# A Dynamic Analysis of an Endogenous Growth Model with Leisure

Salvador Ortigueira*
*Centro de Investigación Económica, ITAM*
*Ave. Camino Sta. Teresa 930*
*10700 Mexico D.F.*

January, 1997

**Abstract**

The main goal in this paper is to analyze an economic model of endogenous growth where human capital accumulation acts as the engine propelling economic activity. The added ingredient in our model is that agents derive utility from consumption and leisure, where leisure is defined as the amount of time devoted to those activities augmented by the level of education. Under regular conditions we show that there is a unique globally stable balanced growth path. We also provide a characterization of the behavior of our economic variables along the transition.

*Key words:* Qualified Leisure; Endogenous Growth; Equilibrium Dynamics.

*JEL classification:* E20; J22; 041.

*I am grateful to Manuel S. Santos for his suggestions and criticisms. Remaining errors are my own responsability. Phone: (525) 628 4197; Fax: (525) 628 4058; E-mail: ortigue(at)master.ster.itam.mx.

## 1. Introduction

The main aim in this paper is to analyze an economic model of endogenous growth where human capital accumulation acts as the engine propelling economic activity. The added ingredient in our model is that consumers' preferences are of the type proposed by Becker (1965). Agents derive utility from consumption and leisure, and leisure is determined by the amount of time devoted to these activities factored by the level of human capital. The idea behind this formulation is that agents having a greater amount of human capital can obtain a higher level of utility. This is related to models with home production [e.g., Benhabib, Rogerson and Wright (1991)]. Some empirical evidence in favor of this hypothesis can be found in Michael (1973). This type of preferences are also used by Heckman (1976) in a life-cycle model of labor supply, earnings and consumption.

The time-allocation decision plays a crucial role in the theory of endogenous growth. Indeed, the long-run rate of growth becomes determined by the time devoted to schooling activities. The inclusion of leisure in this class of models has been challenging in many respects. Important aspects of economic models as existence, uniqueness and stability of the long-run equilibrium may be affected by leisure considerations. Additionally, the dynamic behavior along the transition of worked hours, consumption and other economic variables also change if leisure is an endogenous variables.

In this respect, Ladrón-de-Guevara *et al.* (1997) present an endogenous growth model with unqualified leisure in the utility function [i.e., leisure is not adjusted by the level of human capital], and show that multiple balanced growth paths may arise. Moreover, under certain conditions, the unstable balanced path turns out to be non-optimal. In the taxation literature the presence of leisure as a decision variable has also been proved to play a major role in the determination of the growth effects of different taxation policies [e.g., Rebelo (1991) and Stokey and Rebelo (1995)]. In this paper we show that the consequences of including human capital-adjusted leisure in the utility function may drastically differ from those of the unqualified leisure case. Thereby, the election of the particular modelization of leisure is meaningful in growth models. In this research, however, we do not attempt to assess which formulation is more appropriate to study growth-related

issues; our concern is limited to figure out the dynamic implications derived from the presence of qualified leisure.

As a matter of expositional clarity we briefly review the present status of the theory related to our endogenous growth model. Concerning long-run growth, in the original endogenous growth model without leisure, [e. g., Uzawa(1965) and Lucas(1988)] if externalities are not present the long-run equilibrium is defined by a unique physical-to-human capital ratio. The rate of growth corresponding to this steady-state solution is determined by parameters on preferences, population growth and the productivity in the human capital sector. Among these parameters, the productivity in the educational sector, the rate of population growth and the elasticity of intertemporal substitution for consumption have a positive effect on the rate of growth, whereas the value of the discount rate acts in the opposite direction.

The analysis of stability has been first addressed by Caballé and Santos (1993). Appealing to techniques from the Maximum Principle and from Dymanic Programming, the authors show that the steady-state equilibrium is globally stable. Moreover, if the steady-state solution is perturbed, say, by a positive shock on physical capital, the dynamic process approaching the ray of steady states falls into one of the following three categories: (a) *Paradoxical growth case*, the economy converges to a steady state with lower amounts of both kinds of capital. (b) *Exogenous growth case*, the economy converges to the initial steady state. (c) *Normal growth case*, the economy converges to a steady state with higher amounts of physical and human capital.

As it is to be expected, in this simple framework without leisure, the evolution of worked hours determines each of these three growth cases. Caballé and Santos (1993), Chamley (1993) and Mulligan and Sala-i-Martin (1993) offer good insights as to why an increase in physical capital may affect the amount of optimal working time. On one hand, an increase in physical capital makes labor more productive leading to higher wages. On the other hand, the abundance of physical capital in the economy increases the relative price of human capital. The relative magnitude of these two opposite forces will then determine the initial impact on the allocation of time between the two activities,

production and schooling. Surprisingly, the rule separating the three growth cases is simple. If the inverse of the elasticity of intertemporal substitution for consumption is greater (lower) than the elasticity of the marginal productivity of labor with respect to physical capital, then the normal (paradoxical) case is obtained. In the border-line case in which these values are equal, the economy belongs to the exogenous growth case.

The presence of qualified leisure in the utility function brings about remarkable differences with respect to the analysis of Caballé and Santos (1993) and Chamley (1993) on the transitional dynamics of the Uzawa-Lucas framework. For instance, the rule defining the three growth cases involves all parameters in the economy. Likewise, the paradoxical case becomes now more plausible.

The outline of the paper is the following. Section 2 presents the model and establishes uniqueness of the steady-state equilibrium. Section 3 is devoted to study the stability properties of the stationary solution. We prove global stability via a Liapounov function of the type proposed by Cass and Shell (1976). The analysis on transitional dynamics is presented in section 4. Here, we study the qualitative behavior of leisure, worked hours, education and consumption along the transition path. This analysis is completed in Section 5 with the aid of some numerical computations. Section 6 presents our concluding remarks.

## 2. The Model

We assume a competitive economy which is closed to international markets and is populated by identical and infinitely lived agents. Population at date $t$ is denoted by $N(t)$, which grows at the constant, exogenous rate, $n$. Preferences over per capita consumption, $\tilde{c}(t)$, and efficiency units of leisure, $l(t)\tilde{h}(t)$, are represented by a $\mathcal{C}^2$, concave, increasing utility function $U(\tilde{c}(t), l(t)\tilde{h}(t))$. Individuals have a unit of time at each period which is allocated among leisure activities, $l(t)$, producing goods, $u(t)$, and education, $e(t)$.

The technology in the model is similar to that proposed by Caballé and Santos (1993), which subject to minor considerations generalizes the technology presented in Lucas

(1988). The output sector produces an aggregate good by means of physical capital and labor (in efficiency units). This aggregate good can be consumed or accumulated as physical capital. The technology in this sector is represented by a linearly homogeneous, $\mathcal{C}^2$, concave, increasing function $F(\tilde{K}(t), \tilde{L}(t))$, where $\tilde{L}(t) = N(t)u(t)\tilde{h}(t)$. We furthermore assume that both factors are essential, and Inada conditions are satisfied.

$$\lim_{\tilde{K}\to 0} F_{\tilde{L}}(\tilde{K}, \tilde{L}) = \infty, \lim_{\tilde{L}\to 0} F_{\tilde{K}}(\tilde{K}, \tilde{L}) = \infty, \text{ and } F(0, \bar{L}) = F(\bar{K}, 0) = 0$$

The technology for the accumulation of human capital is given by the process, $\dot{\tilde{h}}(t) = \tilde{h}(t)G(1 - u(t) - l(t))$, where $G(\cdot)$ is a $\mathcal{C}^2$, concave and increasing function.

In the economy discussed here, neither externalities nor distortions are present, and so every competitive equilibrium will yield a Pareto Optimum allocation. This allows us to present the market problem as a central planner's program. Thus, it is written as follows,

$$\begin{aligned} V(\tilde{K}(0), \tilde{h}(0)) \quad &= \quad \max_{\tilde{c}(t), l(t), u(t)} \int_0^\infty e^{-\rho t} N(t) U(\tilde{c}(t), l(t)\tilde{h}(t)) dt \qquad (P0) \\ \text{subject to} & \\ & \dot{\tilde{K}}(t) = F(\tilde{K}(t), N(t)u(t)\tilde{h}(t)) - N(t)\tilde{c}(t) \\ & \dot{\tilde{h}}(t) = \tilde{h}(t)G(1 - u(t) - l(t)) \\ & 0 \leq u(t) \leq 1,\ 0 \leq l(t) \leq 1,\ u(t) + l(t) \leq 1, \\ & \tilde{c}(t) \geq 0,\ \tilde{K}(t) \geq 0,\ \tilde{h}(t) \geq 0 \\ & N(t) = N(0)e^{nt},\ \ n > 0,\ \ N(0) = 1 \\ & \tilde{K}(0), \tilde{h}(0) \text{ given} \end{aligned}$$

**Definition 2.1:** *A balanced path (or steady-state equilibrium) for this economy is an optimal solution* $\{\tilde{c}(t), \tilde{k}(t), \tilde{h}(t),\ l(t), u(t)\}$ *to problem (P0) for some initial conditions* $\tilde{k}(0)$ *and* $\tilde{h}(0)$*, such that they grow at a constant rate, and the output-capital ratio is constant.*

Given the assumption of CRS, it is well known that at a steady-state solution per capita streams $\tilde{c}(t), \tilde{k}(t), \tilde{h}(t)$ grow at a common rate, say $\nu$. In order to characterize the

transitional dynamics of endogenous variables we transform the model by defining the following new variables:

$$\begin{aligned} K(t) &= \tilde{K}(t)e^{-(\nu+n)t} \\ h(t) &= \tilde{h}(t)e^{-\nu t} \\ c(t) &= \tilde{c}(t)e^{-\nu t} \end{aligned}$$

This transformation was first proposed by Caballé and Santos (1993) to obtain non-growing variables at the steady state.

Regarding preferences we assume a constant elasticity of substitution utility function on consumption and qualified leisure, i.e.,

$$U(c,lh) = \frac{1}{1-\sigma}\left(c^{\gamma}(lh)^{1-\gamma}\right)^{1-\sigma} \qquad \text{for } \sigma > 0,\ \sigma \neq 1,\ 0 < \gamma \leq 1.$$

$$U(c,lh) = \gamma \log c + (1-\gamma)\log(lh) \qquad \text{for } \sigma = 1,\ \ 0 < \gamma \leq 1.$$

This functional form implies a constant elasticity of intertemporal substitution, $\sigma^{-1}$, for consumption and qualified leisure jointly, and a unit intratemporal elasticity of substitution between consumption and leisure. These properties are usually required to match the observed growth facts and guarantee the existence of a steady state. Thus, these functional forms have been extensively used in the real business cycle literature [e.g., Kydland and Prescott (1982), Prescott (1986)] and in the literature of household production [e.g., Benhabib, Rogerson and Wright (1991)].

Under these assumptions on preferences and technologies, the transformed optimization problem is the following

$$V(k(0),h(0)) = \max_{c(t),l(t),u(t)} \int_0^{\infty} e^{-(\rho-n-(1-\sigma)\nu)t}\left[\frac{1}{1-\sigma}\left(c(t)^{\gamma}(l(t)h(t))^{1-\gamma}\right)^{1-\sigma}\right]dt \quad (P1)$$

subject to

$$\dot{k}(t) = F(k(t),u(t)h(t)) - (\nu+n)k(t) - c(t) \tag{2.1}$$

$$\dot{h}(t) = h(t)G(1-u(t)-l(t)) - \nu h(t) \tag{2.2}$$

$$0 \leq u(t) \leq 1,\ 0 \leq l(t) \leq 0,\ 0 \leq u(t)+l(t) \leq 1,$$

$$c(t) \geq 0, \;\; k(t) \geq 0, \;\; h(t) \geq 0$$
$$k(0) = \tilde{K}(0) \text{ and } h(0) = \tilde{h}(0) \text{ given.}$$

The Hamiltonian to this problem is:

$$H(c(t), k(t), h(t), l(t), u(t), \varepsilon_1(t), \varepsilon_2(t)) =$$
$$= e^{-(\rho - n - (1-\sigma)\nu)t} \left[ \frac{(c(t)^{\gamma}(l(t)h(t))^{1-\gamma})^{1-\sigma}}{1-\sigma} \right]$$
$$+\varepsilon_1(t) \left( F[k(t), u(t)h(t)] - (\nu + n)k(t) - c(t) \right) + \varepsilon_2(t) \left( h(t)G[(1 - u(t) - l(t)] - \nu h(t) \right).$$

It is well understood [Santos (1990)] that the value function $V(\cdot, \cdot)$ defined in (P1) is concave and twice continuously differentiable at interior solutions. From Arrow's theorem [see Kamien and Schwartz (1981), for instance] we know that if the maximized Hamiltonian is concave in state variables, given co-state variables, then the first order conditions are necessary and sufficient to characterize the maximum. Even though this theorem may be difficult to check in general, it turns out that it is easy to show that it is fulfilled for the special case where $G(1 - u - l)$ is linear. The first order conditions for the current value Hamiltonian[1] are the following

$$\gamma c(t)^{(1-\sigma)\gamma - 1} (l(t)h(t))^{(1-\sigma)(1-\gamma)} = \eta_1(t) \tag{2.3}$$
$$(1-\gamma) c(t)^{(1-\sigma)\gamma} (l(t)h(t))^{-\sigma-(1-\sigma)\gamma} = \eta_2(t) G'(1 - u(t) - l(t)) \tag{2.4}$$
$$\eta_1(t) F_2(k(t), u(t)h(t)) = \eta_2(t) G'(1 - u(t) - l(t)) \tag{2.5}$$
$$\dot{\eta}_1(t) = [\rho + \sigma\nu - F_1(k(t), u(t)h(t))]\eta_1(t) \tag{2.6}$$
$$\dot{\eta}_2(t) = [\rho - n + \sigma\nu - G'(1 - u(t) - l(t))l(t) - G'(1 - u(t) - l(t))u(t) - G(1 - u(t) - l(t))]\eta_2(t) \tag{2.7}$$

Adding the following transversality condition to these f.o.c., gives the set of necessary and sufficient conditions to characterize an optimum,

$$\lim_{t \to \infty} e^{-(\rho - n - (1-\sigma)\nu)t} \left( \eta_1(t)k(t) + \eta_2(t)h(t) \right) = 0,$$

[1] The current value Hamiltonian $\mathcal{H}$ is defined as $e^{(\rho - n - (1-\sigma)\nu)t} H$, then $\eta_1(t) = \varepsilon_1(t) e^{(\rho - n - (1-\sigma)\nu)t}$ and $\eta_2(t) = \varepsilon_2(t) e^{(\rho - n - (1-\sigma)\nu)t}$.

this implies that $\rho - n - (1-\sigma)\nu > 0$

*Steady-State Analysis*

If steady-state conditions are assumed on (2.1)-(2.7), it is possible to determine the long-run value of the following endogenous variables $\{\left(\frac{c}{k}\right)^*, \left(\frac{h}{k}\right)^*, l^*, u^* \text{ and } \nu\}$. From (2.3) it is obtained that in steady state $\frac{\dot{\eta}_1}{\eta_1} = 0$. Also, from (2.5), $\frac{\dot{\eta}_1}{\eta_1} = \frac{\dot{\eta}_2}{\eta_2}$, then we know that,

$$\rho + \sigma\nu = F_1(k, uh) \tag{2.8}$$

$$\rho - n + \sigma\nu = G'(1-u-l)l + G'(1-u-l)u + G(1-u-l) \tag{2.9}$$

Inserting (2.3) and (2.4) into (2.5), yields

$$\frac{1-\gamma}{\gamma} = \frac{k}{c} l \frac{h}{k} F_2(k, uh) \tag{2.10}$$

Finally, the feasibility restrictions (2.1) and (2.2) imply that at a steady-state equilibrium

$$\frac{c}{k} = F_1(k, uh) + F_2(k, uh) u \frac{h}{k} - \nu - n \tag{2.11}$$

$$\nu = G(1-u-l) \tag{2.12}$$

The five equations system (2.8)-(2.12) can be solved for the five endogenous variables. Unlike the model with non-qualified leisure in the utility function, presented in Ladrón-de-Guevara *et al.* (1997), we can prove here an uniqueness result.

**Theorem 2.1:** *Consider the optimization problem defined in (P1), where $F(\cdot,\cdot)$ is a $C^2$, linearly homogeneous, strictly increasing and concave function that satisfies Inada-type conditions and $F_{KK}(\cdot,\cdot) < 0$, $F_{LL}(\cdot,\cdot) < 0$. $G(\cdot,\cdot)$ is a $C^2$, strictly increasing and concave function. Then there exists at most one interior steady-state solution.*

Proof: We shall establish that the equations system given in (2.8)–(2.12) has at most one interior solution. First, we show that for fixed $l$, equations (2.8), (2.9), (2.11) and (2.12) yield at most a unique solution for $\frac{c}{k}$, $\frac{h}{k}$, $u$ and $\nu$. Hence, these variables can be expressed as functions on $l$, and so (2.10) can be thought of as an equation on $l$. We next show that this last equation has at most one interior solution.

From equation (2.9) and (2.12) we see that for $l$ fixed,

$$\rho - n = (1-\sigma)G(1-l-u) + G'(1-l-u)l + G'(1-l-u)u$$

we can obtain the value for $u$. Because the concavity of $G(\cdot)$, it exists at most a unique value of $u$ satisfying the previous equation. Let $u(l)$ be the function determining $u$ for each $l$. Then,

$$\rho - n = (1-\sigma)G(1-l-u(l)) + G'(1-l-u(l))l + G'(1-l-u(l))u(l)$$

Differentiating this equation and after a simple manipulation, we obtain

$$u'(\cdot) = -\frac{\sigma G'(\cdot) - G''(\cdot)l - G''(\cdot)u(\cdot)}{\sigma G'(\cdot) - G''(\cdot)l - G''(\cdot)u(\cdot)} = -1.$$

From equation (2.12), we can write $\nu$ as a function on $l$ with $\nu'(\cdot) = -G'(\cdot)[1+u'(\cdot)] = 0$.

Now from (2.8), and the linearly homogeneity of $F(\cdot)$, we can write, $\rho + \sigma\nu = F_1(1, u\frac{h}{k}) = f_1(u\frac{h}{k})$. Then, denoting $x = \frac{h}{k}$, and differentiating, $f_1'(\cdot)[u'(\cdot)x(\cdot) + u(\cdot)x'(\cdot)] = 0$, which implies $x'(\cdot) = \frac{x(\cdot)}{u(\cdot)} > 0$.

Finally, from (2.11) and denoting $q = \frac{c}{k}$, we can write,

$$q(l) = f_1[u(l)x(l)] + f_2[u(l)x(l)]u(l)x(l) - \nu - n$$

where $f_2(ux) = F_2[1, u\frac{h}{k}]$. Differentiating,

$$q'(l) = (f_1'(\cdot) + f_2'(\cdot)u(\cdot)x(\cdot) + f_2(\cdot))\,[u'(\cdot)x(\cdot) + u(\cdot)x'(\cdot)] = 0.$$

With all this information we write equation (2.10) as follows,

$$\left(\frac{1-\gamma}{\gamma}\right) q = lx(l)f_2(u(l)x(l))$$

The left hand side does not depend on $l$, then differentiating the right hand side, and taking into account the sign of $u'(l), \nu'(l)$ and $x'(l)$ yields,

$$[x(\cdot) + lx'(\cdot)]f_2(\cdot) > 0$$

we know that it is increasing on $l$, hence there exists at most a unique solution for $l$. The theorem is proved.

## 3. Stability Properties of the Steady-State Equilibrium

In this section we study the stability properties of the steady-state equilibrium. First, we show that the unique interior stationary solution to the optimization problem (P1) is globally stable. Our strategy of proof is to construct a Liapounov function, and such construction is based upon the concavity and second order differentiability of the value function. Second, we discuss local stability by linearizing the dinamical system generated by the model. From this local analysis we shall conclude that convergence to the steady-state physical-to-human capital ratio, $\kappa^*$, implies convergence to strictly positive levels $k^*, h^*$. Thus, we rule out convergence to the non-interior steady state $k = 0, h = 0$, from every pair of interior initial conditions.

Cass and Shell (1976), Brock and Scheinkman (1976) and Rockafellar (1976) establish sufficient conditions on the Hamiltonian and the intertemporal discount rate to insure global stability in general problems of optimal control. These conditions consist in establishing an upper bound, which depends on the parameters of the model, on the discount rate. In our model we can prove global stability independently of the size of $\rho$.

**Definition 3.1:** Let be $\kappa^* = \left(\frac{k}{h}\right)^*$ the physical-human capital ratio in the steady-state equilibrium. We say that this solution is asymptotically globally stable if for all $\kappa(0)$, $X(t, \kappa(0)) \to \kappa^*$ as $t \to \infty$; where $X(t, \kappa(0))$ is the solution to $\kappa(t)$ from $\kappa(0)$.

**Definition 3.2:** $L : R_{++} \times R_{++} \to R$ is a Liapounov function if it is defined on the state space and then

*(i)* $L$ is a positive function.

*(ii)* $L$ is continuously differentiable.

*(iii)* $L$ has a unique minimum at the steady-state solution.

*(iv)* $L$ is decreasing along the optimal trajectory.

Once a Liapounov function has been defined, we shall apply the Liapounov theorem. Roughly, this theorem states that if there exists a function satisfying *i-iv*, then the sta-

tionary solution is asymptotically globally stable. As it is well known, there is no method of finding Liapounov functions. Here we make use of the function proposed by Cass and Shell (1976), and we show that it is a Liapounov function to our optimization problem (P1). Thus, we conclude that $\kappa^*, \eta^*$ is asymptotically globally stable, where $\eta^* = \left(\frac{\eta_1}{\eta_2}\right)^*$.

In order to gain tractability we write the optimization problem (P1) in such a way that it contains a unique state variable, $\kappa(t)$, and the control variables $z(t) = \frac{c(t)}{h(t)}$, $l(t)$ and $u(t)$.

$$\begin{aligned} h_0^{1-\sigma} v(\kappa_0) &= h_0^{1-\sigma} \max_{z(t),l(t),u(t)} \int_0^\infty e^{-(\rho-n)t+(1-\sigma)\int_0^t G(1-u(s)-l(s))ds} \left[\frac{(z(t)^\gamma l(t)^{1-\gamma})^{1-\sigma}}{1-\sigma}\right] dt \qquad (P2) \\ \text{subject to} \quad & \dot{\kappa}(t) = F(\kappa(t), u(t)) - n\kappa(t) - z(t) - G(1-u(t)-l(t))\kappa(t) \\ & \kappa(0)\text{given} \end{aligned}$$

where $v(\cdot)$ is the value function. Notice, however, that in this optimization problem the discount rate is a function of the control variables, and so the maximum principle no longer applies. To avoid this we propose the following change of variable that will allow us to use the standard techniques in optimal control problems.

Define

$$\hat{\rho}(t) = (\rho - n)t - (1-\sigma)\int_0^t G(1-u(s)-l(s))ds$$

and differentiating with respect to $t$

$$d\hat{\rho}(t) = [\rho - n - (1-\sigma)G(1-u(t)-l(t))]dt = \phi(t)dt$$

Then, we normalize $h_0 = 1$, and write the problem $(P2)$ as

$$\begin{aligned} v(\kappa) &= \max_{z(t),l(t),u(t)} \int_0^\infty e^{-\hat{\rho}(t)} \frac{(z(t)^\gamma l(t)^{1-\gamma})^{1-\sigma}}{(1-\sigma)\phi(t)} d\hat{\rho}(t) \\ \text{subject to} \quad & \frac{d\kappa(t)}{d\hat{\rho}(t)} = \frac{F(\kappa(t), u(t)) - n\kappa(t) - z(t) - G(1-u(t)-l(t))\kappa(t)}{\phi(t)} \\ & \kappa(0)\text{given} \end{aligned}$$

The current value Hamiltonian to this problem is

$$\mathcal{H} = \frac{(z(t)^\gamma l(t)^{1-\gamma})^{1-\sigma}}{(1-\sigma)\phi(t)} + \eta(t)\left[\frac{F(\kappa(t), u(t)) - n\kappa(t) - z(t) - G(1-u(t)-l(t))\kappa(t)}{\phi(t)}\right]$$

Optimality implies that

$$\begin{aligned}\frac{d\eta(t)}{d\hat{\rho}(t)} &= \eta(t) - \mathcal{H}^0_\kappa(\kappa(t),\eta(t)) \\ \frac{d\kappa(t)}{d\hat{\rho}(t)} &= \mathcal{H}^0_\eta(\kappa(t),\eta(t))\end{aligned}$$

where $\mathcal{H}^0(\cdot,\cdot)$ is the maximized Hamiltonian defined as

$$\mathcal{H}^0(\kappa,\eta) = \max_{z,l,u}\left\{\frac{(z(t)^\gamma l(t)^{1-\gamma})^{1-\sigma}}{(1-\sigma)\phi(t)} + \eta(t)\left[\frac{F(\kappa(t),u(t)) - n\kappa(t) - z(t) - G(1-u(t)-l(t))\kappa(t)}{\phi(t)}\right]\right\}$$

This function is usually interpreted as the maximized value of the national product to the relative prices, $\eta(t)$, given the relative endowment of factors $\kappa$. It is readily shown that this function is concave in $\kappa$ and convex in $\eta$. These properties are derived from the relationship between the value function and the maximized Hamiltonian. See Rockafellar (1976) and Lau (1976) for a systematic study of the properties of $\mathcal{H}^0(\kappa,\eta)$.

The steady-state solution $\kappa^*,\eta^*$, are those values satisfying the following conditions

$$\begin{aligned}\mathcal{H}^0_\eta(\kappa^*,\eta^*) &= 0 \\ \mathcal{H}^0_\kappa(\kappa^*,\eta^*) &= \eta^*\end{aligned}$$

Thus, in order to show the global asymptotic stability of $\kappa^*,\eta^*$ we prove that the following function is a Liapounov function

$$L(\kappa(t),\eta(t)) = -[\kappa(t)-\kappa^*][\eta(t)-\eta^*]$$

Hence, we have to check that

*i)* $L(\kappa(t),\eta(t)) \geq 0$ for all $\kappa(t),\eta(t)$, and with strict inequality if $\kappa(t) \neq \kappa^*$ and $\eta(t) \neq \eta^*$.

*ii)* $\dot{L}(\kappa(t),\eta(t)) \leq 0$ for all $\kappa(t),\eta(t)$, and with strict inequality if $\kappa(t) \neq \kappa^*$ and $\eta(t) \neq \eta^*$.

First, *i)* is directly obtained from the concavity of the value function and from $\eta(t) = v'(\kappa(t))$, that is to say,

$$L(\kappa(t), \eta(t)) = -[\kappa(t) - \kappa^*][v'(\kappa(t)) - v'(\kappa^*)] \geq 0$$

Part *ii)* implies that

$$\dot{L}(\kappa(t), \eta(t)) = -\left[(\kappa(t) - \kappa^*)\dot{\eta}(t) + (\eta(t) - \eta^*)\dot{\kappa}(t)\right] \leq 0 \tag{3.1}$$

To prove *ii)* it is sufficient to show that $\kappa(t) > \kappa^*$ implies $\dot{\eta}(t) > 0$, and $\kappa(t) < \kappa^*$, implies $\dot{\eta}(t) < 0$, that is, $\eta(t)$ follows a stable process. Then, it is obtained that $\dot{L}(\kappa(t), \eta(t)) < 0$ for all $\kappa(t) \neq \kappa^*$ and $\eta(t) \neq \eta^*$ and that $\dot{L}(\kappa^*, \eta^*) = 0$. Also, since by Theorem 2.1 the steady-state equilibrium is unique, we can restrict the study of the stability of the relative prices process to a neighborhood of its long-run value, $\eta^*$. Likewise, and without loss of generality, we shall assume that the variation in $\kappa(t)$ is due to an increase in $k(t)$.

Under these conditions, we know by the concavity of the value function, $v(\kappa(t))$, that the increase in $k(t)$ reduce the relative prices. Our proposition is that after the shock, $\dot{\eta}(t) > 0$, and so there exists convergence. Making use of equations (2.6) and (2.7) we can derive the law of motion to $\eta$ as

$$\frac{\dot{\eta}(t)}{\eta(t)} = -F_1(k(t), u(t)h(t)) + n + G'(1 - u(t) - l(t))(u(t) + l(t)) + G(1 - u(t) - l(t))$$

differentiating with respect to $k(t)$, and evaluating at the stationary equilibrium, we have

$$\frac{d}{dk}\left(\frac{\dot{\eta}}{\eta}\right)^* = -F_{11}(k^*, u^*h^*) - F_{12}(k^*, u^*h^*)h^*\frac{\partial u^*}{\partial k} - G''(1 - u^* - l^*)(u^* + l^*)\left[\frac{\partial u^*}{\partial k} + \frac{\partial l^*}{\partial k}\right] \tag{3.2}$$

where $\frac{\partial u^*}{\partial k}$ and $\frac{\partial l^*}{\partial k}$ denote the derivative of $u(t)$ and $l(t)$ with respect to $k(t)$, evaluated at the steady state equilbrium. Now, the strategy consists in showing that equation (3.2) can not be zero or negative. First, assume that $\frac{d}{dk}\left(\frac{\dot{\eta}}{\eta}\right)^* = 0$; this means that the value function, $v(\kappa)$, is linear and that $\eta$ does not change with $\kappa$. If relative prices remain unchanged after the increase in $k$, we know, by the Rybczynski theorem, that the production goes up

in the sector intensive in physical capital, and goes down in the sector intensive in labor. As in our model the physical capital is not an input in the human capital technology, we have that $\dot{h}(t) < 0$. In this situation $\frac{\partial u^*}{\partial k} + \frac{\partial l^*}{\partial k} > 0$. Then, we know by theorem 4.3 in the next section 4, that in this case (paradoxical growth), $\frac{\partial u^*}{\partial k}\frac{k^*}{u^*} < 1 - \frac{\sigma}{\beta^*}$.[2] By the homogeneity of degree one of $F(\cdot,\cdot)$, we can write equation (3.2) as

$$\frac{d}{dk}\left(\frac{\dot{\eta}}{\eta}\right)^* = -F_{11}(k^*, u^*h^*)\left[1 - \frac{\partial u^*}{\partial k}\frac{k^*}{u^*}\right] - G''(1-u^*-l^*)(u^*+l^*)\left[\frac{\partial u^*}{\partial k} + \frac{\partial l^*}{\partial k}\right] \quad (3.3)$$

If we write $\frac{\partial u^*}{\partial k}\frac{k^*}{u^*} = 1 - \frac{\sigma}{\beta^*} - A$, where $A > 0$, and insert in (3.3) we obtain

$$\frac{d}{dk}\left(\frac{\dot{\eta}}{\eta}\right)^* = -F_{11}(k^*, u^*h^*)\left[\frac{\sigma}{\beta^*} + A\right] - G''(1-u^*-l^*)(u^*+l^*)\left[\frac{\partial u^*}{\partial k} + \frac{\partial l^*}{\partial k}\right] > 0$$

which is a contradiction.

Assume now that $\frac{d}{dk}\left(\frac{\dot{\eta}}{\eta}\right)^* < 0$. This situation means that the value function is strictly concave. Relative prices go down after the increase in $k$ and then $\dot{\eta} < 0$; then, $\eta \to 0$ and $\kappa \to \infty$. Consider initially that the divergence in $\kappa$ consists in $\dot{k}(t) > 0$ and $\dot{h}(t) \geq 0$, then, $\frac{\partial u^*}{\partial k} + \frac{\partial l^*}{\partial k} \leq 0$. By the laws of motion to $\eta_1(t)$ and $\eta_2(t)$

$$\frac{\dot{\eta}_1(t)}{\eta_1(t)} = \rho + \sigma\nu - F_1(k(t), u(t)h(t)) \quad (3.4)$$

$$\frac{\dot{\eta}_2(t)}{\eta_2(t)} = \rho - n + \sigma\nu - G'(1-u(t)-l(t)) - G(1-u(t)-l(t)) \quad (3.5)$$

we obtain from (3.4) that $\dot{\eta}_1(t) > 0$. From (3.5), and given that $u(t) + l(t) \leq u^* + l^*$, we have that $\dot{\eta}_2(t) \leq 0$. Thus, necessarily $\dot{\eta}(t) > 0$, which is a contradiction with our initial assumption.

Consider finally that the divergence in $\kappa$ consists in $\dot{k}(t) \gtrless 0$ and $\dot{h}(t) < 0$. In this case the economy belongs to the paradoxical case, and then we can find a contradiction appealing to the same arguments as in the previous case, $\frac{d}{dk}\left(\frac{\dot{\eta}}{\eta}\right)^* = 0$.

[2] Where $\beta^*$ is defined as the elasticity of the marginal productivity of labor with respect to the physical capital at the steady-state equilibrium, $\frac{F_{LK}(k^*,u^*h^*)k^*}{F_L(k^*,u^*h^*)}$.

We have proved that $\frac{d}{dk}\left(\frac{\dot{\eta}}{\eta}\right)^* > 0$, and so the price process is stable. Finally, a simple inspection of (3.1) shows the established result.

Now, we shall focus on the study of the local stability of the long-run equilibrium. The idea behind local stability theory is based on the two following elements. First, the existence of a local stable manifold crossing the steady-state solution. This point emerges from the study of the eigenvalues of the linearization of the dynamic system for $k(t), h(t), c(t), l(t)$ and $u(t)$. The number of negative eigenvalues determines the qualitative behavior around a neighborhood of the steady-state equilibrium. Second, given initial values for state variables, optimal values for control variables belong to the local stable manifold.

We concentrate here in discussing the first point. From the differentiation of equation (2.5) and making use of (2.6) and (2.7), it is obtained an expression to the dynamics of $u$. Likewise, differentiating (2.3) and (2.4), and using (2.5) we derive the dynamics to $c$ and $l$. From feasibility restrictions, (2.1) and (2.2), we know the dynamics to $k$ and $h$. Following the same arguments as in Caballé and Santos (1993) we can see that at least one eigenvalue of the linearization of $\dot{c}(t)$, $\dot{k}(t)$, $\dot{h}(t)$, $l(t)$ and $u(t)$ is zero. This follows from the fact that the set of steady states is a manifold of dimension one. Writing down the linearized dynamical system to the previously defined variables $\kappa, z, l$ and $u$, the zero eigenvalue is eliminated. Thus, the local stable manifold theorem says that if all four eigenvalues are negatives, then the stable manifold of the nonlinear system is an open neighborhood of $(\kappa^*, z^*, l^*, u^*)$ in $R_{++} \times R_{++} \times (0,1) \times (0,1)$; if $0 < p < 4$ eigenvalues are negative and the others are positive, then it is a p-dimensional manifold that contains $(\kappa^*, z^*, l^*, u^*)$; and if all eigenvalues are positive, it is the single point $(\kappa^*, z^*, l^*, u^*)$.

It follows from the global asymptotic stability of $(\kappa^*, z^*, l^*, u^*)$, shown in this section, that eigenvalues are not all positive. Otherwise, optimal trajectories would never converge to the steady-state equilibrium. Also, if the number of negative eigenvalues is greater than 1, then there would be a continuum of equilibrium paths converging to the steady-state solution. That is, given $\kappa(0)$, initial values for $z, l$ and $u$ on a converging path would not be unique. However, in our model, equilibrium solves a concave optimization

problem, and then, fixed initial conditions, the equilibrium is unique. It follows that we have a unique negative eigenvalue that defines a unique equilibrium path converging exponentially towards the long-run equilibrium.[3] Then, following Santos (1990), we know that this negative eigenvalue plus a zero eigenvalue correspond to the derivative of the policy function $Dg(k^*, h^*)$, and, hence, there exists a one-dimensional stable manifold from which optimal paths converge to $k^*, h^*$. Since the policy function is homogeneous of degree zero, this result is valid to every steady state $k^*, h^*$.

Let $v = (v_1, v_2, v_3, v_4)$ be the eigenvector associated to the negative eigenvalue, $\lambda_1$, then the linear stable manifold, $S$, crossing $(\kappa^*, z^*, l^*, u^*)$ can be parameterized by the initial condition for $\kappa$ as

$$\begin{aligned} S \;=\; & \{(\kappa, z, u, l) \in R_{++} \times R_{++} \times (0,1) \times (0,1) \;\; | \; | \; \kappa - \kappa^* \; | < \epsilon, q = q^* + \frac{v_2}{v_1}(\kappa - \kappa^*), \\ & u = u^* + \frac{v_3}{v_1}(\kappa - \kappa^*), l = l^* + \frac{v_4}{v_1}(\kappa - \kappa^*)\} \end{aligned}$$

Furthermore, trajectories on $S$ approach the steady-state equilibrium at the rate given by $\lambda_1$.

The linear stable manifold is a powerful device to study the transitional dynamics around the steady-state equilibrium. In some cases it becomes simple and direct to use. However, in this paper we have found simpler studying transitional dynamics exploiting the properties of the value function.

Ortigueira and Santos (1997) consider a special case of this model with a Cobb-Douglas production function and a linear technology for human capital accumulation. For these commonly used production functions the authors derive analytically the rate of convergence along the linear stable manifold. They find that this rate depends in a quasilinear fashion on the technological parameters of the model. The productivity of the human capital technology and the population growth increase the speed of convergence, whereas the produtivity of the physical capital in the goods sector has a negative effect.

[3]Note that there exists necessarily one negative eigenvalue. Otherwise the stable manifold, in a $(\dot{\kappa}, \kappa)$-plane, should present a zero slope (evaluated at $\kappa^*$), and this is in contradiction with the strict concavity of the value function showed above.

## 4. Dynamic Behavior Along the Transition

In this section we present some results characterizing the direction of convergence towards the ray of steady states. To this aim, we assume that the economy is at a given steady state and it is perturbed with an increase on the amount of physical capital. As it was mentioned above, we shall distinguish the following cases: paradoxical, exogenous and normal growth. Under the paradoxical case, the economy reduces the rate of investment in human capital and reaches a new steady state with lower amounts of both capitals. In the exogenous case the rate of investment in human capital does not reacts to the increase in physical capital and so the economy goes back to the same steady state. Finally, under the normal case, the investment in human capital jumps up and the economy attains a higher steady state. Briefly, the analysis of this section allows us to figure out the conditions on parameters values that determine the permanent effects derived from temporary shocks on physical capital. Also, it may be of some interest if we are concerned in empirical work testing the growth case that real economies follow.

In the Uzawa-Lucas framework, the effects of endowment shocks on human capital investment have been studied by Caballé and Santos (1993), Chamley (1993), Mulligan and Sala-i-Martin (1993) and Faig (1993). In this framework, important parameters in the model as the discount rate, the productivity in the educational sector or the rate of population growth are irrelevant to determine the qualitative behavior of h. We show in this section that this is a consequence of leaving leisure considerations out of the model.

In our model, an increase in physical capital causes an optimal adjustment in consumption and time allocation variables. However, solely leisure time and working time are the relevant variables to explain the human capital behavior. On one hand, the increase in physical capital rises the wage rate paid in the output sector. Then, there exist incentives to spend more time working in this sector. As the wage rate is the price of the leisure time, representative consumer find it now more expensive (negative substitution effect). On the other hand, the increase in physical capital reduces its relative price making less attractive its production, and leading to a decrease of the working time. Also, the abundance of physical capital encourages the time devoted to leisure activities (posi-

tive income effect). What determines the response of the investment in human capital to the increase in physical capital is the sum of the variations in working and leisure time. The following theorem presents the condition which separates the three growth cases: exogenous, normal and paradoxical growth.

**Theorem 4.1:** *Consider the optimization problem given by (P1) under the same assumptions as in Theorem 2.1. Let be* $\beta^* = \frac{F_{LK}(k^*,u^*h^*)k^*}{F_L(k^*,u^*h^*)}$, *where* $(k^*, u^*h^*)$ *is an interior steady state. Then*

*(a) In the paradoxical case,* $\sigma < \beta^* \left( \frac{1+\gamma \frac{l^*}{u^*}}{1+\beta^* \gamma \frac{l^*}{u^*}} \right)$

*(b) In the exogenous case,* $\sigma = \beta^* \left( \frac{1+\gamma \frac{l^*}{u^*}}{1+\beta^* \gamma \frac{l^*}{u^*}} \right)$

*(c) In the normal case,* $\sigma > \beta^* \left( \frac{1+\gamma \frac{l^*}{u^*}}{1+\beta^* \gamma \frac{l^*}{u^*}} \right)$

*where* $l^*$ and $u^*$ *are steady-state values.*

Proof: Part (a). In the paradoxical case the growth rate of human capital decreases with the positive shock on physical capital, then by equation (2.2) $\frac{\partial u}{\partial k} + \frac{\partial l}{\partial k} > 0$. Taking *logs* in (2.3)-(2.5), differentiating with respect to $k$ and evaluating at the steady state it is obtained that

$$\left[1 - \frac{G''}{G'}\left(\frac{u}{\beta} - \left(\frac{(1-\sigma)\gamma - 1}{\sigma}\right) l\right)\right]\left[\frac{\partial u}{\partial k} + \frac{\partial l}{\partial k}\right] =$$
$$= \frac{\partial \eta_1}{\partial k}\frac{k}{\eta_1}\frac{u}{k\beta} - \frac{\partial \eta_2}{\partial k}\frac{k}{\eta_2}\frac{u}{k\beta} + \frac{u}{k} - \left(\frac{(1-\sigma)\gamma}{\sigma}\right)\frac{\partial \eta_1}{\partial k}\frac{k}{\eta_1}\frac{l}{k} - \left(\frac{1-(1-\sigma)\gamma}{\sigma}\right)\frac{\partial \eta_2}{\partial k}\frac{k}{\eta_2}\frac{l}{k}.$$

We also know that in the new steady state there must be a lower amount of human capital. From equation (2.7) it is obtained that $\frac{d}{dk}\left(\frac{\dot{\eta}_2}{\eta_2}\right)^* < 0$, then the human capital price jumps up with the shock, $\frac{\partial \eta_2^*}{\partial k} > 0$. By the homogeneity of degree $1-\sigma$ of the value function, and provided that $(\eta_1(t), \eta_2(t)) = DV(k(t), h(t))$, [see Benveniste and Scheinkman (1982)], we can write,

$$\frac{\partial \eta_1}{\partial k}\frac{k}{\eta_1} = -\sigma - \frac{\partial \eta_2}{\partial k}\frac{h}{\eta_1}$$

Inserting this information in the previous expression for $\frac{\partial u}{\partial k}+\frac{\partial l}{\partial k}$, we obtain after some algebraic manipulations that at a steady state $k^*, h^*$,

$$\left[1-\frac{G''}{G'}\left(\frac{u^*}{\beta^*}-\left(\frac{(1-\sigma)\gamma-1}{\sigma}\right)l^*\right)\right]\left[\frac{\partial u^*}{\partial k}+\frac{\partial l^*}{\partial k}\right]=$$
$$\left(1-\frac{\sigma}{\beta^*}+(1-\sigma)\gamma\frac{l^*}{u^*}\right)-\frac{\partial \eta_2^*}{\partial k}\left[\frac{h^*}{\eta_1^*}\frac{1}{\beta^*}+\frac{k^*}{\eta_2^*}\frac{1}{\beta^*}-\left(\frac{(1-\sigma)\gamma}{\sigma}\right)\frac{h^*}{\eta_1^*}\frac{l^*}{u^*}+\left(\frac{1-(1-\sigma)\gamma}{\sigma}\right)\frac{k^*}{\eta_2^*}\frac{l^*}{u^*}\right]$$

As the first expression in brackets is always positive, $\frac{\partial u^*}{\partial k}+\frac{\partial l^*}{\partial k}$ is positive if the right hand side is positive. If $\sigma \geq 1$ the last term in brackets is greater than zero, and given that in the paradoxical case $\frac{\partial \eta_2^*}{\partial k}>0$, the last term in the right hand side is positive. Then, the first term must be necessarily positive; and to this be true $\sigma<\beta^*\left(\frac{1+\gamma\frac{l^*}{u^*}}{1+\beta^*\gamma\frac{l^*}{u^*}}\right)$.

If $\sigma<1$, the last term in brackets may be positive or negative. When it is positive the previous argument applies. To complete the proof assume that this term is negative, then it means that,

$$(1-\sigma)\gamma\frac{l^*}{u^*}\beta^*-\sigma\left(\frac{k^*}{h^*}\frac{\eta_1^*}{\eta_2^*}+1\right)-(1-(1-\sigma)\gamma)\frac{k^*}{h^*}\frac{\eta_1^*}{\eta_2^*}\frac{l^*}{u^*}\beta^*>0,$$

Hence,

$$\sigma<\beta^*\left(\frac{\gamma\frac{k^*}{h^*}\frac{\eta_1^*}{\eta_2^*}\frac{l^*}{u^*}+\gamma\frac{l^*}{u^*}-\frac{k^*}{h^*}\frac{\eta_1^*}{\eta_2^*}\frac{l^*}{u^*}}{\gamma\frac{l^*}{u^*}\beta^*+\frac{k^*}{h^*}\frac{\eta_1^*}{\eta_2^*}+1+\gamma\frac{k^*}{h^*}\frac{\eta_1^*}{\eta_2^*}\frac{l^*}{u^*}\beta}\right)<\beta^*\left(\frac{1+\gamma\frac{l^*}{u^*}}{1+\beta^*\gamma\frac{l^*}{u^*}}\right)$$

Part (b). In the exogenous growth case the rate of growth of the human capital remains unchanged after the positive shock in physical capital. Then by equation (2.2) we know that in this case $\frac{\partial u^*}{\partial k}+\frac{\partial l^*}{\partial k}=0$. Also, we know from (2.7) that $\frac{\partial \eta_2^*}{\partial k}=0$. Hence, in this case

$$\frac{\partial \eta_1}{\partial k}\frac{k}{\eta_1}=-\sigma \tag{4.1}$$

Inserting (4.1) into the previous expression to $\frac{\partial u}{\partial k}+\frac{\partial l}{\partial k}=0$, and rearranging terms we obtain that,

$$\sigma=\beta^*\left(\frac{1+\gamma\frac{l^*}{u^*}}{1+\beta^*\gamma\frac{l^*}{u^*}}\right)$$

Part (c): This part can be proved following the same arguments as in parts (a) and (b). The theorem is proved.

**Corollary 4.2:** *Consider the optimization problem given by (P1). If consumers do not derive utility from leisure, i.e.,* $\gamma = 1$*, then,*

*(a) In the paradoxical case,* $\beta^* > \sigma$

*(b) In the exogenous case,* $\beta^* = \sigma$

*(c) In the normal case,* $\beta^* < \sigma$

Proof: Since $\gamma = 1$, the optimal level for leisure is zero and $\frac{l^*}{u^*} = 0$. The result follows directly from theorem 4.1.

Theorem 4.1 establishes the dynamic behavior for the human capital after positive shocks of physical capital. It shows that the paradoxical case becomes more likely than in the model without leisure in the utility function. Moreover, the exogenous growth case may occurs for values of $\sigma$ greater than $\beta$ and lower than 1. However, if we take the value of $\sigma$ reported in the empirical literature, this parameter seems to be no lower than 1. Then, even in our model, the normal case is the only plausible one.

The following three theorems show the elasticities of time allocations and consumption respect to the physical capital.

**Theorem 4.3:** *Consider optimization problem (P1). Then, under the same assumptions as in theorem 2.1, we have*

*(a) In the paradoxical case,* $\frac{\partial u^*}{\partial k}\frac{k^*}{u^*} < 1 - \frac{\sigma}{\beta^*}$

*(b) In the exogenous case,* $\frac{\partial u^*}{\partial k}\frac{k^*}{u^*} = 1 - \frac{\sigma}{\beta^*}$

*(c) In the normal case,* $\frac{\partial u^*}{\partial k}\frac{k^*}{u^*} > 1 - \frac{\sigma}{\beta^*}$

Proof: From equation (2.5) we can write the elasticity of $u$ with respect to $k$, evaluated at the steady-state equilibrium, as:

$$\frac{\partial u^*}{\partial k}\frac{k^*}{u^*} = 1 - \frac{\sigma}{\beta^*} - \frac{1}{\beta^*}\frac{\partial \eta_2}{\partial k}\left(\frac{k^*}{\eta_2^*}\frac{h^*}{\eta_1^*}\right) + \frac{1}{\beta^*}\left(\frac{G''(1-u^*-l^*)}{G'(1-u^*-l^*)}\right)\left(\frac{\partial u^*}{\partial k} + \frac{\partial l^*}{\partial k}\right)k^*$$

Because in the exogenous case $\frac{\partial \eta_2^*}{\partial k} = 0$ and $\frac{\partial u^*}{\partial k} + \frac{\partial l^*}{\partial k} = 0$, part (b) follows directly. Since in the normal case $\frac{\partial \eta_2^*}{\partial k} < 0$ and $\frac{\partial u^*}{\partial k} + \frac{\partial l^*}{\partial k} < 0$, then by the concavity of $G(\cdot)$ it is observed that last two terms are positive and so the result to the normal case is proved. Similar arguments prove part (a).

**Theorem 4.4:** *Consider optimization problem (P1). Then, under the same assumptions as in theorem 2.1, we have*

*(a) In the paradoxical case,* $\frac{\partial l^*}{\partial k}\frac{k^*}{l^*} > -\left(1 - \frac{\sigma}{\beta^*}\right)\frac{u^*}{l^*}$

*(b) In the exogenous case,* $\frac{\partial l^*}{\partial k}\frac{k^*}{l^*} = -\left(1 - \frac{\sigma}{\beta^*}\right)\frac{u^*}{l^*} = (1-\sigma)\gamma$

*(c) In the normal case,* $\frac{\partial l^*}{\partial k}\frac{k^*}{l^*} < -\left(1 - \frac{\sigma}{\beta^*}\right)\frac{u^*}{l^*}$

REMARK: Note that in the exogenous case we are able to write $-(1 - \frac{\sigma}{\beta^*})\frac{u^*}{l^*} = (1-\sigma)\gamma$ from the condition defining such case stated in the theorem 4.1. However, this equality is no longer valid when we are dealing with the paradoxical or the normal case.

Proof: The proof is trivial. It follows directly from theorem 4.3. Consider, say, case (b); in the exogenous growth case, $\frac{\partial u^*}{\partial k} + \frac{\partial l^*}{\partial k} = 0$, then by theorem 4.3 we know that $\frac{\partial u^*}{\partial k} = \left(1 - \frac{\sigma}{\beta^*}\right)\frac{u^*}{k^*}$; therefore it must be true that $\frac{\partial l^*}{\partial k} = -\left(1 - \frac{\sigma}{\beta^*}\right)\frac{u^*}{k^*}$. Parts (a) and (c) follows from similar arguments.

**Theorem 4.5:** *Consider optimization problem (P1). Then, under the same assumptions as in theorem 2.1, we have*

*(a) In the paradoxical case,* $\frac{\partial c^*}{\partial k}\frac{k^*}{c^*} > \sigma - \left(1 - \frac{\sigma}{\beta^*}\right)\frac{u^*}{l^*}$

*(b) In the exogenous case,* $\frac{\partial c^*}{\partial k}\frac{k^*}{c^*} = \sigma - \left(1 - \frac{\sigma}{\beta^*}\right)\frac{u^*}{l^*} = \sigma + (1-\sigma)\gamma$

*(c) In the normal case,* $\frac{\partial c^*}{\partial k}\frac{k^*}{c^*} < \sigma - \left(1 - \frac{\sigma}{\beta^*}\right)\frac{u^*}{l^*}$

Proof: From equations (2.3)-(2.5), taking *logs* and differentiating with respect to $k$ it is obtained that

$$\frac{\partial c^*}{\partial k}\frac{k^*}{c^*} = -\frac{\partial u^*}{\partial k}\frac{k^*}{u^*}\beta^* + \beta^* + \frac{\partial l^*}{\partial k}\frac{k^*}{l^*}$$

Then, part (b) follows directly from theorems 3.3 and 3.4. Parts (a) and (c) can be proved appealing to similar arguments.

Theorems 4.3, 4.4 and 4.5 summarize the sensibility of $u(t), l(t)$ and $c(t)$ with respect to the physical capital around the steady-state equilibrium.

It follows from (2.3)–(2.5) that leisure goes up in the normal case whenever $\sigma \leq 1$, and therefore we have that worked hours go down. That is, leisure is a normal good, the wealth effect associated with the increase in $k$ dominates the substitution effect derived from the increase in the relative price of leisure time. If $\sigma > 1$ this result is no longer true in general. Under this latter condition, leisure may goes down with $k$ and worked hours may go up. This result is heavily dependent on the concavity of $G$, i.e., on the dependence of the marginal productivity in the educational sector on the schooling time. The greater is the concavity of $G$ the lower (in absolute value) the sensitivity of $u$ with respect to $k$, and greater the sensitivity of the leisure time.

## 5. Numerical Computations

In this section we present the magnitude of the elasticities of worked hours, leisure and consumption with respect to capital stocks evaluated at the steady-state equilibrium. We also explore the sensitivity of these elasticities to changes in preferences parameters $\sigma$ and $\gamma$, and technology parameters $\beta$ and $\delta$.

We use a benchmark model with a Cobb-Douglas production function in the output sector and a concave function in the educational sector; that is

$$F(k, uh) = Ak^{\beta}(uh)^{1-\beta} \text{ and } G(1-u-l) = \delta(1-u-l)^{\alpha}$$

where $0 < \beta < 1, \delta > 0$ and $0 < \alpha < 1$. The benchmark values for this economy are the following,

$$\rho = 0.05,\ \gamma = 0.45,\ \sigma = 1.5,\ A = 3,\ \beta = 0.4,\ \delta = 0.07,\ \alpha = 1 \text{ and } n = 0.01$$

The corresponding steady-state equilibrium is: $z^* = 4.0573,\ \kappa^* = 23.8663,\ l^* = 0.4528,\ u^* = 0.2615,$ and $\nu = 0.02$.

The next two figures present the elasticities of worked hours, leisure and consumption with respect to the physical capital as functions of preferences and technologies parameters. These elasticities are evaluated at the steady-state equilibrium presented above.

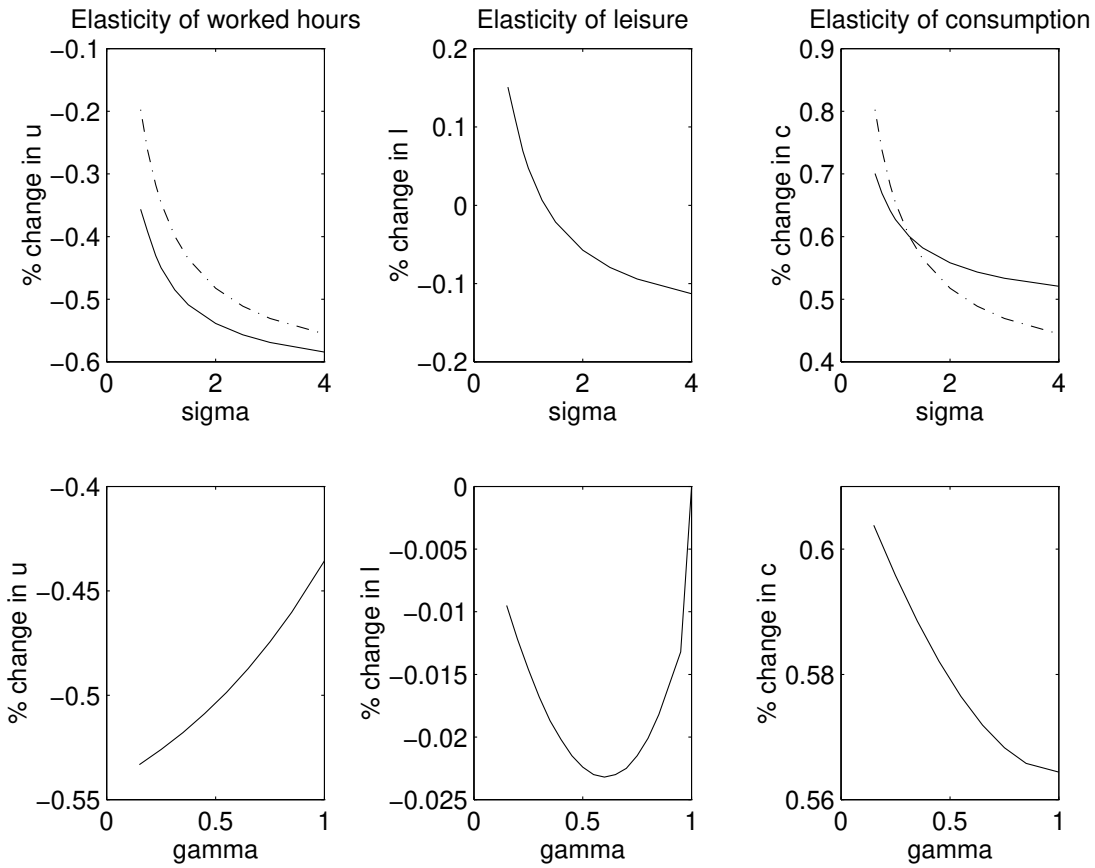


**Figure 1.** Elasticities of worked hours, leisure and consumption with respect to the physical capital as functions of the parameters defining the utility function. The broken line corresponds to the case without leisure in the utility function, that is, with $\gamma = 1$.

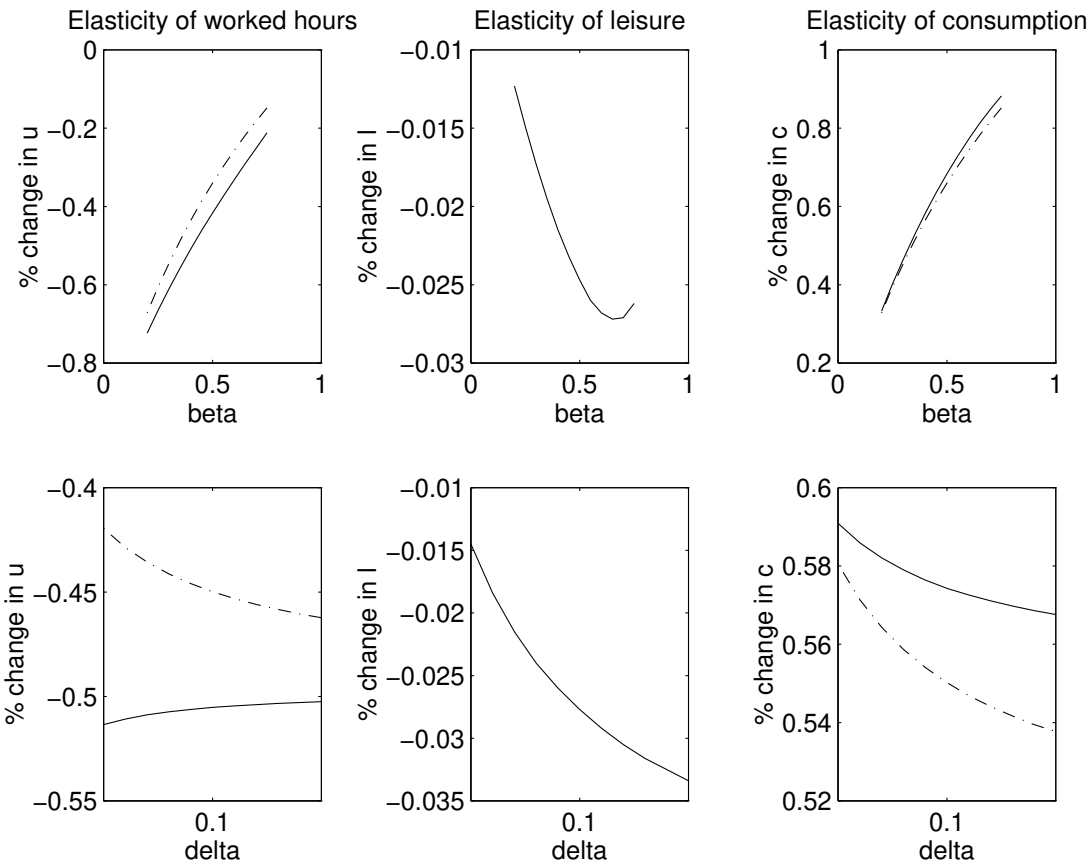


**Figure 2.** Elasticities of worked hours, leisure and consumption with respect to the physical capital as functions of the parameters defining both output and education technologies. The broken line corresponds to the case without leisure in the utility function.

For realistic parameters values the presence of qualified leisure has a minor impact on the elasticity of worked hours and consumption with respect to the physical capital. [4] From the previous exercise it can be seen that worked hours are slightly more sensitive to state variables in the model with qualified leisure. Finally, as we pointed out in the previous section, we see that $\sigma$ is a key parameter in the determination of the dynamic behavior of the leisure time. In our benchmark economy, for values of $\sigma$ around 1.25 the

[4] Note that as the steady-state equilibrium is defined by the ratio of state variables, $\kappa^*$, the elasticities with respect to the human capital have the opposite sign to those with respect to $k$.

policy function for leisure is roughly flat, which correspond to the case of an exogenous determination of leisure at the steady-state level given by $l^*$.

## 6. Concluding Remarks

In this paper we have analyzed a general model of endogenous growth with physical and human capital in which the representative consumer derives utility from consumption and qualified leisure. Considering general production functions we have shown that, unlike alternative modelizations of the leisure activities, there exists a unique ray of steady states. Our study constitutes a complete characterization of the equilibrium dynamics paths both in the long run as well as during the transitional period.

Our modelization of leisure in the utility function departs from the presented in Ladrón-de-Guevara *et al.* (1997) by considering human capital as well as time devoted to leisure-related activities to measure the effective amount of leisure. From a technical point of view, this modelization, together with regular conditions on both production functions, insures the concavity of our optimization problem. Hence, we are able to make use of the concavity and differentiabililty properties of the value function in order to characterize the equilibrium dynamics.

In the setting presented in this paper, the weight of leisure in the utility function (parameter $\gamma$) does not affect the long-run rate of growth. However, our results on transitional dynamics show that these parameters play a major role in the determination of the long-run consequences derived from temporary endowment shocks. Actually, some results obtained in the model without leisure may be reversed in the present model. In particular, the exogenous and paradoxical cases become here more likely than in the Uzawa-Lucas setting.

In this sense, our model must be seen as a generalization of the Uzawa-Lucas framework addressed to study issues such as: fiscal policy, labor economy, rate of convergence, etc. In this line, the analysis in Section 4 on the response of leisure, working time and consumption to physical capital shocks may become useful to evaluate the short-run impact of different policies.

# References


[1] Becker, G., "A Theory of the Allocation of Time," *Economic Journal* 75 (1965), 493-517.

[2] Benhabib, J., Rogerson, R. and R. Wright, " Homework in Macroeconomics: Household Production and Aggregate Fluctuations," *Journal of Political Economy* 99 (1992), 1166-1187.

[3] Benveniste, L. M. and J. A. Scheinkman, " Duality Theory for Dynamic Optimization Models of Economics: The Continuous Time Case," *Journal of Economic Theory* 27 (1982), 1-19.

[4] Brock, W. A. and J. A. Scheinkman, "Global Asymptotic Stability of Optimal Control Systems with Applications to the Theory of Economic Growth," *Journal of Economic Theory* 12 (1976), 164-190.

[5] Caballé, J. and M. S. Santos, "On Endogenous Growth with Physical and Human Capital Accumulation," *Journal of Political Economy* 101 (1983), 1042-1068.

[6] Cass, D. and K. Shell, "The Structure and Stability of Competitive Dynamical Systems," *Journal of Economic Theory* 12 (1976), 31-70.

[7] Chamley, C., "Externalities and Dynamics in Models of "Learning or doing."," *International Economic Review* 34 (1983), 583-609.

[8] Faig, M., "A Simple Economy with Human Capital: Transitional Dynamics, Technology Shocks, and Fiscal Policies," Mimeo, University of Toronto (1993).

[9] Heckman, J. J., "A Life-Cycle Model of Economics, Learning, and Consumption," *Journal of Political Economy* 84 (1976), S11-S44.

[10] Kurz, M., "The General Instability of a Class of Competitive Growth Processes," *Review of Economic Studies* 35 (1968), 155-174.

[11] Kydland, F. E. and E. C. Prescott, "Time to build and Aggregate Fluctuations," *Econometrica* 50 (1982), 1350-1370.

[12] Ladrón-de-Guevara, A., Ortigueira, S. and M. S. Santos, "A Two–Sector Model of Endogenous Growth with Leisure," Working Paper, Centro de Investigación Económica, ITAM (1997).

[13] Lau, S., "A Characterization of the normalized restricted profit function," *Journal of Economic Theory* 12 (1976), 131-163.

[14] Levhari, D. and N. Liviatan, "On Stability in the Saddle-Path Sense," *Journal of Economic Theory* 4 (1972), 88-93.

[15] Lucas, R. E., "On the Mechanics of Economic Development," *Journal of Monetary Economy* 22 (1988), 3-42.

[16] Michael, R. T., "The Effect of Education on Efficiency in Consumption," New York: National Bureau of Economic Research, (1973).

[17] Mulligan, C. B. and X. Sala-i-Martin, "Transitional Dynamics in Two-Sector Models of Endogenous Growth," *Quarterly Journal of Economics* CVIII (1993), 739-775.

[18] Ortigueira, S. and M. S. Santos, "On the Speed of Convergence in Endogenous Growth Models," Forthcoming in *American Economic Review,* (1997).

[19] Prescott, E. C., "Theory Ahead of Business Cycle Measurement," *Federal Reserve Bank of Minneapolis Quarterly Review* 10 (1986), 9-22.

[20] Rebelo, S., "Long-Run Policy Analysis and Long-Run Growth," *Journal of Political Economy* 99 (1991), 500-521.

[21] Rockafellar, R. T., "Saddle Points of Hamiltonian Systems in Convex Lagrange Problems Having a Nonzero Discount Rate," *Journal of Economic Theory* 12 (1976), 71-113.

[22] Santos, M.S., "On the Policy Function in Continuous Time Economic Models," Mimeo, Universidad Carlos III de Madrid (1990).

[23] Stokey, N. and S. Rebelo, "Growth Effects of Flat-Rate Taxes," *Journal of Political Economy* 103 (1995), 519-550.

[24] Uzawa, H., "Optimal Technical Change in an Aggregative Model of Economic Growth," *International Economic Review* 6 (1965), 18-31.