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Effects of Financial Innovations on Market Volatility when Beliefs are Heterogeneous*

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Abstract

Consider a model with two logarithmic utility maximizer agents that observe aggregate consumption but disagree about its expected rate of change. Markets are incomplete. We first show that volatility of the interest rate in an economy with additional information is higher. In a complete markets economy (with no additional information) there are two differences in volatility with the previous. The first one, due to the increase in volatility of the wealth share of the individuals is always positive. The second difference is the change in covariance between the components of the interest rate volatility and can be positive or negative.

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1. Introduction

Financial innovation seems to have taken off in the 80's. A growing body of literature tries to analyze both causes and effects of financial innovation. Allen and Gale (1994) offer an extense survey. Of particular importance is the relationship between financial innovation and volatility. As a result of the stock market crash of 1987, a debate opened about the responsibility of new derivative securities on the volatility of the market. It has been argued that the proliferation of new derivative securities increases the volatility of the market. Although several empirical studies were conducted as a result of the 1987 crash, only recently theoretical models are addressing this issue. Miller (1991) discusses some results that seem to be inconclusive. This paper is an attempt to address that question in a theoretical framework with volatility arising because of the heterogeneity of beliefs.

Precedents to the current studies on financial innovation are the classic work of Hart (1975) and Geneakoplos and Polemarchakis (1986). They study welfare effects of introducing a new security in an incomplete markets framework. Allen and Gale (1988) study the optimal design of a security. In their model, volatility stems from the heterogeneity in endowments and degrees of risk aversion. Introducing new securities is costly. Duffie and Jackson (1989) study the introduction of an optimal futures contract with transaction costs. Their goal is to characterize the set of Nash equilibria. Allen and Gale (1991) study an economy with risk-neutral entrepreneurs, risk-averse consumers and costly introduction of securities. In their setting, incentives to innovate depend on the way new securities affect prices of the existing firms. Shortselling rules play a decisive role. Duffie and Rahi (1995) have an extensive survey on the design of new securities. Citanna (1996) studies the effects of financial innovation on volatility from a welfare point of view.

In this paper we use a pure exchange economy version of the model in Detemple and Murthy (1994) in order to investigate the effects of financial innovation and changes in information structure on the volatility of the equilibrium interest rate; therefore, our model is a heterogeneous beliefs continuous time version of Lucas (1978). Aggregate consumption/endowment is exogenous and paid to the owner(s) of the productive technology as a dividend. There are two logarithmic utility maximizer agents that observe the dividend process but disagree about the parameters that describe the dynamics of the dividend process. They live in different probability spaces in order to reconcile the realizations of the dividend process with their priors. As information arrives, they update their beliefs in a bayesian way. As in Cox, Ingersoll and Ross (1985), in equilibrium, the investors must hold the totality of the

productive technology. Furthermore, they must consume the totality of the aggregate endowment. In equilibrium, the volatility of the stock that represents ownership of the productive technology is the same as the volatility of the output of the exogenous endowment process. Volatility of the interest rate, however, obtains as a result of the asymmetry of beliefs. This is what we will call "market volatility". Unlimited borrowing and shortselling are allowed.

Besides Detemple and Murthy (1994), previous models where agents have different priors include Abel (1990), Harris and Raviv (1991) and Varian (1985). Other recent models based on some type of heterogeneity across agents are Dumas (1989), Duffie and Constantinides (1996). We also mention the branch of literature started by Kyle (1985); some intertemporal models in that line are Back (1992), He and Wang (1995) and Wang (1993, 1994). Basak (1996) uses a model similar to ours that gives rise to sunspot equilibria.

1.1. Summary of Results and Structure of the Paper

In the setting we consider equilibrium interest rate is always volatile due to the heterogeneity of beliefs which has two effects: a direct effect through the volatility of the conditional means of the agents and an indirect effect through the volatility of the distribution of wealth (or the shares of aggregate consumption). Since the information structure considered leads to homogeneous conditional variances the direct effect is tied to the common sensitivity of agents to news (update of beliefs). The indirect effect is a wealth-weighted average (or a consumption shares-weighted average) of the conditional means. We start with the incomplete information/incomplete markets setting and then consider two additional settings. In the first of them individuals receive some additional information (a signal). Individuals also have heterogeneous beliefs about this additional information but it affects the way they update their beliefs. We show that with additional information local volatility is always higher. It is also clear that since individuals have more information they will update their beliefs faster and will converge faster to the true value. Arguably, volatility might be lower that it would have been without the additional information after some time. Then we consider a complete markets setting. The additional security does not provide more information than the signal in the additional information setting. We "design" the security in such a way that its volatility is the same as the volatility of the signal. This security has an effect on volatility that can be decomposed in two parts. The first part always means an increase of the existing volatility through an increase in the volatility of the share of wealth of the individuals; the second part is the effect of the change in wealth share in the covariance between wealth share and information update. This second component can be positive or negative. If the additional information and the security are bundled together, volatility always goes up.

The paper is organized as follows: in Section II we introduce the continuous time version of the Lucas (1978) model and the assumptions of our particular model. In

Section III we derive optimal policies and equilibrium returns. In Section IV we compute the interest rate volatility in each type of specification. In Section V we compute the change in volatility and analyze its reasons. We close the paper with some conclusions.

2. The Economy

We present a pure exchange version of the intertemporal two-heterogeneous agents model of Detemple and Murthy (1994).

2.1. Agents of the Economy

In this economy there are two rational price-taker individuals that try to maximize expected lifetime utility deriving from consumption. They are identical in their preferences, that we parametrize as follows,

$$\int_0^\tau e^{-\delta t} \log c_t dt \tag{1}$$

where δ is a discount process that we assume to be constant[1] and c is the consumption process; optimal consumption of each individual will be represented by c^i, $i = O, P$; we represent by π^i, $i = O, P$ the optimal investment strategy of each agent in the risky securities (to be defined later); wealth of the individuals is given by X^i, $i = O, P$; an admissible process c^i, $i = O, P$ is a consumption process defined in $[0, T]$ such that $c^i \in \mathcal{L}$, $i = O, P$; and admissible investment process is a process π^i, $i = O, P$ defined in $[0, T]$ such that $\pi^i \in \mathcal{L}^2$, $i = O, P$.

These individuals (indexed by O and P) differ in their beliefs. Specifically, the agents do not agree about the dynamics followed by the aggregate consumption/dividend process that we introduce later. According to their beliefs about this process we will have two different types of agents that we will call "optimist" and "pessimist", respectively.

Ideally, we would like to parametrize individuals with heterogeneous preferences as well[2]. However, the assumption that the individuals are identical in their preferences will allow us to evaluate the importance of the asymmetry in beliefs: an economy with two identical individuals with homogeneous information and beliefs is equivalent to a representative agent economy as studied in Huang (1987) and Duffie and Zame (1990). A general version of our model is presented in Detemple and Murthy (1994).

[1] Logarithmic utilities are frequently used in general equilibrium models because they induce myopia and that permits to derive closed form solution of the demand functions.

[2] See Dumas (1989) on this problem

2.2. Consumption Good

In this economy there is a single consumption good in exogenous supply p and whose dynamics are explained by the following stochastic differential equation,

$$\frac{dp_t}{p_t} = \mu dt + \rho_1 dW_t^1 + \rho_2 dW_t^2 \tag{2}$$

where μ and $\rho^\top = (\rho_1\, \rho_2)$ are exogenous constant parameters[3] and $W = (W_1, W_2)^\top$ is a two-dimensional standard Brownian motion process under the true probability measure P that explains the dynamics of this economy. All possible realizations of W span the state space that we denote by Ω. $\mathcal{F}^W$ is the filtration generated by W. We assume that this consumption good is the output of a "productive technology" whose ownership is represented by the stock that we introduce later. Furthermore, this commodity is the numeraire of the economy. Asset prices are expressed in units of this consumption good.

2.3. Information

The agents of this economy do not observe the true Brownian motion process W. They have heterogeneous beliefs about the dynamics of the consumption good process. One of the agents, that we will call "optimist" for reasons that will be obvious later, lives in an informational setting defined by the triplet $(\Omega, \mathcal{F}^{W^O}, P^O)$ where Ω represents the state space, $\mathcal{F}^{W^O}$ is the filtration generated by W^O (the "innovation" process observed by this individual), W^O is a Brownian motion process with respect to a probability measure P^O, continuous with respect to P.

Similarly with respect to the other individual, that we call "pessimist" and lives in an informational setting defined by $(\Omega, \mathcal{F}^{W^P}, P^P)$ with similar interpretation.

However, both individuals observe the path of the consumption good. Therefore, $\mathcal{F}^{W^O} = \mathcal{F}^{W^P} = \mathcal{F}^W$.

Now we consider three different settings. In the first setting individuals face a situation where the risky stock that pays the consumption good and the bond are the only securities available and the aggregate endowment is the only source of information. Then we consider another setting with additional information. For example, a statistic (in addition to the existing information) is computed and publicly released. The third setting is similar, but also a contingent claim (with no additional informational content) is traded.

2.3.1. Basic information

We assume that the aggregate endowment process of (2) is observed. The first agent (the "optimist") believes that the drift of the dividend/aggregate consumption process

[3] We will use the notation "$\top$" to denote transpose.

is μ^O. More explicitly, this investor considers that the "true" drift belongs to a normal distribution with initial mean μ_0^O and initial standard deviation $\nu(0)$[4]. However, the agent knows the exact volatility structure of the dividend process ρ.

This specification is a subset of the "Incomplete Information" case in a continuous time setting considered by Detemple (1986, 1991), Dothan and Feldman (1986) and Gennotte (1986). This is a classic filtering problem: as information arrives (realizations of the dividend process) the investor will update beliefs about the drift in a bayesian way. μ^O is a stochastic process whose dynamics we introduce later. Those dynamics are the result of bayesian update. μ^O converges to μ (although it will not reach μ in a finite time).

Therefore, the "innovation" process observed by this individual is (see Liptser and Shiryayev, 1977)

$$dW_t^{OB} = \frac{1}{\sqrt{\rho^\top \rho}} \left(\frac{dp_t}{p_t} - \mu_t^O \right) \tag{3}$$

From equation (2) it is clear that we can write the previous expression as

$$dW_t^{OB} = \frac{\rho^\top}{\sqrt{\rho^\top \rho}} (dW_t + \lambda_t^O) \tag{4}$$

with $\lambda^\top = (\lambda_{1t}^O, \lambda_{2t}^O)$. For the beliefs of the individual to be consistent with the observed values we require that

$$\rho^\top \lambda_t^O = \mu - \mu_t^O \tag{5}$$

The second agent, however, considers that the "true" drift belongs to a normal distribution with initial mean μ_0^P and standard deviation $\nu(0)$. (To simplify the problem, we assume that the variance of the distribution of the drift is the same for both agents). This agent also knows ρ. The "innovation" process observed by this individual is

$$dW_t^{PB} = \frac{1}{\sqrt{\rho^\top \rho}} \left(\frac{dp_t}{p_t} - \mu_t^P \right) \tag{6}$$

We can express the previous equation as in (4) with λ^P.

We also require that

$$\rho^\top \lambda_t^P = \mu - \mu_t^P \tag{7}$$

For convenience, we assume (this justifies that we call our agents "optimist" and "pessimist")[5]

$$\mu_0^O > \mu > \mu_0^P \tag{8}$$

[4]The standard deviation resulting from bayesian update is a deterministic function of time. This is the reason why we represent the time indicator in parenthesis.

[5]The fact that optimist agents remain optimist at all times is a consequence of the information structure considered. In more general contexts an optimist may become pessimist and conversely.

This asymmetry of beliefs about the dynamics of the aggregate consumption/dividend process will translate into disagreement about the dynamics of the existing securities. We will be more specific in the next section about the markets.

2.3.2. Additional Information

We consider a second setting with additional information. For example, besides the endowment process described before, an index or a statistic is computed and publicly released. We represent this statistic by s. Its dynamics are given by

$$\frac{ds_t}{s_t} = \epsilon dt + \sigma_{21} dW_t^1 + \sigma_{22} dW_t^2 \tag{9}$$

where ϵ and σ_{21} and σ_{22} are constant. The parameter ϵ is not observed and $\epsilon \neq \mu$. Equation (9) provides information about the unobserved Brownian motion processes W^1 and W^2 which affect the aggregate endowment process and not about the unobserved drift μ. We assume that the agents also have asymmetric beliefs about the dynamics of the signal. Therefore, the "optimist" believes that the dynamics of the signal are

$$\frac{ds_t}{s_t} = \epsilon_t^O dt + \sigma_{21} dW_t^{O1} + \sigma_{22} dW_t^{O2} \tag{10}$$

where $(W^O)^\top$ is the two-dimensional "innovation" process in this setting and it is such that

$$dW_t^O = dW_t + \lambda_t^O \tag{11}$$

This places a second constraint on λ^O,

$$\sigma_{21}\lambda_{1t}^O + \sigma_{22}\lambda_{2t}^O = \epsilon - \epsilon_t^O \tag{12}$$

At a given moment t this agent is uncertain about the true drifts of the aggregate consumption and signal processes and she believes they are normally distributed with means (μ_t^O, ϵ_t^O) and variance-covariance matrix Υ,

$$\Upsilon = \begin{pmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{pmatrix} \tag{13}$$

where, in order to be consistent, $\nu = \nu_{11}$. The variance-covariance matrix is a deterministic function of time.

Similarly for the "pessimist"

$$\frac{ds_t}{s_t} = \epsilon_t^P dt + \sigma_{21} dW_t^{P1} + \sigma_{22} dW_t^{P2} \tag{14}$$

with

$$dW_t^P = dW_t + \lambda_t^P \tag{15}$$

that yields

$$\sigma_{21}\lambda_{1t}^P + \sigma_{22}\lambda_{2t}^P = \epsilon - \epsilon_t^P \tag{16}$$

The "pessimist" believes that the true drifts of the aggregate consumption and signal processes are normally distributed with means (μ_t^P, ϵ_t^P) and variance-covariance matrix Υ (we assume that the variance-covariance matrices are the same for both agents).

2.4. Financial Markets

We will consider two or three financial assets that give rise to the incomplete or complete markets specification. We have to stress that we use this terminology (complete/incomplete) for convenience purposes. In fact, in the basic information economy there is one innovation process: the market is then complete relative to this information structure. In the economy with additional information there are two innovation processes (two-dimensional innovation process). The market with three securities (two of them risky) is again complete relative to this information structure. Thus, "adding" an information process makes a complete market incomplete. "Adding" the second security completes the market in the "additional information" economy.

Altogether, then, we have the three economies announced in the introduction: one risky security and no additional information, one risky security with additional information and the two risky securities economy.

2.4.1. Incomplete Markets

First, there is a (locally) riskfree bond or bank account that satisfies the following dynamics,

$$dB_t = r_t B_t\, dt \;; B_0 = 1 \tag{17}$$

where r is the stochastic interest rate. This security is present in both the complete and incomplete markets specifications.

Then, there is one risky security. We will denote by S the price of this security, the stock that represents ownership of the "productive technology". Owners of that stock are entitled to their share of the dividend p and the capital gains. The sum of these two concepts is what we will call "gain process", that we denote by G. In other words, the return of investing in the risky security is given by

$$\frac{dG_t}{G_t} = \alpha_t dt + (\sigma_1)^\top dW_t = \frac{dS_t}{S_t} + \frac{p_t}{S_t}dt \tag{18}$$

where $(\sigma_1)^\top = (\sigma_{11}\, \sigma_{12})$. The parameters α and σ_1 and the dynamics of S are endogenous to the model. As we will show, in equilibrium, σ_1 is constant (we then drop the time indicator). This security is also present in both market specifications.

The investors will disagree about the dynamics of the "productive technology". The first investor (the "optimist") thinks that the gain process of the stock offers an expected return α^O such that

$$\alpha_t^O + \sigma_{11}\lambda_1^O + \sigma_{12}\lambda_2^O = \alpha_t \tag{19}$$

This agents knows the exact volatility structure of this security.

Similarly, the second investor (the "pessimist") thinks that the gain process of the stock on the "productive technology" offers an expected return α^P, such that,

$$\alpha_t^P + \sigma_{11}\lambda_1^P + \sigma_{12}\lambda_2^P = \alpha_t \tag{20}$$

This agents also knows the exact volatility structure of this process.

Note that our model implies that agents recognize that expected returns are the result of the equilibrium, but they disagree about the equilibrium value.

2.4.2. Complete Markets

We now consider an economy with additional information and a second risky security (that effectively completes markets) and that we will call "contingent claim". This is a security whose payoff depends on a linear combination of the signal and the aggregate endowment (that are observable by both agents). This security completes the markets. In order to keep our setting as simple as possible we are going to assume that *the contingent claim is designed in such a way that its volatility structure is the same as the volatility of the signal s* (this simplifies the notation but does not mean any additional restriction: once markets are complete any volatility structure can be replicated). The price of the contingent claim is also publicly released but the priors of the agents are such that no additional information is drawn. Both individuals know the volatility structure of the contingent claim. Equilibrium will then determine the expected return of the contingent claim. This is exactly as in Cox, Ingersoll and Ross (1985) but here individuals do not observe the true dynamics of the contingent claim. They observe the price of the contingent claim and they have priors about the expected return of the contingent claim. We denote the price of the contingent claim by C. The dynamics of the contingent claim as observed by the "optimistic" investor will be

$$\frac{dC_t}{C_t} = \gamma_t^O dt + \sigma_{21} dW_t^{O1} + \sigma_{22} dW_t^{O2} \tag{21}$$

Similarly, for the pessimistic investor,

$$\frac{dC_t}{C_t} = \gamma_t^P dt + \sigma_{21} dW_t^{P1} + \sigma_{22} dW_t^{P2} \tag{22}$$

Since the contingent claim does not add any information to that already provided by the signal the beliefs of the individuals about the drift of the contingent claim will

have to be consistent with the beliefs of the individuals about the drift of the signal, that is,

$$\gamma - \gamma_t^O = \sigma_{21}\lambda_{1t}^O + \sigma_{22}\lambda_{2t}^O = \epsilon - \epsilon_t^O \tag{23}$$

and

$$\gamma - \gamma_t^P = \sigma_{21}\lambda_{1t}^P + \sigma_{22}\lambda_{2t}^P = \epsilon - \epsilon_t^P \tag{24}$$

The true dynamics of the contingent claim (unobserved by both individuals) are

$$\frac{dC_t}{C_t} = \gamma_t dt + \sigma_{21} dW_t^1 + \sigma_{22} dW_t^2 \tag{25}$$

Where γ is the true drift of the contingent claim, to be determined in equilibrium. Equations (23) and (24) guarantee that (21) and (22) are compatible with (25).

Remark 1. *We want markets to be complete under the three security structure. This amounts to requiring that σ, the matrix formed by stacking σ_1 and σ_2 be non-singular. As we shall see in equilibrium, this requires imposing restrictions on ρ and σ_2.*

The primitives of our model are the parameters that explain the dynamics of the aggregate consumption process, μ and ρ, the initial beliefs of the investors about the drift of the aggregate consumption process, μ_0^O and μ_0^P, the initial beliefs about the signal ϵ_t^O and ϵ_t^P (that we indexed at t since we assume that it is then introduced) as well as their initial variance-covariance matrix $\Upsilon(t)$ and, finally, the diffusion vector of the contingent claim σ_2 (this is as in Cox, Ingersoll and Ross, 1985).

2.5. Equilibrium

These individuals have free access to the financial markets, and there are no transaction costs (with the restriction that final wealth has to be nonnegative). The notion of equilibrium that we use is a normal extension of the equilibrium in the single agent model of Lucas(1978): in equilibrium, the individuals consume the totality of the aggregate consumption process and hold the totality of the production technology that represents the total wealth of the economy. The riskfree security and the contingent claim are in zero net supply. Therefore, we look for equilibrium interest rate r, expected return α and diffusion σ_1, of the "productive technology" and expected return of the contingent claim γ which clear the markets.

Finally, note that for $\mu^O = \mu^P$, i.e., homogeneous beliefs, we could construct a representative agent (Huang, 1987 and Duffie and Zame, 1990) and we would be back in the single agent case of Lucas (1978).

3. Optimal Policies and Equilibrium Returns

3.1. Optimal Policies

We introduce the optimal policy -investment and consumption- of a rational price-taker individual. This problem was first solved by Merton (1971) in a markovian setting, where the parameters of the economy -interest rate and drift and diffusion of securities returns- are constant. More recently, Cox and Huang (1990) and Karatzas, Lehoczky and Shreve (1987) have introduced martingale techniques that allow us to consider the optimal investor problem for more general processes. Martingales methods for economies with incomplete information are discussed in Karatzas and Xue (1991). The case of logarithmic utility with incomplete information is treated in Detemple and Murthy (1994). Using these techniques, it is straightforward to show that the solution provided in Merton (1971) extends to the case we are considering[6].

3.1.1. Complete Markets

We first consider the case when both the stock and the contingent claim trade. We introduce some notation. We define the *market price of risk,* that will be different across investors and we denote as θ^O and θ^P (for the "optimist" and "pessimist" respectively, (the C refers to the complete markets setting),

$$\theta_t^{OC} = (\sigma)^{-1} \begin{pmatrix} \alpha_t^O - r_t \\ \gamma_t^O - r_t \end{pmatrix} \tag{26}$$

$$\theta_t^{PC} = (\sigma)^{-1} \begin{pmatrix} \alpha_t^P - r_t \\ \gamma_t^P - r_t \end{pmatrix} \tag{27}$$

Lemma 1. *Consider the complete markets economy described in Section 2. Denote by X^O and X^P the respective wealth levels of the investors and by c^O, c^P, π^O and π^P their respective consumption and investment as defined in section 2.1. Their optimal policies are given by*

$$c_t^O = \frac{\delta}{1 - e^{-\delta(\tau - t)}} X_t^O \tag{28}$$

$$c_t^P = \frac{\delta}{1 - e^{-\delta(\tau - t)}} X_t^P \tag{29}$$

$$\pi_t^O = (\sigma^\top)^{-1} \theta_t^O X_t^O \tag{30}$$

$$\pi_t^P = (\sigma^\top)^{-1} \theta_t^P X_t^P \tag{31}$$

PROOF: See Cox and Huang (1990) and Karatzas, Lehoczky and Shreve (1987).

□

[6] This is a peculiarity of the logarithmic case with constant subjective discount rate. In the case of a power running utility the solution of Merton (1971) does not extend to the nonmarkovian setting. For more see Ocone and Karatzas (1991).

3.1.2. Incomplete Markets with Additional Information

Partial equilibrium with continuous trading in an incomplete markets setting has been studied by He and Pearson (1991) and Karatzas, Lehoczky, Shreve and Xu (1991). Their idea is as follows: suppose that the utility maximizer investor has indeed the possibility to invest in the "contingent claim" that completes the market, i.e., an asset whose standard deviation vector is not in the span of the standard deviation vectors of the real asset. For a given expected return on the asset, the investor will choose a certain investment level. Suppose that the expected return on the contingent claim increases: intuitively, the investor would like to buy more of that asset. The opposite result will obtain for a smaller expected return. It is conceivable that for a very large expected return the investor will optimally maintain a long position in this asset and for a very small expected return -maybe negative and large in absolute value- the investor will optimally maintain a short position in that asset. In between there will be a point at which the investor will optimally maintain a neutral position. The way to solve the problem is to assume that the investor believes that the expected return on the contingent claim is such that the optimal investment policy amounts to a zero balance in the security; then we compute the optimal investment policy in the existing real asset. We express the result as a lemma.

Lemma 2. *Consider a logarithmic utility maximizer agent that has the possibility to invest in a (locally) riskfree security that pays an interest rate r and two risky securities with expected returns α and γ and standard deviation vectors $(\sigma_1)^\top = (\sigma_{11}\,\sigma_{12})$ and $(\sigma_2)^\top = (\sigma_{21}\,\sigma_{22})$ respectively. The expected return $\gamma^i, i = (O, I)$ at which the investor will optimally invest nothing in the second security is,*

$$\gamma_t^i = r_t + \frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1}(\alpha_t^i - r_t) \tag{32}$$

PROOF: Recall the optimal investment policy from equations (30) and (31). Solve for $\pi_{2t}^i = 0$, where π_2^i is the second component of the two-dimensional vector π^i, defined in Section 2.1 and the result follows.

□

Now we state the result about optimal policies in a corollary.

Corollary 1. *Consider the incomplete markets economy with additional information of Section 2. Define θ^{OI} and θ^{PI} as in equations (26) and (27) but replacing γ^O and γ^P by the corresponding "shadow-prices" as computed in Lemma 2 . Optimal policies are as in Lemma 1 but using θ^{OI} and θ^{PI}.*

PROOF: Obvious from Lemma 1.

□

3.1.3. Basic Information Setting

As we pointed at the beginning of Section 2.4, in this case agents in fact face complete markets with respect to their information. We define the market prices of risk in this setting as

$$\theta_t^{OB} = \frac{1}{\sqrt{(\sigma_1)^\top \sigma_1}}(\alpha_t^O - r_t) \tag{33}$$

$$\theta_t^{PB} = \frac{1}{\sqrt{(\sigma_1)^\top \sigma_1}}(\alpha_t^P - r_t) \tag{34}$$

Corollary 2. *Consider the basic information economy of Section 2. Optimal policies are as in Lemma 1 but using θ^{OB} and θ^{PB}.*

PROOF: As in Lemma 1.

□

3.2. Equilibrium Returns

Recall our notion of equilibrium: the sum of consumptions is equal to the aggregate consumption/dividend process; the total net demand of the investors is equal to the value of the productive technology; total net demand for both the contingent claim and the bond is zero. We present the equilibrium results in the following theorem.

Theorem 1. *Consider the economy presented in Section 2 in all three versions. In equilibrium,*

$$r_t = \delta - (\rho)^\top \rho + \mu_t^O \frac{c_t^O}{p_t} + \mu_t^P \frac{c_t^P}{p_t} \tag{35}$$

$$\alpha = \delta + \mu \tag{36}$$

$$\sigma_1 = \rho \tag{37}$$

In the complete markets version, the equilibrium drift of the contingent claim is

$$\gamma_t = r_t + (\sigma_1)^\top \sigma_2 + \epsilon - \left(\frac{c_t^O}{p_t} \epsilon_t^O + \frac{c_t^P}{p_t} \epsilon_t^P \right) \tag{38}$$

Interest rate dynamics will be different across the different economies considered since the distribution of consumption follows different processes.

PROOF: See the Appendix.

□

Remark 2. *Equations (35)-(37) are independent of the beliefs of the investors about the dynamics of the contingent claim.*

Remark 3. *Consider the economy of Section 2. The equilibrium interest rate and expected return and standard deviation of the productive technology will be (locally) the same as those that would obtain in a complete markets setting, expressed by equations (35)-(37).*

The structure of equilibrium is unchanged (but the equilibrium trajectories of consumption processes do change). The security that completes markets will not have an immediate effect on the equilibrium values of the parameters of the economy. However, it will affect the investment strategies of the investors and, therefore, their wealth paths. That will affect the dynamics of the equilibrium values. In fact, that is the basis of the difference in interest rate volatilities of next Section.

The asymmetry of beliefs about the dynamics of the aggregate consumption process is reflected in the equilibrium interest rate. The dynamics of the "productive technology" are not affected by the beliefs of the investors[7]. The main consequence of this result is that the difference between the equity premium in this setting and the equity premium in a classic setting with homogeneous beliefs is determined by the difference in the equilibrium interest rate level. In particular, a lower interest rate will yield a higher equity premium. This analysis has been exploited by part of the literature (see Abel, 1990) in an attempt to explain the "equity premium puzzle" first pointed by Mehra and Prescott (1985). The state variable in this model will be the weighted average of beliefs. Interest rates will be determined by this state variable -and so will be the equity premium-.

4. Interest Rate Volatility

In this Section we compute the volatility of the interest rate so that we can compare the results in the different market settings.

A key component of the interest rate volatility is the update of beliefs of the investors. Since equilibrium interest rates depend on the beliefs of the investors, the way they update their beliefs will affect the dynamics of the equilibrium interest rate. Moreover, the way individuals update their beliefs will depend on the information available. Therefore, in the economy we are considering there will be two ways of updating beliefs depending on the informational setting. Market completeness is irrelevant since, once we introduce the statistic, the security that completes markets does not add information. The updating rule in each setting is given in the next lemma.

[7]This is characteristic of logarithmic utilities. The problem is that more general parametrizations of the preferences -as power utility- become quickly untractable.

Lemma 3. *Consider the economy described in Section 2. In the "basic information" setting the agents update their beliefs about the drift of the aggregate consumption process according to the following stochastic differential equations,*

$$d\mu_t^O = \frac{\nu(t)}{\sqrt{(\sigma_1)^\top \sigma_1}} dW_t^{OB} \tag{39}$$

$$d\mu_t^P = \frac{\nu(t)}{\sqrt{(\sigma_1)^\top \sigma_1}} dW_t^{PB} \tag{40}$$

where W^{OB} and W^{PB} are as in equations (3) and (6) and ν satisfies the following deterministic differential equation,

$$d\nu(t) = -\nu(t)dt \tag{41}$$

with starting value $\nu(0)$. In the "additional information" setting (and, therefore, in the complete markets setting) the agents update their beliefs about the drifts of the aggregate consumption process and signal according to the following stochastic differential equations,

$$\begin{pmatrix} d\mu_t^O \\ d\epsilon_t^O \end{pmatrix} = \Upsilon(t)(\sigma)^{-1} dW_t^O \tag{42}$$

$$\begin{pmatrix} d\mu_t^P \\ d\epsilon_t^P \end{pmatrix} = \Upsilon(t)(\sigma)^{-1} dW_t^P \tag{43}$$

where W^{OA} and W^{PA} are the processes in equations (10) and (14) and Υ satisfies the following deterministic differential equation,

$$d\Upsilon(t) = -\Upsilon(t)dt \tag{44}$$

with starting value $\Upsilon(0)$.

PROOF: Substitute the corresponding parameters of our structure in the equations presented by Detemple (1986), p. 386.

□

Recall our notation θ^{OI} and θ^{PI}, the respective "shadow market price of risk" (that is, replacing the γ of the market price of risk by the "shadow price", in the economy with additional information) and θ^{OC} and θ^{PC} the subjective market price of risk in the complete markets setting. Recall also our notation θ^{OB} and θ^{PB} from equations (33) and (34). We can now derive the dynamics of the interest rate. We are only interested in the stochastic component. We state the result as a corollary.

Corollary 3. *The standard deviations of the interest rate process under the "basic information" setting, the incomplete markets with "additional information" setting and the complete markets setting (that we denote by σ^{rB}, σ^{rI} and σ^{rC} respectively) are given by,*

$$\sigma_t^{rB} = \frac{c_t^O}{p_t}\mu_t^O(\theta_t^{OB}\frac{\rho}{\sqrt{(\rho)^\top\rho}} - \rho) + \frac{c_t^P}{p_t}\mu_t^P(\theta_t^{PB}\frac{\rho}{\sqrt{(\rho)^\top\rho}} - \rho) + \frac{\nu(t)}{(\sigma_1)^\top\sigma_1}\sigma_1 \quad (45)$$

$$\sigma_t^{rI} = \frac{c_t^O}{p_t}\mu_t^O(\theta_t^{OI} - \rho) + \frac{c_t^P}{p_t}\mu_t^P(\theta_t^{PI} - \rho) + \Upsilon(t)(\sigma^\top)^{-1}\begin{pmatrix}1\\0\end{pmatrix} \quad (46)$$

$$\sigma_t^{rC} = \frac{c_t^O}{p_t}\mu_t^O(\theta_t^{OC} - \rho) + \frac{c_t^P}{p_t}\mu_t^P(\theta_t^{PC} - \rho) + \Upsilon(t)(\sigma^\top)^{-1}\begin{pmatrix}1\\0\end{pmatrix} \quad (47)$$

PROOF: This is a straightforward application of Ito's Lemma, using the equilibrium interest rate as computed in equation (35) and the optimal consumption and updating rules of equations (39)-(44).

□

Observe that the standard deviation of the interest rate has two components: the first component depends on the volatility of the proportion of the aggregate endowment consumed by each individual -in our setting with logarithmic utility this is the same as proportion of total wealth owned by each agent- and the second on the volatility of the priors of the agents as a result of their bayesian update. The volatility formulas of Corollary 2 can be written in terms of the difference in beliefs among agents (see Detemple and Murthy, 1994, Proposition 4.2).

5. Changes in Market Volatility

In this Section we study the difference in the volatility of the interest rate process (that is the representative equilibrium value since the drift and variance of the stock process are not affected by the asymmetry of beliefs as a result of the myopic preferences) between the three economies considered.

5.1. Addition of Information

We compare the incomplete markets "basic information" setting with the incomplete markets with "additional information" setting where agents observe the signal s. The agents will have also heterogeneous beliefs with respect to the drift of the signal ϵ. We assume that the investors' beliefs about drift of aggregate consumption and signal are jointly normally distributed with mean (μ^O, ϵ^O) for the optimistic agent and (μ^P, ϵ^P) for the pessimistic and variance-covariance matrix Υ (for both agents). We will denote the elements of this matrix as ν_{ij}. In order to be consistent, it has to be the case that

at the moment t of introduction of the signal, $\nu(t) = \nu_{11}(t)$. We state the change in volatility in the following theorem,

Theorem 2. *The difference in volatility of the interest rate between the basic information setting and the setting with an additional signal defined in Section 2 is always positive and given by*

$$(\sigma^{rI})^\top \sigma^{rI} - (\sigma^{rB})^\top \sigma^{rB} = \left(\frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1} \right)^2 \frac{(\sigma_1)^\top \sigma_1}{\mid \sigma\sigma^\top \mid} (\nu_{11})^2 \left(1 - \frac{\frac{\nu_{12}}{\nu_{11}}}{\frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1}} \right)^2 \tag{48}$$

PROOF: In the Appendix.

□

The previous expression is always nonnegative. In general, the introduction of the signal will increase the volatility of the interest rate. The economic intuition of this result is straightforward: individuals receive more information and, as a result of it, will update their beliefs faster. The increase will be larger the larger the difference between the ratio of the covariance of the estimates of the drift of the signal and the variance of the estimate of the drift of the aggregate consumption process, $\frac{\nu_{12}}{\nu_{11}}$ and the ratio of the covariance of the drift of the aggregate consumption process (recall that in equilibrium $\rho = \sigma_1$) and the drift of the signal, $\frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1}$. Only if these ratios are equal volatility will not increase. In this case the individual will not learn anything about the drift of the aggregate consumption process.

However this is a "local" result. The signal translates into immediate greater volatility, although, since individuals update faster, they will converge faster and, overtime, this will mean a drop in volatility.

5.2. Market Completion

We now compare the complete markets economy with the incomplete markets with additional information economy. There is no difference in information between these two settings, the difference is the securities that can be traded. As we pointed before, we will assume that *the security is designed in such a way that its volatility is identical to the volatility of the existing signal.* Individuals will disagree about the equilibrium value of the drift of the contingent claim, but they will disagree in a way that is consistent with their disagreement about the drift of the signal on which the contingent claim contract depends.

Remark 4. *Since we are assuming logarithmic preferences, this result does not depend on the completion of markets. We assume completion in order to keep the notation as sparse as possible, but a setting with three Brownian motion processes and addition of the contingent claim would yield the same result.*

We state the result in the following theorem:

Theorem 3. *Consider the economies described in Section 2. The difference in volatility of the equilibrium interest rate between the incomplete markets setting with a signal and the complete markets setting (but with the same information) is given by the following expression,*

$$
\begin{aligned}
&(\sigma^{rC})^\top \sigma^{rC} - (\sigma^{rI})^\top \sigma^{rI} = \\
&\left(\frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1}\right)^2 \frac{(\sigma_1)^\top \sigma_1}{\mid \sigma\sigma^\top \mid} \left(\frac{c^O}{p}\frac{c^P}{p}\right)(\mu^O - \mu^P)^2 \times \\
&\left[\left(\frac{c^O}{p}\frac{c^P}{p}\right)(\mu^O - \mu^P)^2 \left(1 - \frac{\frac{\gamma^O - \gamma^P}{\mu^O - \mu^P}}{\frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1}}\right)^2 + 2\nu_{11}\left(1 - \frac{\frac{\nu_{12}}{\nu_{11}}}{\frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1}}\right)\left(1 - \frac{\frac{\gamma^O - \gamma^P}{\mu^O - \mu^P}}{\frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1}}\right)\right]
\end{aligned}
\tag{49}
$$

where γ^O and γ^P are the values given by equations (23) and (24).

PROOF: See the Appendix.

□

The change in volatility has two components. The first component is always positive and is larger the larger the discrepancy between the ratio of the difference in beliefs with respect to the drift of the contingent claim and the difference in beliefs with respect to the drift of the aggregate consumption process (and, therefore, the drift of the stock), $\frac{\gamma^O - \gamma^P}{\mu^O - \mu^P}$ and the ratio of the covariance of the securities and the variance of the existing security, $\frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1}$. The intuition is straightforward: if the security is traded, the agents will change their portfolios and the share of wealth of the agents becomes more volatile. Only if the disagreement were as large as the risk -previous ratios equal- individuals would not readjust their portfolios.

The second component is the cross-effect: change in volatility due to the change in covariance between the update of beliefs and the share of wealth. Clearly, a necessary condition for this term to be positive is that the ratios of covariance of the estimates to variance of the estimate of the drift of the aggregate consumption process $\frac{\nu_1 2}{\nu_1 1}$ and difference in beliefs about the drift of the contingent claim and difference in beliefs about the aggregate consumption process $\frac{\gamma^O - \gamma^P}{\mu^O - \mu^P}$ be both larger or smaller than the ratio of covariance of the contingent claim and stock to variance of the stock, $\frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1}$.

Note that these ratios are both deterministic (this is obvious for $\frac{\nu_{12}}{\nu_{11}}$ and straightforward to show for $\frac{\gamma^O - \gamma^P}{\mu^O - \mu^P}$). However, their values will depend in a complex way on initial conditions and parameters, therefore it does not seem possible to make a general statement about them. We also point that it is not clear if the increase in

wealth share volatility is a permanent effect or introduction of the contingent claim will lead overtime to a more stable wealth share[8].

The difference in volatility between the basic information setting and the complete markets setting (contingent claim and information bundled together) is always positive. We state the result in the following corollary.

Corollary 4. *The difference in volatility between the basic information setting and the setting with both the signal and the contingent claim is always positive, as expressed by the following value,*

$$\begin{aligned} & (\sigma^{rC})^\top \sigma^{rC} - (\sigma^{rB})^\top \sigma^{rB} \\ = & \left(\frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1}\right)^2 \frac{(\sigma_1)^\top \sigma_1}{\mid \sigma\sigma^\top \mid} \times \\ & \left[\left(\frac{c^O}{p}\frac{c^P}{p}\right)(\mu^O - \mu^P)^2 \left(1 - \frac{\frac{\gamma^O - \gamma^P}{\mu^O - \mu^P}}{\frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1}}\right) + \nu_{11}\left(1 - \frac{\frac{\nu_{12}}{\nu_{11}}}{\frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1}}\right)\right]^2 \end{aligned} \tag{50}$$

PROOF: Obvious from Theorem 2 and 3.

□

6. Conclusions

We construct a pure exchange version of the Detemple and Murthy (1994) equilibrium model with two investors with heterogeneous beliefs. As a difference with the homogeneous beliefs case, the interest rate of the economy is volatile. Volatility has two components. The first component is the result of the update of information. The second component is the weighted average (where weights are the share of wealth) of the beliefs of the individuals. We show that a signal that adds information will have a positive (local) effect on the volatility of the interest rate. In a different setting, a security that completes markets will have two effects on volatility: one positive (the weights of the individuals become more volatile) and one undetermined (effect on covariance between both components). The drawback of our model is the assumption about logarithmic utilities that, as it is well known, induce myopia. Ideally, the strategies of our agents would involve more refined hedging. Unfortunately, utility specifications with that feature generate problems that are untractable. Another point that deserves further consideration is the update rule of the representative agents. Alternative update rules based on the distance to the beliefs of other investors,

[8]The volatility results seem to be valid for more general information structures, i.e., if μ and ϵ where also stochastic, adapted to the filtration generated by W^1 and W^2. The current setting simplifies the notation.

for example, would generate more complex dynamics with possibly different effects. Finally, it is worth stressing that we only study the immediate effect on volatility of introducing the information. Arguably, the agents of the economy will update "faster" their priors and their beliefs will converge faster. Overtime it will have a smoothing effect by forcing the heterogeneous beliefs –that produce the volatility– to converge faster.

A Appendix

A1. Proof of Theorem 1

In order to keep the notation as simple as possible, *in this proof we will use the notation γ^O and γ^P to represent both the drift of the contingent claim observed by the individuals in the complete markets specification and the "shadow-prices" of the incomplete markets specification.* Therefore, we will denote by θ^O and θ^P the expressions in equations (26) and (27) for both the complete and the incomplete markets with additional information economies. At the end of the proof we extend the results to the basic information economy. From the definition of θ^O,

$$\theta_t^O = (\sigma)^{-1} \begin{pmatrix} \alpha_t^O - r_t \\ \gamma_t^O - r_t \end{pmatrix} = (\sigma)^{-1} \begin{pmatrix} \alpha_t - r_t \\ \gamma_t - r_t \end{pmatrix} + (\sigma)^{-1} \begin{pmatrix} \alpha_t^O - \alpha_t \\ \gamma_t^O - \gamma_t \end{pmatrix}$$

but from equations (12) and (16),

$$(\sigma)^{-1} \begin{pmatrix} \alpha_t^O - \alpha_t \\ \gamma_t^O - \gamma_t \end{pmatrix} = -\lambda_t^O$$

Similarly for θ^P. Therefore,

$$\theta_t^O = \theta_t - \lambda_t^O \tag{51}$$
$$\theta_t^P = \theta_t - \lambda_t^P \tag{52}$$

Now, consider the equilibrium in the consumption good markets,

$$p_t = c_t^O + c_t^P$$

We introduce the exponential martingale η,

$$\eta_t = e^{-\int_0^t (\theta_s)^\top dW_s - \frac{1}{2}\int_0^t (\theta_s)^\top \theta_s ds}$$

From Cox and Huang (1990) and Karatzas, Lehoczky and Shreve (1987), the optimal consumption policy of the agents is,

$$c_t^* = \frac{1}{y e^{\int_0^t (\delta - r_s) ds} \eta_t} \tag{53}$$

where y is a constant (the lagrange multiplier).

Take equation (53): if we differentiate both hand-sides,

$$\begin{aligned} p_t(\mu dt + \rho^\top dW_t) &= c_t^O \left((r_t - \delta)dt + (\theta_t^O)^\top \theta_t^O dt + (\theta_t^O)^\top dW_t^O \right) \\ &+ c_t^P \left((r_t - \delta)dt + (\theta_t^P)^\top \theta_t^P dt + (\theta_t^P)^\top dW_t^P \right) \end{aligned} \tag{54}$$

We replace dW^O and dW^P by the expressions in equations (11) and (15). From the terms in dW in the previous equation, we get

$$\theta_t^O c_t^O + \theta_t^P c_t^P = \rho p_t$$

From equations (51) and (52) and the fact that $c^O + c^P = p$,

$$\theta_t = \rho + \frac{c_t^O}{p_t}\lambda_t^O + \frac{c_t^P}{p_t}\lambda_t^P \tag{55}$$

From the terms in dt in equation (54) and taking into account (51), and (52) we get,

$$\begin{aligned} r_t &= \delta + \mu - \frac{c_t^O}{p_t}(\theta_t^O)^\top \theta_t^O - \frac{c_t^P}{p_t}(\theta_t^P)^\top \theta_t^P - \frac{c_t^O}{p_t}(\theta_t^O)^\top \lambda_t^O - \frac{c_t^P}{p_t}(\theta_t^P)^\top \lambda_t^P \\ &= \delta + \mu - \frac{c_t^O}{p_t}(\theta_t - \lambda_t^O)^\top (\theta_t - \lambda_t^O) - \frac{c_t^P}{p_t}(\theta_t - \lambda_t^P)^\top (\theta_t - \lambda_t^P) \\ &\quad -\frac{c_t^O}{p_t}(\theta_t - \lambda_t^O)^\top \lambda_t^O - \frac{c_t^P}{p_t}(\theta_t - \lambda_t^P)^\top \lambda_t^P \\ &= \delta + \mu - (\theta_t)^\top \theta_t + \frac{c_t^O}{p_t}(\theta_t)^\top \lambda_t^O + \frac{c_t^P}{p_t}(\theta_t)^\top \lambda_t^P \end{aligned}$$

Using the equilibrium value of θ in equation (55),

$$\begin{aligned} r_t &= \delta + \mu - (\theta_t)^\top \rho \\ &= \delta + \mu - (\rho)^\top (\rho + \frac{c_t^O}{p_t}\lambda_t^O + \frac{c_t^P}{p_t}\lambda_t^P) \end{aligned}$$

and using the restrictions of equations (12) and (16),

$$r_t = \delta + \mu - (\rho)^\top \rho + (\mu_t^O - \mu)\frac{c_t^O}{p_t} + (\mu_t^P - \mu)\frac{c_t^P}{p_t}$$

and equation (35) follows immediately.

We now consider the equilibrium in the financial markets. According to our notion of equilibrium, the linear technology represents the wealth of the economy,

$$S_t = X_t^O + X_t^P \tag{56}$$

and the total net supply of the linear technology has to be equal to optimal demands of the investors, while the contingent claim and the riskfree asset are in zero net supply,

$$\pi_t^O + \pi_t^P = \begin{pmatrix} S_t \\ 0 \end{pmatrix}$$

or, according to equations (28)–(31),

$$\begin{aligned}\pi_t^O + \pi_t^P &= (\sigma^\top)^{-1}\theta_t^O X_t^O + (\sigma^\top)^{-1}\theta_t^P X_t^P \\ &= (\sigma^\top)^{-1}(\theta_t - \lambda_t^O)X_t^O + (\sigma^\top)^{-1}(\theta_t - \lambda_t^P)X_t^P \\ &= \begin{pmatrix} S_t \\ 0 \end{pmatrix}\end{aligned}$$

Multiplying both hand-sides of the last equality by $\sigma^\top \frac{1}{S}$,

$$\theta_t = \sigma_1 + \frac{X_t^O}{S_t}\lambda_t^O + \frac{X_t^P}{S_t}\lambda_t^P$$

From equation (55),

$$\rho + \frac{c_t^O}{p_t}\lambda_t^O + \frac{c_t^P}{p_t}\lambda_t^P = \sigma_1 + \frac{X_t^O}{S_t}\lambda_t^O + \frac{X_t^P}{S_t}\lambda_t^P$$

From equation (56) and the consumption function of each agent, it is immediate to conclude that S_t is proportional to p_t. Simple derivation then leads to (36) and (37).

With respect to the basic information economy, consider the subjective market premia of risk θ_t^{OB} and θ_t^{PB} of equations (33) and (34) and the equivalent value with the true underlying parameter,

$$\theta_t^B = \frac{1}{\sqrt{(\sigma_1)^\top \sigma_1}}(\alpha_t - r_t) \tag{57}$$

Using the same reasoning as at the beginning of this proof, and equations (5) and (7) it is straightforward to show that

$$\theta_t^{OB} = \theta_t^B - \frac{(\rho)^\top}{\sqrt{(\rho)^\top \rho}}\lambda_t^O \tag{58}$$

$$\theta_t^{PB} = \theta_t^B - \frac{(\rho)^\top}{\sqrt{(\rho)^\top \rho}}\lambda_t^P \tag{59}$$

Using this fact we can replicate the proof of the two-dimensional innovation process replacing in (54) the innovation processes dW^O and dW^P by its value from equations (3) and (6).

A2. A technical Lemma

We introduce the following technical result that we will use later.

Lemma 4. *The standard deviations of the interest rate under the different settings computed in Section 4 can be also expressed as (we drop the time indicators)*

$$\begin{aligned}
\sigma^{rC} &= \frac{c^O}{p}\frac{c^P}{p}(\mu^O-\mu^P)^2\sigma^{-1}\begin{pmatrix} 1 \\ \frac{\gamma^O-\gamma^P}{\mu^O-\mu P} \end{pmatrix} + \Upsilon(\sigma^\top)^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\sigma^{rI} &= \frac{c^O}{p}\frac{c^P}{p}(\mu^O-\mu^P)^2\sigma^{-1}\begin{pmatrix} 1 \\ \frac{\sigma_1^\top\sigma_2}{\sigma_1^\top\sigma_1} \end{pmatrix} + \Upsilon(\sigma^\top)^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}^\top \\
\sigma^{rB} &= \frac{c^O}{p}\frac{c^P}{p}(\mu^O-\mu^P)^2\sigma^{-1}\begin{pmatrix} 1 \\ \frac{\sigma_1^\top\sigma_2}{\sigma_1^\top\sigma_1} \end{pmatrix} + \frac{\nu(t)}{(\sigma_1)^\top\sigma_1}\sigma_1
\end{aligned}$$

PROOF: From equations (51) and (55)

$$\begin{aligned}
\theta^{OC} &= \sigma_1 + \frac{c^O}{p}\lambda^O + \frac{c^P}{p}\lambda^P - \lambda^O \\
&= \sigma_1 + \frac{c^P}{p}(\lambda^P - \lambda^O)
\end{aligned}$$

Recall also equation (37). Then,

$$\theta^{OC} - \rho = \frac{c^P}{p}(\lambda^P - \lambda^O)$$

From Theorem 1 and equations (5), (7), (12), (16), (23) and (24), it is straightforward to show that,

$$\lambda^O = (\sigma)^{-1}\begin{pmatrix} \mu-\mu^O \\ \gamma-\gamma^O \end{pmatrix} \tag{60}$$

$$\lambda^P = (\sigma)^{-1}\begin{pmatrix} \mu-\mu^P \\ \gamma-\gamma^P \end{pmatrix} \tag{61}$$

Then,

$$\lambda^P - \lambda^O = (\sigma)^{-1}\begin{pmatrix} \mu^O-\mu^P \\ \gamma^O-\gamma^P \end{pmatrix}$$

and

$$\theta^{OC} - \rho = \frac{c^P}{p}\sigma^{-1}\begin{pmatrix} \mu^O-\mu^P \\ \gamma^O-\gamma^P \end{pmatrix}$$

Similarly,

$$\begin{aligned}
\theta^{PC} - \rho &= \frac{c^O}{p}(\lambda^O - \lambda^P) \\
&= \frac{c^O}{p}\sigma^{-1}\begin{pmatrix} \mu^O-\mu^P \\ \gamma^O-\gamma^P \end{pmatrix}
\end{aligned}$$

Then,

$$\begin{aligned}
& \frac{c^O}{p}\mu^O(\theta^{OC}-\rho)+\frac{c^P}{p}\mu^P(\theta^{OP}-\rho) \\
= & \frac{c^O}{p}\mu^O\frac{c^P}{p}\sigma^{-1}\begin{pmatrix}\mu^O-\mu^P\\ \gamma^O-\gamma^P\end{pmatrix}+\frac{c^P}{p}\mu^P\frac{c^O}{p}\sigma^{-1}\begin{pmatrix}\mu^P-\mu^O\\ \gamma^P-\gamma^O\end{pmatrix} \\
= & \frac{c^O}{p}\frac{c^P}{p}\sigma^{-1}\left(\mu^O\begin{pmatrix}\mu^O-\mu^P\\ \gamma^O-\gamma^P\end{pmatrix}-\mu^P\begin{pmatrix}\mu^O-\mu^P\\ \gamma^O-\gamma^P\end{pmatrix}\right) \\
= & \frac{c^O}{p}\frac{c^P}{p}\sigma^{-1}\begin{pmatrix}(\mu^O-\mu^P)^2\\ (\gamma^O-\gamma^P)(\mu^O-\mu^P)\end{pmatrix}
\end{aligned}$$

and the result about σ^{rC} follows.

With respect to the incomplete markets with additional information case, we denote by γ^{OS} and γ^{PS} the "shadow" drift of the contingent claim for each of the agents (drift of the contingent claim at which it would not be traded by the agents). Recall that

$$\gamma^{OS}=r+\frac{(\sigma_1)^\top\sigma_2}{(\sigma_1)^\top\sigma_1}(\alpha^O-r)$$

Then, by definition,

$$\begin{aligned}
\theta^{OI} &= \sigma^{-1}\begin{pmatrix}\alpha^O-r\\ \frac{(\sigma_1)^\top\sigma_2}{(\sigma_1)^\top\sigma_1}(\alpha^O-r)\end{pmatrix} \\
&= (\alpha^O-r)\sigma^{-1}\begin{pmatrix}1\\ \frac{(\sigma_1)^\top\sigma_2}{(\sigma_1)^\top\sigma_1}\end{pmatrix} \\
&= \left(\mu^O+(\sigma_1)^\top\sigma_1-\frac{c^O}{p}\mu^O-\frac{c^P}{p}\mu^P\right)\sigma^{-1}\begin{pmatrix}1\\ \frac{(\sigma_1)^\top\sigma_2}{(\sigma_1)^\top\sigma_1}\end{pmatrix} \\
&= \left((\sigma_1)^\top\sigma_1+\frac{c^P}{p}(\mu^O-\mu^P)\right)\sigma^{-1}\begin{pmatrix}1\\ \frac{(\sigma_1)^\top\sigma_2}{(\sigma_1)^\top\sigma_1}\end{pmatrix}
\end{aligned}$$

where we have relied on the equilibrium values of α^O and r.

Similarly, it is straightforward to show that

$$\theta^{PI}=\left((\sigma_1)^\top\sigma_1+\frac{c^O}{p}(\mu^P-\mu^O)\right)\sigma^{-1}\begin{pmatrix}1\\ \frac{(\sigma_1)^\top\sigma_2}{(\sigma_1)^\top\sigma_1}\end{pmatrix}$$

Then,

$$\begin{aligned}
& \frac{c^O}{p}\mu^O(\theta^{OI}-\rho)+\frac{c^P}{p}\mu^P(\theta^{PI}-\rho) \\
= & \frac{c^O}{p}\mu^O\left((\sigma_1)^\top\sigma_1+\frac{c^P}{p}(\mu^O-\mu^P)\right)\sigma^{-1}\begin{pmatrix}1\\ \frac{(\sigma_1)^\top\sigma_2}{(\sigma_1)^\top\sigma_1}\end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{c^P}{p}\mu^P\left((\sigma_1)^\top\sigma_1+\frac{c^O}{p}(\mu^P-\mu^O)\right)\sigma^{-1}\begin{pmatrix}1\\ \frac{(\sigma_1)^\top\sigma_2}{(\sigma_1)^\top\sigma_1}\end{pmatrix}\\
&-\left(\frac{c^O}{p}\mu^O-\frac{c^P}{p}\mu^P\right)\rho\\
=\;&\frac{c^O}{p}\frac{c^P}{p}(\mu^O-\mu^P)^2\sigma^{-1}\begin{pmatrix}1\\ \frac{(\sigma_1)^\top\sigma_2}{(\sigma_1)^\top\sigma_1}\end{pmatrix}\\
&+\left(\frac{c^O}{p}\mu^O+\frac{c^P}{p}\mu^P\right)\left(\sigma^{-1}\begin{pmatrix}(\sigma_1)^\top\sigma_1\\ (\sigma_1)^\top\sigma_2\end{pmatrix}-\rho\right)
\end{aligned}
$$

But recall that, in equilibrium, $\rho=\sigma_1$. Then, the expression in the last parenthesis of the last equation disappears since

$$
\begin{pmatrix}(\sigma_1)^\top\sigma_1\\ (\sigma_1)^\top\sigma_2\end{pmatrix}=\sigma\sigma_1=\sigma\rho \tag{62}
$$

With respect to σ^{rB}, the volatility in the basic information economy, note that,

$$
\begin{aligned}
\theta^{OB}&=(\alpha^O-r)\frac{1}{\sqrt{(\sigma_1)^\top\sigma_1}}\\
&=(\mu^O+(\sigma_1)^\top\sigma_1-\frac{c^O}{p}\mu^O-\frac{c^P}{p}\mu^P)\frac{1}{\sqrt{(\sigma_1)^\top\sigma_1}}\\
&=\left((\sigma_1)^\top\sigma_1+\frac{c^P}{p}(\mu^O-\mu^P)\right)\frac{1}{\sqrt{(\sigma_1)^\top\sigma_1}}
\end{aligned}
$$

Similarly,

$$
\theta^{PB}=\left((\sigma_1)^\top\sigma_1+\frac{c^O}{p}(\mu^P-\mu^O)\right)\frac{1}{\sqrt{(\sigma_1)^\top\sigma_1}}
$$

Then, (recall that $\rho=\sigma_1$),

$$
\begin{aligned}
&\frac{c^O}{p}\mu^O\left(\theta^{OB}\frac{\sigma_1}{\sqrt{(\sigma_1)^\top\sigma_1}}-\sigma_1\right)+\frac{c^P}{p}\mu^P\left(\theta^{PB}\frac{\sigma_1}{\sqrt{(\sigma_1)^\top\sigma_1}}-\sigma_1\right)\\
=\;&\frac{c^O}{p}\mu^O\sigma_1\left(\frac{(\sigma_1)^\top\sigma_1+\frac{c^P}{p}(\mu^O-\mu^P)}{(\sigma_1)^\top\sigma_1}-1\right)+\frac{c^P}{p}\mu^P\sigma_1\left(\frac{(\sigma_1)^\top\sigma_1+\frac{c^O}{p}(\mu^P-\mu^O)}{(\sigma_1)^\top\sigma_1}-1\right)\\
=\;&\frac{c^O}{p}\frac{c^P}{p}\mu^O(\mu^O-\mu^P)\frac{\sigma_1}{(\sigma_1)^\top\sigma_1}+\frac{c^O}{p}\frac{c^P}{p}\mu^P(\mu^P-\mu^O)\frac{\sigma_1}{(\sigma_1)^\top\sigma_1}\\
=\;&\frac{c^O}{p}\frac{c^P}{p}\frac{\sigma_1}{(\sigma_1)^\top\sigma_1}(\mu^O-\mu^P)^2
\end{aligned}
$$

From equation (62),

$$\frac{\sigma_1}{(\sigma_1)^\top \sigma_1} = \sigma^{-1} \begin{pmatrix} 1 \\ \frac{\sigma_1^\top \sigma_2}{\sigma_1^\top \sigma_1} \end{pmatrix} \tag{63}$$

and substituting in equation (45) the result about σ^{rB} follows.

□

Remark 5. *From (63) it is obvious that the expression* $\sigma^{-1} \begin{pmatrix} 1 \\ \frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1} \end{pmatrix}$ *does not depend on the elements of* σ_2 *as one would expect from the fact that the second security is still not traded. However, it makes computations easier.*

A3. Proof of Theorem 2

From Lemma 4 we have

$$\begin{aligned} & (\sigma^{rI})^\top \sigma^{rI} \\ = \; & \frac{c^O}{p}\frac{c^P}{p}^2 (\mu^O - \mu^P)^4 \begin{pmatrix} 1 \\ \frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1} \end{pmatrix}^\top (\sigma\sigma^\top)^{-1} \begin{pmatrix} 1 \\ \frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1} \end{pmatrix} \\ & + 2\frac{c^O}{p}\frac{c^P}{p}(\mu^O - \mu^P)^2 \begin{pmatrix} 1 \\ \frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1} \end{pmatrix}^\top (\sigma\sigma^\top)^{-1} (\Upsilon)^\top \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ & + \begin{pmatrix} 1 \\ 0 \end{pmatrix}^\top (\Upsilon)^\top (\sigma\sigma^\top)^{-1} \Upsilon \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned} \tag{64}$$

Similarly,

$$\begin{aligned} & (\sigma^{rB})^\top \sigma^{rB} \\ = \; & \frac{c^O}{p}\frac{c^P}{p}^2 (\mu^O - \mu^P)^4 \begin{pmatrix} 1 \\ \frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1} \end{pmatrix}^\top (\sigma\sigma^\top)^{-1} \begin{pmatrix} 1 \\ \frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1} \end{pmatrix} \\ & + 2\frac{c^O}{p}\frac{c^P}{p}(\mu^O - \mu^P)^2 \begin{pmatrix} 1 \\ \frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1} \end{pmatrix}^\top (\sigma^\top)^{-1} \sigma_1 \frac{\nu_{11}}{(\sigma_1)\top \sigma_1} \\ & + \frac{\nu_{11}}{(\sigma_1)^\top \sigma_1} \end{aligned} \tag{65}$$

Then,

$$\begin{aligned} & (\sigma^{rI})^\top \sigma^{rI} - (\sigma^{rB})^\top \sigma^{rB} \\ = \; & 2\frac{c^O}{p}\frac{c^P}{p}(\mu^O - \mu^P)^2 \left[\begin{pmatrix} 1 \\ \frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1} \end{pmatrix}^\top (\sigma\sigma^\top)^{-1} (\Upsilon)^\top \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \frac{(\sigma_1)^\top \sigma_2}{(\sigma_1)^\top \sigma_1} \end{pmatrix}^\top (\sigma^\top)^{-1} \sigma_1 \frac{\nu_{11}}{(\sigma_1)\top \sigma_1} \right] \end{aligned}$$

$$
\begin{aligned}
&+\begin{pmatrix}1\\0\end{pmatrix}^{\top}(\Upsilon)^{\top}(\sigma\sigma^{\top})^{-1}\Upsilon\begin{pmatrix}1\\0\end{pmatrix}-\frac{\nu_{11}}{(\sigma_1)^{\top}\sigma_1}\\
=\;& 2\frac{c^O}{p}\frac{c^P}{p}(\mu^O-\mu^P)^2\left[\frac{\nu_{11}}{\mid\sigma\sigma^{\top}\mid}\left((\sigma_2)^{\top}\sigma_2-\frac{((\sigma_1)^{\top}\sigma_2)^2}{(\sigma_1)^{\top}\sigma_1}\right)-\begin{pmatrix}1\\ \frac{(\sigma_1)^{\top}\sigma_2}{(\sigma_1)^{\top}\sigma_1}\end{pmatrix}^{\top}(\sigma^{\top})^{-1}\sigma^{\top}\begin{pmatrix}1\\0\end{pmatrix}\frac{\nu_{11}}{(\sigma_1)^{\top}\sigma_1}\right]\\
&+\frac{1}{\mid\sigma\sigma^{\top}\mid((\sigma_1)^{\top}\sigma_1)}\left(\nu_{11}((\sigma_1)^{\top}\sigma_2)-\nu_{12}((\sigma_1)^{\top}\sigma_1)\right)^2\\
=\;& 2\frac{c^O}{p}\frac{c^P}{p}(\mu^O-\mu^P)^2\left[\frac{\nu_{11}}{\mid\sigma\sigma^{\top}\mid}\left((\sigma_2)^{\top}\sigma_2-\frac{((\sigma_1)^{\top}\sigma_2)^2}{(\sigma_1)^{\top}\sigma_1}\right)-\frac{\nu_{11}}{(\sigma_1)^{\top}\sigma_1}\right]\\
&+\left(\frac{(\sigma_1)^{\top}\sigma_2}{(\sigma_1)^{\top}\sigma_1}\right)^2\frac{(\sigma_1)^{\top}\sigma_1}{\mid\sigma\sigma^{\top}\mid}(\nu_{11})^2\left(1-\frac{\frac{\nu_{12}}{\nu_{11}}}{\frac{(\sigma_1)^{\top}\sigma_2}{(\sigma_1)^{\top}\sigma_1}}\right)^2
\end{aligned}
$$

and the claim follows since the expression in brackets is zero.

A4. Proof of Theorem 3

From Lemma 4 we have

$$
\begin{aligned}
&(\sigma^{rC})^{\top}\sigma^{rC}\\
=\;&\frac{c^O}{p}\frac{c^{P\,2}}{p}(\mu^O-\mu^P)^4\begin{pmatrix}1\\ \frac{\gamma^O-\gamma^P}{\mu^O-\mu^P}\end{pmatrix}^{\top}(\sigma\sigma^{\top})^{-1}\begin{pmatrix}1\\ \frac{\gamma^O-\gamma^P}{\mu^O-\mu^P}\end{pmatrix}\\
&+2\frac{c^O}{p}\frac{c^P}{p}(\mu^O-\mu^P)^2\begin{pmatrix}1\\ \frac{\gamma^O-\gamma^P}{\mu^O-\mu^P}\end{pmatrix}^{\top}(\sigma\sigma^{\top})^{-1}(\Upsilon)^{\top}\begin{pmatrix}1\\0\end{pmatrix}\\
&+\begin{pmatrix}1\\0\end{pmatrix}^{\top}(\Upsilon)^{\top}(\sigma\sigma^{\top})^{-1}\Upsilon\begin{pmatrix}1\\0\end{pmatrix}
\end{aligned}
$$

From this expression and equation (64) we get,

$$
\begin{aligned}
&(\sigma^{rC})^{\top}\sigma^{rC}-(\sigma^{rM})^{\top}\sigma^{rM}\\
=\;&\frac{c^O}{p}\frac{c^{P\,2}}{p}(\mu^O-\mu^P)^4\frac{(\sigma_1)^{\top}\sigma_1}{\mid\sigma\sigma^{\top}\mid}\left(\frac{\gamma^O-\gamma^P}{\mu^O-\mu^P}-\frac{(\sigma_1)^{\top}\sigma_2}{(\sigma_1)^{\top}\sigma_1}\right)^2\\
&+2\frac{c^O}{p}\frac{c^P}{p}\nu_{11}\frac{(\sigma_1)^{\top}\sigma_1}{\mid\sigma\sigma_{\top}\mid}\left(\frac{\gamma^O-\gamma^P}{\mu^O-\mu^P}-\frac{(\sigma_1)^{\top}\sigma_2}{(\sigma_1)^{\top}\sigma_1}\right)\begin{pmatrix}0\\1\end{pmatrix}^{\top}(\sigma\sigma^{\top})^{-1}\Upsilon\begin{pmatrix}1\\0\end{pmatrix}\\
=\;&\frac{c^O}{p}\frac{c^{P\,2}}{p}(\mu^O-\mu^P)^4\frac{(\sigma_1)^{\top}\sigma_1}{\mid\sigma\sigma^{\top}\mid}\left(\frac{\gamma^O-\gamma^P}{\mu^O-\mu^P}-\frac{(\sigma_1)^{\top}\sigma_2}{(\sigma_1)^{\top}\sigma_1}\right)^2\\
&+2\frac{c^O}{p}\frac{c^P}{p}(\mu^O-\mu^P)^2\nu_{11}\frac{(\sigma_1)^{\top}\sigma_1}{\mid\sigma\sigma_{\top}\mid}\left(\frac{\nu_{12}}{\nu^{11}}-\frac{(\sigma_1)^{\top}\sigma_2}{(\sigma_1)^{\top}\sigma_1}\right)\left(\frac{\gamma^O-\gamma^P}{\mu^O-\mu^P}-\frac{(\sigma_1)^{\top}\sigma_2}{(\sigma_1)^{\top}\sigma_1}\right)\\
=\;&\frac{c^O}{p}\frac{c^{P\,2}}{p}(\mu^O-\mu^P)^4\frac{(\sigma_1)^{\top}\sigma_1}{\mid\sigma\sigma^{\top}\mid}\left(\frac{(\sigma_1)^{\top}\sigma_2}{(\sigma_1)^{\top}\sigma_1}\right)^2\left(1-\frac{\frac{\gamma^O-\gamma^P}{\mu^O-\mu^P}}{\frac{(\sigma_1)^{\top}\sigma_2}{(\sigma_1)^{\top}\sigma_1}}\right)^2
\end{aligned}
$$

$$+2\frac{c^O}{p}\frac{c^P}{p}(\mu^O-\mu^P)^2\frac{(\sigma_1)^\top\sigma_1}{\mid\sigma\sigma^\top\mid}\left(\frac{(\sigma_1)^\top\sigma_2}{(\sigma_1)^\top\sigma_1}\right)^2\nu_{11}\left(1-\frac{\frac{\nu_{11}}{\nu_{12}}}{\frac{(\sigma_1)^\top\sigma_2}{(\sigma_1)^\top\sigma_1}}\right)\left(1-\frac{\frac{\gamma^O-\gamma^P}{\mu^O-\mu^P}}{\frac{(\sigma_1)^\top\sigma_2}{(\sigma_1)^\top\sigma_1}}\right)$$

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