. In this case the total output function f is given as follows: $f(K) = F(K, 1) + K(1 - \delta)$.

A4 $f(0) = 0$, $f' > 0$ *and* $f'' < 0$

2.2 Decision Problem

The decision problem that an individual agent faces is a version of the problem studied in the literature which attempts to formalize Friedman's (1957) permanent-income hypothesis. The theoretical literature on this decision problem includes the work of Miller (1975), Schechtman and Escudero (1977), Mendelson and Amihud (1982), Sotomayor (1984), Chamberlain and Wilson (1984) and many others. The mathematical structure in these papers is also similar to that employed by Deaton and Laroque (1992) to analyze commodity storage problems. Recently, a number of investigators have used computational methods to highlight quantitative aspects of the consumption, saving, asset pricing, wealth distribution, portfolio choice and business cycle properties of models that have as a component some version of this decision problem. See Deaton (1991, 1992), Carroll (1992, 1997), Huggett (1993), Aiyagari (1994), Krusell and Smith (1994, 1997), Castaneda et al (1994), Rios-Rull (1995), den Haan (1997), Obiols-Homs (1997) and the references cited therein for a guide to this literature.

In the version of this decision problem considered in this paper, an agent experiences random variation over time in the efficiency of his/her labor endowment. An agent faces a fixed real wage w per efficiency unit of labor and a fixed real interest rate r at which it can borrow and lend as long as asset holdings stay above a borrowing limit, $\underline{k}$. An agent maximizes expected utility. This decision problem is stated below in the language of dynamic programming. The individual state variable $x = (k, e)$ of a particular agent contains the current period values of asset holding k and the labor endowment shock e. The individual state x lies in the individual state space $X = [\underline{k}, \infty) \times E$.

Expectations in the dynamic programming problem are taken with respect to the probability measure P defined on endowment shocks. A solution to this problem is given by optimal decision rules for consumption $c(x)$ and next period asset holding $k(x)$ as well as an optimal value function $V(x)$ satisfying the following equation:

$$V(x) = \max u(c) + \beta E[V(k', e')]$$
$$(c, k') \in \Gamma(x; w, r, \underline{k})$$

The period budget set $\Gamma(x; w, r, \underline{k})$ allows an agent to divide current period resources between consumption c and next period asset holdings k'. Current period resources equal the value of the current period labor endowment ew and the value of asset holdings $k(1+r)$. In addition, an agent faces a borrowing limit $\underline{k}$ in that next period asset holdings must lie above this level.

$$\Gamma(x; w, r, \underline{k}) = \{(c, k') : c + k' \leq ew + k(1+r), c \geq 0, k' \geq \underline{k}\}$$

We put two restrictions on the borrowing limit $\underline{k}$, given that w and $(1 + r) > 0$. First, agents face a liquidity constraint in that the borrowing limit $\underline{k}$ is set at any level above the level associated with solvency. The limit associated with solvency is the level $\underline{k}$ such that when assets are at this level and when the agent receives the lowest endowment shock $\underline{e}$ all resources would be used to maintain assets at the borrowing limit (i.e. $\underline{k} = \underline{e}w + \underline{k}(1+r)$) and, hence, consumption is zero. Second, $\underline{k}$ must lie below the level of the capital stock per person, K, in the economy. The restrictions above are summarized as follows:

$$\underline{k} < \underline{e}w + \underline{k}(1+r) \text{ and } \underline{k} < K$$

2.3 Properties of the Decision Problem

Lemma 1 below lists some properties that optimal value functions $V(x)$ and optimal decision rules $c(x)$ and $k(x)$ must satisfy. We use these properties in proving the main results of the paper. In particular, we use the necessary condition from Lemma 1 (2)(iii) in much of the paper. This necessary condition is stated in terms of the right-hand derivative of the value function with respect to its first argument, $V_1^+(x)$. It states that the marginal utility gain to additional asset holdings equals the maximum of its use in current consumption or its use in asset accumulation. Lemma 1 (3) establishes that for low enough interest rates there are values of the individual state x such that the borrowing limit will be strictly binding. We will say that the borrowing limit strictly binds when $V_1^+(x) > \beta(1+r)E[V_1^+(k(x), e')]$. This just states that the best use of an increment in assets is for current consumption rather than for asset accumulation. Lemma 1 (4) establishes that for low enough interest rates individual agents will always receive labor endowment shocks which lead them to reduce asset holdings if they were not already at the borrowing limit.

To prove Lemma 1 we use the following standard results for this type of problem. First, a unique, continuous and bounded function $V(x)$ exists that solves the decision problem stated above. Second, V is strictly increasing and strictly concave in asset holdings k, given strictly positive factor prices w and $(1+r)$.

Lemma 1: Assume A1-3. Assume that factor prices are strictly positive and that agents face a liquidity constraint.

(1) $c(x)$ and $k(x)$ are continuous.

(2) (i) $V_1^+(x)$ exists and is strictly decreasing in both arguments.

(ii) $V_1^+(x)$ is bounded

(iii) $V_1^+(x) = max\{u'(c(x))(1+r), \beta(1+r)E[V_1^+(k(x), e')]\}$

(3) If there is idiosyncratic uncertainty and $\beta(1+r) \leq 1$, then there exists

a set $B = [\underline{k}, a) \times A$ *where* $a > \underline{k}$ *and* $P(A) > 0$ *such that the borrowing limit strictly binds for all* $x \in B$.

(4) If there is idiosyncratic uncertainty and $\beta(1+r) \leq 1$, *then for any* $k_1 > \underline{k}$ *there exists* $\epsilon, \delta > 0$ *and* $A \in \mathcal{E}$ *where* $P(A) > 0$ *such that* $k(k, e) \leq k_1 - \epsilon$ *for all* $k \in [k_1, k_1 + \delta)$ *and for all* $e \in A$.

Proof: See the Appendix.

In the analysis that follows we make use of the necessary condition in Lemma 1 (2)(iii) rather than the more familiar necessary condition $u'(c(x)) \geq \beta(1+r)E[u'(c(k(x), e'))]$. This latter condition follows from the former when $V_1^+(x) = u'(c(x))(1+r)$. This would follow from Mirman and Zilcha (1975, Lemma 1) at least when both asset holding and consumption are off corners. From the argument in the proof of Lemma 1 (2)(ii) it is clear that $u'(0) = \infty$ is sufficient for $V_1^+(x) = u'(c(x))(1+r)$. As this need not hold when $u'(0) < \infty$, we use the more general necessary condition in all of our arguments.

2.4 Equilibrium

In this section we provide a definition of steady-state equilibrium. This equilibrium concept requires that the distribution of agents across individual states x does not change over time even though individual agents can move within the distribution. The distribution is described mathematically by a probability measure y defined on $(X, \mathcal{X})$, where $X = [\underline{k}, \infty) \times E$ is the individual state space and $\mathcal{X}$ are the Borel sets of X.[7] To state the equilibrium concept used in this paper we define a transition function $P(x, B) \equiv P(\{e' : (k(x), e') \in B\})$ on the sets $B \in \mathcal{X}$. The transition function $P(x, B)$ gives the probability that an agent with state x this period will have a state in the set B next period.

[7]The treatment of agent heterogeneity is similar to that in Lucas and Prescott (1974), Foley and Hellwig (1975) and Lucas (1980).

Definition: A steady-state equilibrium with a liquidity constraint is $(c(x), k(x), K, w, r, \underline{k}, y)$ *such that*

(1) $c(x)$ *and* $k(x)$ *are optimal decision rules, given* $(w, r, \underline{k})$.

(2) Factor Prices are Competitively Determined:

$1 + r = f'(K)$ *and* $w = f(K) - f'(K)K$

(3) Resource Feasibility:

(i) $K = \int_X k dy$ *(ii)* $\int_X e dy = 1$ *and (iii)* $\int_X (c(x) + k(x))dy = f(K)$

(4) Distributions Are Time Invariant:

$y(B) = \int_X P(x, B)dy$ *for all sets* $B \in \mathcal{X}$

(5) Liquidity Constraint Restrictions:

$\underline{k} < \underline{e}w + \underline{k}(1 + r)$ *and* $\underline{k} < K$

3 Results

The main results of the paper are contained in two theorems. This section states the main results, presents the logic used to prove these results and provides several comments indicating how the results can be extended to apply to economies with heterogeneity in preferences and endowment processes. All other discussion of Theorems 1 and 2 is deferred to the next section.

Theorem 1 states that the real interest rate r is always lower in economies with idiosyncratic uncertainty and a liquidity constraint than in economies without idiosyncratic uncertainty. As the steady-state capital stock increases as the interest rate decreases, Theorem 1 states that aggregate precautionary saving is a necessary feature of an equilibrium with idiosyncratic uncertainty and a liquidity constraint. This is true regardless of the third derivative properties of the period utility function, regardless of how much idiosyncratic earnings uncertainty is present and regardless of the precise level of the liquidity constraint.

The logic used to prove Theorem 1 is as follows. Any statistic of the aggregate state of the economy y cannot change over time in a steady-state equilibrium. Applied to economies with no idiosyncratic uncertainty, this logic implies that $\beta(1+r) = 1$. Otherwise, the marginal utility of consumption when summed over agents or, more generally, the marginal gain in utility to additional asset holdings when summed over agents would change over time. This is shown by integrating the necessary condition from Lemma 1 (2)(iii) using the steady-state distribution y. In economies with idiosyncratic uncertainty, the same argument shows that $\beta(1+r) \leq 1$. However, in the presence of idiosyncratic uncertainty, a steady state with $\beta(1+r) = 1$ is not possible as optimal smoothing behavior implies that a positive mass of agents would draw down asset holdings so that the liquidity constraint would be strictly binding. This would imply that marginal utility when averaged over the economy would shrink over time.

It should be noted that the trivial steady state with zero capital is ignored. Thus, all results implicitly concern steady states with positive capital.

Theorem 1: Assume A1-4.

(i) Consider the model economy without idiosyncratic uncertainty. In a steady-state equilibrium with a liquidity constraint $\beta(1+r) = 1$.

(ii) Consider the model economy with idiosyncratic uncertainty. In a steady -state equilibrium with a liquidity constraint $\beta(1+r) < 1$.

Proof: See the Appendix.

Comment 1. Theorem 1 (ii) can easily be extended to economies with heterogeneity in preferences and endowment processes. The reason is that the logic used to prove Theorem 1 (ii) can be applied separately to each of a finite number of agent types. Here an agent type is characterized by their preferences and their endowment processes. When discount factors differ across types, then the result is that with idiosyncratic uncertainty

$\beta(1+r) < 1$ using the highest discount factor β associated with any agent type.

Comment 2. Becker (1980) and Hernandez (1991) characterize steady states in similar models but where agents differ in preferences and endowments and do not experience idiosyncratic uncertainty. They prove that in steady state $\beta(1+r) = 1$, where β is the highest discount factor associated with any agent type. The argument used in Theorem 1 (i) could easily be modified to handle economies with preference and endowment heterogeneity and thus to offer a different proof of this result.

The second result of the paper provides a complete characterizaton of aggregate precautionary saving in the presence of a liquidity constraint. Theorem 2 states that aggregate precautionary saving occurs if and only if the liquidity constraint is endogenously binding for a positive mass of agents. Theorems 1 and 2 then jointly imply that binding liquidity constraints are a necessary feature of any equilibrium with idiosyncratic uncertainty and a liquidity constraint. The reader will recall that the liquidity constraint is binding for an agent when $V_1^+(x) > \beta(1+r)E[V_1^+(k(x), e')]$ and is not binding when $V_1^+(x) = \beta(1+r)E[V_1^+(k(x), e')]$.

The logic used to prove Theorem 2 is quite simple. The logic is that the condition for aggregate precautionary saving ($\beta(1+r) < 1$) is precisely the condition that the liquidity constraint binds for a positive mass of agents in steady state. This follows from integrating the above necessary conditions over the population using the steady-state distribution y.

> *Theorem 2: Assume A1-4. In a steady-state equilibrium with a liquidity constraint* $\beta(1+r) < 1$ *if and only if the liquidity constraint binds for a positive mass of agents.*

Proof: See the Appendix.

Comment 3. Theorem 2 can be restated to apply to economies with a positive mass of each of a finite number of agent types, where type refers to both preferences and endowment processes as discussed in Comment 1. The theorem could then be stated as follows: $\beta_i(1+r) < 1$ for each agent type i if and only if the liquidity constraint binds for a positive mass of each agent type. The proof is exactly the same.

4 Discussion

In this section we discuss which features of the model economy are critical for producing aggregate precautionary saving. Our discussion takes the form of answers to several natural questions that one might like answers to: (1) Is a borrowing limit key?, (2) Are infinitely-lived agents key? and (3) Is a general equilibrium model key?

Is a Borrowing Limit Key?

It is clear that there must be borrowing limits in any meaningful dynamic model of individual decision making or else all consumption plans lie in the budget set. So the interesting issue is what type of borrowing limits might one want to consider rather than what would happen in the absence of borrowing limits. Given that the point of the precautionary savings literature is to investigate consumer behavior in the presence of uninsurable, idiosyncratic uncertainty, it would not make sense to consider borrowing limits that depend on an individual's shock history in such a way so as to effectively offer complete insurance. This would simply make aggregate precautionary saving zero by construction. Furthermore, as the environment facing an agent is stationary, it seems natural to focus on borrowing limits that are independent of time as well as history. For these reasons this paper focuses on borrowing limits that are independent of time and shock history (i.e. $k' \geq \underline{k}$).

From this perspective, the interesting question is not 'Is a borrowing limit key?' but rather 'What happens at different values of $\underline{k}$?'. Theorem 1 of this paper answers this question for all values of $\underline{k}$ that satisfy the condition $\underline{k} < \underline{e}w + \underline{k}(1+r)$. This case has been termed the case of a liquidity constraint. This case covers all borrowing limits of interest except the extreme case of a solvency constraint discussed in section 2.2. A solvency constraint allows agents to borrow up to the point where consumption could potentially be zero when an agent receives the worst income shock and is already at the borrowing limit: $\underline{k} = \underline{e}w + \underline{k}(1+r)$. One situation with a solvency constraint occurs when agents receive zero earnings with positive probability and $\underline{k} = 0$.

We do not have theoretical results for the solvency constraint case. However, Krusell and Smith (1994) report computational results suggesting that there is aggregate precautionary saving in the particular economies with a solvency constraint that they have considered.[8]

Are Infinitely-Lived Agents Key?

Given the well-known equivalence of an infinitely-lived agent with a sequence of finitely-lived agents comprising the generations of an altruistically linked family, it is clear that the results of this paper also apply to at least some situations where agents live for finitely many periods. Thus, a narrow answer to the question is that infinite lifetimes are not absolutely fundamental to our results on aggregate precautionary saving.

A more interesting question would ask when aggregate precautionary saving occurs in models with finitely-lived agents having no altruistic links. This is the focus of the overlapping generations framework. One can easily con-

[8] It is easy to show that $\beta(1+r) = 1$ in the case of no idiosyncratic uncertainty and a solvency constraint. In the case of idiosyncratic uncertainty and a solvency constraint the integral that we have focused on, $\int_X V_1^+(x)dy$, could potentially be infinite. If infinite, then our approach is silent on the value of the interest rate. If finite, then $\beta(1+r) = 1$ and there cannot be aggregate precautionary saving. We leave this issue for future research.

struct examples with overlapping generations of two-period-lived agents with quadratic preferences where the addition of labor endowment uncertainty in the second period of life (holding second period mean endowment constant for each agent) leaves unchanged all steady-state aggregates. This works as long as all agents are interior on their saving decision (i.e. borrowing limits do not bind) before and after the introduction of idiosyncratic uncertainty. Thus, with finite lifetimes agents need not optimally choose to hit their borrowing limits for at least some labor earning processes. This is a key difference with infinitely-lived agent models with the stationary labor income processes considered here. In these models optimal smoothing dictates that agents eventually hit their borrowing limits for any settings of this limit corresponding to a liquidity constraint.

In summary, aggregate precautionary saving is not a necessary feature of an equilibrium with idiosyncratic uncertainty in at least some overlapping generations models where borrowing limits do not bind. Thus, a broader answer to the question that this section addresses would be that finite lifetimes and a lack of altruistic links across generations can be quite important for questions of aggregate precautionary saving.[9]

Is a General Equilibrium Model Key?

To address this question one can focus on analyzing aggregate precautionary saving in a partial equilibrium context where factor prices are set at strictly positive but arbitrary values. As the results of this analysis are independent of the precise level of the real wage, there are then three cases to consider: Case 1: $\beta(1+r) < 1$, Case 2: $\beta(1+r) = 1$ and Case 3: $\beta(1+r) > 1$.

In case 1 it is clear that all agents are eventually at the borrowing limit

[9]To the best of our knowledge, there are currently no general theoretical results on aggregate precautionary saving in overlapping generations models where agents face earnings uncertainty. However, Abel (1985) and Fuster (1997) provide theoretical results when there is uncertainty over the length of a lifetime.

(i.e. $k = \underline{k}$) when there is no idiosyncratic uncertainty. The aggregate net capital owned by these agents is therefore $K = \underline{k}$. With idiosyncratic uncertainty it is clear that aggregate net capital must be at least as large, holding the borrowing limit constant. Infact, aggregate net capital will be strictly larger under two circumstances. First, a given borrowing limit may no longer be feasible in the presence of uncertainty. In this case aggregate capital must be strictly larger simply because uncertainty acts to raise the set of feasible borrowing limits. Second, aggregate capital will be strictly larger, holding the borrowing limit equal, in any situation in which smoothing consumption is beneficial. This is the case as agents can only smooth consumption by holding assets. The only situtation in which consumption smoothing will not be beneficial is when the interest rate is sufficently low so that agents find it optimal to maintain asset holdings at the borrowing limit when hit by any possible endowment shock.

In case 2 any distribution of capital across agents is a steady-state distribution when agents face no idiosyncratic uncertainty. In this case it is always optimal to consume the interest on asset holdings. The value of K is therefore indeterminate. When agents face idiosyncratic uncertainty the results of Sotomayor (1984, Theorems 2.8 and 4.5) guarantee that when $\beta(1+r) \geq 1$ the asset holdings of any individual will go to infinity with probability 1. This result suggests that no steady state distribution would exist with idiosyncratic uncertainty. In fact, this is precisely what was shown in the proof of Theorem 1 for any interest rate satisfying $\beta(1+r) \geq 1$. Thus, there is no theoretical result on aggregate precautionary saving, as we have defined it, as steady states do not exist when the interest rate satisfies $\beta(1+r) = 1$.

One can now see that case 3 runs into the same incomparability problem as in case 2. However, in case 3 the problem is even more dramatic as steady states do not exist even in the absence of idiosyncratic uncertainty. Hence, the only way to talk about aggregate precautionary saving is to make path comparisons. This is beyond the scope of this paper.

In summary, the only situation that permits steady-state comparisons of idiosyncratic and no idiosyncratic uncertainty is the case where $\beta(1+r) < 1$. For these interest rates the partial equilibrium comparisons yield a similar conclusion as was previously obtained in Theorem 1. This conclusion is that there is at least as much capital in steady state with idiosyncratic uncertainty than without whenever steady states can be compared. Furthermore, there will be strictly more capital in steady state provided that the interest rate is such that there is some gain to holding assets to smooth consumption.

5 Conclusion

The conventional wisdom among many economists is that the phenomena of precautionary saving is theoretically related to third derivative properties of expected utility representations of preferences. This paper shows that the conventional wisdom is mistaken for the question of the theoretical determinants of *aggregate* precautionary saving. This paper focuses on growth models populated by infinitely-lived agents who face a liquidity constraint. Within this context it has been shown that (1) aggregate precautionary saving is a necessary implication of an equilibrium where agents experience idiosyncratic uncertainty and face a liquidity constraint and that (2) aggregate precautionary saving coincides with the phenomena of endogenously binding liquidity constraints.

These results have been examined for robustness. We have shown that they hold for any amount of idiosyncratic uncertainty as well as any value of the liquidity constraint that is feasible in equilibrium. These results hold independent of third or higher order derivative conditions, independent of the rate at which agents discount future utility and independent of whether or not there are several types of agents in the economy differing in preferences and endowment processes. Furthermore, the results do not depend on whether one wants to characterize aggregate precautionary saving in a general or partial equilibrium framework. However, the results do depend, just like several important results in capital theory, on whether or not agents are altruistically linked in particular ways. For example, aggregate precautionary saving is not a necessary property of an equilibrium in at least some overlapping generations models as discussed in section 4. Thus, the sign of the third derivative is not irrelevant for aggregate precautionary saving in such models even though it is irrelevant in the models that we investigate. It remains for future work to determine whether our results extend to alternative preference specifications or to situations where labor supply is endogenous.[10]

Given that this paper has concentrated on the theory of aggregate precautionary saving in the presence of liquidity constraints, some readers might wonder whether there is any evidence for precautionary saving or for the existence of liquidity constraints. On the issue of precautionary saving, we refer the reader to the survey of Browning and Lusardi (1996). They review the literature that relates measures of risk (typically the variance of earnings or income) to wealth, to the ratio of wealth to income or to some other variables. Many but not all of the studies reviewed find some association between the two measures. In addition, some of the studies claim that precautionary saving accounts for some positive fraction of accumulated wealth. On the issue of evidence for the existence of liquidity constraints of the type that this paper considers, we refer the reader to the survey in Deaton (1992). After reviewing the empirical work related to the presence of liquidity constraints, Deaton (1992, p.160) states:

> *While I have no doubt that all of the contrary evidence can be explained if we try hard enough, I find the evidence for some form of liquidity constraints both plausible and convincing.*

[10]Future work will also determine whether third derivative conditions are important for the magnitude of aggregate precautionary saving in these models even though they are irrelevant for the issue of whether or not an equilibrium displays aggregate precautionary saving.

References

Abel, A. (1985), " Precautionary Saving and Accidental Bequests," *American Economic Review* **75**, 777-91.

Abel, A, Mankiw, G, Summers, L. and R. Zeckhauser (1989), " Assessing Dynamic Efficiency: Theory and Evidence," *Review of Economic Studies* **56**, 1-20.

Aiyagari, R. (1994), " Uninsured Idiosyncratic Risk and Aggregate Savings," *Quarterly Journal of Economics* **109**, 659-684.

Aiyagari, R. (1995), " Optimal Capital Income Taxation with Incomplete Markets, Borrowing Constraints and Constant Discounting," *Journal of Political Economy* **103**, 1158-75.

Becker, R. (1980), " On the Long-Run Steady State in a Simple Dynamic Model of Equilibrium with Heterogeneous Households," *Quarterly Journal of Economics* **95**, 375-382.

Bewley, T. (1984), " Notes on Stationary Equilibrium with a Continuum of Independently Fluctuating Consumers," Yale University.

Billingsley, P. (1986), " Probability and Measure," John Wiley and Sons; New York; Second Edition.

Browning, M. and Lusardi, A. (1996), " Household Saving: Micro Theories and Micro Facts," *Journal of Economic Literature* **34**, 1797-1855.

Caballero, R. (1991), " Earnings Uncertainty and Aggregate Wealth Accumulation," *American Economic Review* **81**, 859-71.

Carroll, C. (1992), "The Buffer-Stock Theory of Saving: Some Macroeconomic Evidence," *Brookings Papers on Economic Activity* **2**, 61-135.

Carroll, C. (1997), " Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis," *Quarterly Journal of Economics* **112**, 1-56.

Cass, D. (1965), " Optimum Growth in an Aggregative Model of Capital Accumulation," *Review of Economic Studies* **32**, 233-240.

Castaneda, A, Diaz-Gimenez, J. and Rios-Rull, V. (1994), " On the Cyclical Behavior of the Income Distribution," University of Pennsylvania, mimeo.

Clarida, R. (1990), " International Lending and Borrowing in a Stochastic, Stationary Equilibrium," *International Economic Review* **31**, 543-58.

Chamberlain, G. and Wilson, C. (1984), " Optimal Intertemporal Consumption Under Uncertainty," SSRI Discussion Paper 8422; University of Wisconsin.

Deaton, A. (1991), " Saving and Liquidity Constraints," *Econometrica* **59**, 1221-48.

Deaton, A. (1992), "Understanding Consumption," Clarendon Press; Oxford.

Deaton, A. and Laroque, G. (1992), " On the Behavior of Commodity Prices," *Review of Economic Studies* **59**, 1-23.

DEN HAAN, W. (1997), " Solving Dynamic Models with Aggregate Shocks and Heterogeneous Agents," *Macroeconomic Dynamics* **1**, 355-386.

DIAMOND, P. (1965), " National Debt in a Neoclassical Growth Model," *American Economic Review* **55**, 1126-1150.

FOLEY, D. AND HELLWIG, M. (1975), " Asset Management with Trading Uncertainty," *Review of Economic Studies* **42**, 327-46.

FRIEDMAN, M. (1957), " A Theory of the Consumption Function," Princeton University Press.

FUSTER, L. (1997), " Effects of Uncertain Lifetime and Annuity Insurance on Capital Accumulation and Growth," Universitat Pompeu Fabra, mimeo.

HERNANDEZ, A. (1991), " The Dynamics of Competitive Equilibrium Allocations with Borrowing Constraints," *Journal of Economic Theory* **55**, 180-191.

HOPENHAYN, H. AND PRESCOTT, E. (1992), " Stochastic Monotonicity and Stationary Dsitributions for Dynamic Economies," *Econometrica* **60**, 1387-1406.

HUGGETT, M. (1993), " The Risk-Free Rate in Heterogeneous-Agent Incomplete-Insurance Economies," *Journal of Economic Dynamics and Control* **17**, 953-969.

HUGGETT, M. (1996), "Wealth Distribution in Life-Cycle Economies," *Journal of Monetary Economics* **38**, 469-494.

HUGGETT, M. (1997), " The One-Sector Growth Model with Idiosyncratic Shocks: Steady States and Dynamics," *Journal of Monetary Economics* **39**, 385-403.

KENNICKELL, A. AND M. STARR-MCCLUER (1994), " Changes in Family Finances from 1989-92: Evidence from the Survey of Consumer Finances," Federal Reserve Bulletin; 861-882.

KOOPMANS, T. (1965), " On the Concept of Optimal Growth," The Econometric Approach to Development Planning; Chicago: Rand McNally.

KRUSELL, P. AND SMITH, A. (1994), " Income and Wealth Heterogeneity, Aggregate Fluctuations and the Representative Agent," University of Rochester, mimeo.

KRUSELL, P. AND SMITH, A. (1997), " Income and Wealth Heterogeneity, Portfolio Choice, and Equilibrium Asset Returns," *Macroeconomic Dynamics* **1**, 387-422.

KYDLAND, F AND PRESCOTT, E. (1982), " Time to Build and Aggregate Fluctuations," *Econometrica* **50**, 1345-70.

LAITNER, J. (1979), " Bequests, Golden-age Capital Accumulation and Government Debt," *Economica* **46**, 403-14.

LELAND, H. (1968), " Savings and Uncertainty: The Precautionary Demand for Saving," *Quarterly Journal of Economics* **82**, 465-73.

LUCAS, R. AND PRESCOTT, E. (1974), " Equilibrium Search and Unemployment," *Journal of Economic Theory* **7**, 188-209.

LUCAS, R. (1980), " Equilibria in a Pure Currency Economy," in Models of Monetary Economies; editors J. Kareken and N. Wallace; Federal Reserve Bank of Minneapolis.

MENDELSON, H. AND AMIHUD, Y. (1982), " Optimal Consumption Policy Under Uncertainty," *Management Science* **28**, 683-697.

MILLER, B. (1975), " Optimal Consumption with a Stochastic Income Stream," *Econometrica* **42**, 253-266.

MIRMAN, L. (1971), " Uncertainty and Optimal Consumption Decisions," *Econometrica* **39**, 179-185.

MIRMAN, L. AND ZILCHA, I. (1975), " On Optimal Growth Under Uncertainty," *Journal of Economic Theory* **11**, 329-339.

OBIOLS-HOMS, F. (1997), " Incomplete Unemployment Insurance and Aggregate Fluctuations," ECARE; mimeo.

PROJECTOR, D. AND WEISS, G. (1966), " Survey of Financial Characteristics of Consumers," Board of Governors of the Federal Reserve System.

RIOS-RULL, J.V. (1995), "Models With Heterogeneous Agents," in T. Cooley, ed.; Frontiers of Business Cycle Research, Princeton University Press.

ROCKAFELLAR, T. (1970), " Convex Analysis," Princeton University Press; Princeton.

SANDMO, A. (1970), " The Effect of Uncertainty on Saving Decisions," *Review of Economic Studies* **37**, 353-60.

SCHECHTMAN, J. AND ESCUDERO, V. (1977), " Some Results on "An Income Fluctuation Problem"," *Journal of Economic Theory* **16**, 151-166.

SKINNER, J (1988), " Risky Income, Life Cycle Consumption, and Precautionary Savings," *Journal of Monetary Economics* **22**, 237-55.

SOTOMAYOR, M. (1984), " On Income Fluctuations and Capital Gains," *Journal of Economic Theory* **32**, 14-35.

STOKEY, N. AND LUCAS, R. (1989), " Recursive Methods in Economic Dynamics," Harvard University Press.

6 Appendix

Proof of Lemma 1:

(1) The Theorem of the Maximum together with the strict concavity of u and V generates this result.

(2) (i) As the value function is strictly increasing and strictly concave in its first argument, the right-hand derivative exists (Rockafellar (1970, Thm. 23.1)) and is strictly decreasing in k. It is clear that $V(k,e) = V(k',e')$ when $ew + k(1+r) = e'w + k'(1+r)$. Thus, $V_1^+(x)$ is also strictly decreasing in e.

(2) (ii) To prove that $V_1^+(x)$ is bounded we argue as follows. First, Lemma 1(2)(i) implies that $V_1^+(x) \leq V_1^+(\underline{k}, \underline{e})$ for all $x \in X$. Second, Rockafellar (1970, Thm. 29.1) implies that any subgradient of V with respect to k is of the form $\lambda_1(1+r)$, where λ_1 is a 'multiplier' associated with the resource constraint in our dynamic programming problem. Thus, we know that $V_1^+(\underline{k}, \underline{e}) \leq \lambda_1(1+r)$ if a subgradient exists. Finally, Rockafellar (1970, Cor. 28.3.1) states Kuhn-Tucker conditions that are necessary and sufficient for a solution to concave programming problems. These conditions are stated in terms of subdifferentials so the objective function need be concave but not necessarily differentiable.

Corollary 28.3.1 applies to our dynamic programming problem as the budget set has nonempty interior. Since a solution to the dynamic programming problem exists, Corollary 28.3.1 states that there exist finite, positive values $(\lambda_1, \lambda_2, \lambda_3)$ satisfying the following restriction at the optimal levels of (c, k'). In the equation below $h(c,k') \equiv u(c) + \beta E[V(k',e')]$, ∂h denotes the subdifferential and $(\lambda_1, \lambda_2, \lambda_3)$ are multipliers associated with the resource constraint, the liquidity constraint and the non-negativity constraint on consumption.

$$(0,0) \in \partial h(c,k') + \lambda_1(-1,-1) + \lambda_2(0,1) + \lambda_3(1,0)$$

(2) (iii) First show $V_1^+(x) \geq max\{u'(c(x))(1+r), \beta(1+r)E[V_1^+(k(x),e')]\}$. The opposite inequality will then also be shown to hold. Clearly for $\epsilon > 0$, $V(k+\epsilon, e) - V(k,e) \geq u(c(k,e)+\epsilon(1+r)) - u(c(k,e)$. Similarly, $V(k+\epsilon,e) - V(k,e) \geq \beta E[V(k(k,e)+\epsilon(1+r), e')] - \beta E[V(k(k,e),e')]$ Dividing by ϵ and taking limits produces $V_1^+(x) \geq u'(c(x))(1+r)$ and $V_1^+(x) \geq \beta(1+r)E[V_1^+(k(x),e')]$. The second inequality relies on differentiating under the expectation operator. Billingsley (1986, Thm. 16.8) shows that $V_1^+(x)$ bounded is sufficient for this to be valid. Boundedness was established in Lemma 1(2)(ii).

Now show that $V_1^+(x) \leq max\{u'(c(x))(1+r), \beta(1+r)E[V_1^+(k(x),e')]\}$. The following must hold for $\epsilon > 0$:

$$V(k+\epsilon,e) - V(k,e) \leq u'(c(k,e))\Delta c + \beta E[V_1^+(k(k,e),e')]\Delta k'$$

$$V(k+\epsilon,e) - V(k,e) \leq max\{u'(c(k,e), \beta E[V_1^+(k(k,e),e')]\}\epsilon(1+r)$$

The first inequality above follows from the concavity of u and V, after defining $\Delta c \equiv c(k+\epsilon, e) - c(k, e)$ and $\Delta k' \equiv k(k+\epsilon, e) - k(k, e)$. The second inequality follows from the first as the budget constraint implies that $\Delta c + \Delta k' = \epsilon(1+r)$ The result follows from dividing each side of the second inequality by ϵ and taking limits.

(3) Lemma 1 (2)(i) and the fact that there is idiosyncratic uncertainty implies that there is $\hat{e} \in E$ and a set $A = E \cap [0, \hat{e}]$ such that $0 < P(A) < 1$ and such that the strict inequality below holds for all $e \in A$. The two weak inequalities below follow because $\beta(1+r) \leq 1$ and because $V_1^+(x)$ is decreasing in k.

$$V_1^+(\underline{k}, e) > E[V_1^+(\underline{k}, e')] \geq \beta(1+r)E[V_1^+(\underline{k}, e')] \geq \beta(1+r)E[V_1^+(k(\underline{k}, e), e')]$$

Lemma 1 (2)(iii) then implies that the following equation holds for all $e \in A$.

$$V_1^+(\underline{k}, e) = u'(c(\underline{k}, e))(1+r) > \beta(1+r)E[V_1^+(k(\underline{k}, e), e')]$$

The continuity of $u'(c(x))$ then implies that the result holds for $x \in [\underline{k}, a) \times \hat{e}$ for some $a > \underline{k}$. As $V_1^+(x)$ is decreasing in e and as asset holding is at the borrowing constraint, the result must hold for all $x \in B = [\underline{k}, a) \times A$.

(4) We first establish that $k(k, e) < k_1$ on a set of positive probability when $k = k_1$. Suppose by way of contradiction that this does not hold. Thus, $k(k_1, e) \geq k_1$ for all $e \in A$ where $P(A) = 1$. By Lemma 1 (2) and by the fact that asset holdings are off the corner the equality below holds. The inequality holds as $\beta(1+r) \leq 1$ and as $V_1^+(x)$ is decreasing in k.

$$V_1^+(k_1, e) = \beta(1+r)E[V_1^+(k(k_1, e), e')] \leq E[V_1^+(k_1, e')]$$

Given that there is idiosyncratic uncertainty and that $V_1^+(x)$ is strictly decreasing in e, this cannot hold for all $e \in A$. Contradiction.

Define $\epsilon_1 = k_1 - k(k_1, e)$ using any $e \in E$ for which $\epsilon_1 > 0$ and for which any open neighborhood of e lieing in E has positive probability. Set $\epsilon = \epsilon_1/2$. Since $k(x)$ is continuous (Lemma 1 (1)), there is a sufficiently small neighborhood (constructed using the Euclidean metric) around (k_1, e) lieing in X such that $k(x) \leq k_1 - \epsilon$ for all x in that neighborhood. Clearly, this also holds for a neighborhood $[k_1, k_1 + \delta) \times A$ lieing inside the previously constructed neighborhood. The set A can by choosen to have positive probability. □

Proof of Theorem 1:

It will first be shown that in steady-state equilibrium $\beta(1+r) \leq 1$ for both part (i) and (ii) of Theorem 1. This follows from the three equations below:

$$V_1^+(x) \geq \beta(1+r)E[V_1^+(k(x), e')] \equiv \beta(1+r)E[V_1^+(x')|x]$$

$$\int_X V_1^+(x)dy \geq \beta(1+r)\int_X E[V_1^+(x')|x]dy$$

$$1 \geq \beta(1+r)$$

The first equation above holds by Lemma 1 (2) and by taking expectations with respect to x'. The second equation integrates the first using the steady-state distribution y. By Lemma 1 (2) these integrals are finite. The third equation follows from the second after noting two facts. First, Stokey and Lucas (1989, Thm. 8.3) implies that $\int_X E[V_1^+(x')|x]dy = \int_X V_1^+(x')dy^*$, where $y^*(B) \equiv \int_X P(x,B)dy$ for $B \in \mathcal{X}$. Second, the definition of steady-state equilibrium implies that $y^* = y$.

It will now be shown that when there is idiosyncratic uncertainty $\beta(1+r) \neq 1$. The key step is to show that $\beta(1+r) = 1$ implies that there is a set B of positive measure ($y(B) > 0$) on which the borrowing constraint binds. From this fact and from the arguments given above the following four inequalities hold:

$$\int_B V_1^+(x)dy > \beta(1+r)\int_B E[V_1^+(x')|x]dy$$

$$\int_{X/B} V_1^+(x)dy \geq \beta(1+r)\int_{X/B} E[V_1^+(x')|x]dy$$

$$\int_X V_1^+(x)dy > \beta(1+r)\int_X E[V_1^+(x')|x]dy$$

$$\beta(1+r) < 1$$

From Lemma 1 (3) there is a set $B = [\underline{k}, a) \times A \in \mathcal{X}$ such that $x \in B$ implies that $V_1^+(x) > \beta(1+r)E[V_1^+(x')|x]$. We now show that $y(B) > 0$. Define $k_1 \equiv sup\{k : y([k,\infty) \times E) = 1\}$. Note that $y([k_1, a) \times E) > 0$ for any $a > k_1$ by the definition of k_1 and that $y([k_1,a) \times A) = P(A)y([k_1,a) \times E)$ by the independence of shocks. We now prove by way of contradiction that $k_1 = \underline{k}$. Define the set $D = [\underline{k}, k_1) \times E$. Lemma 1 (4) implies that $P(x, D) = 1$ for all $x \in [k_1, a) \times A$, where $P(A) > 0$. Since $P(x,D) = 1$ on a set of positive measure (i.e. $y([k_1,a) \times A) > 0$), we have that $y(D) = \int_X P(x,D)dy > 0$. This proves that $k_1 = \underline{k}$ and that $y([\underline{k},a) \times E) > 0$ for any $a > \underline{k}$. Now as shocks are independent we have that $y(B) \equiv y([\underline{k},a) \times A) = P(A)y([\underline{k},a) \times E) > 0$.

It will now be shown that when there is no idiosyncratic uncertainty $\beta(1+r) \geq 1$. Suppose that $\beta(1+r) < 1$. Define B as the set of $x \in X$ such that $V_1^+(x) > \beta(1+r)V_1^+(k(x),1)$. Define X/B as the set of $x \in X$ such that $V_1^+(x) = \beta(1+r)V_1^+(k(x),1)$. As $V_1(x)$ is decreasing in its first argument, $k(x) < k$ within X/B except on a set of measure zero. Within the set B, $k(x) = \underline{k}$ and thus $k(x) \leq k$. These two results and the fact that $\underline{k} < K$ imply that the inequality below holds.

The equality is, once again, due to Stokey and Lucas (1989, Thm. 8.3) and to the fact that y is time invariant in steady state. This establishes the contradiction.

$\int_X kdy > \int_X k(x)dy \equiv \int_X E[k'|x]dy = \int_X k'dy$ □

Proof of Theorem 2:

We first prove the necessity of a binding liquidity constraint for aggregate precautionary savings by contraposition. Thus, suppose that the constraint does not bind for a positive mass of agents in a steady-state equilibrium. Then the first equation below holds except on a set of measure zero. The second equation below follows by integration as the value of the integral does not depend on the value of the function on a set of measure zero. Lemma 1 (2) guarantees that these integrals are finite. The third equation follows from two facts. First, apply Stokey and Lucas (1989, Thm. 8.3) to get that $\int_X E[V_1^+(x')|x]dy = \int_X V_1^+(x)dy^*$, where $y^*(B) \equiv \int_X P(x,B)dy$. Second, note that in steady-state equilibrium $y^* = y$. The third equation completes the proof by contraposition as the negation of $\beta(1+r) < 1$ is $\beta(1+r) \geq 1$.

$$V_1^+(x) = \beta(1+r)E[V_1^+(x')|x]$$

$$\int_X V_1^+(x)dy = \beta(1+r)\int_X E[V_1^+(x')|x]dy$$

$$1 = \beta(1+r)$$

We now prove the sufficiency of a binding liquidity constraint for aggregate precautionary saving. Thus, suppose that the constraint binds for a positive mass of agents. Let B be the set of states for which the constraint binds and X/B be the set for which the constraint does not bind. We then know that the first two equations hold after integrating the necessary condition from Lemma 1 (2) over the set B and its complement X/B. The third equation below follows by combining the previous two equations.

$$\int_B V_1^+(x)dy > \beta(1+r)\int_B E[V_1^+(x')|x]dy$$

$$\int_{X/B} V_1^+(x)dy = \beta(1+r)\int_{X/B} E[V_1^+(x')|x]dy$$

$$\int_X V_1^+(x)dy > \beta(1+r)\int_X E[V_1^+(x')|x]dy$$

The result follows from this last equation from applying, once again, Stokey and Lucas (1989, Thm. 8.3). □