On the Recoverability of Preferences and Beliefs in Financial Models*

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This draft: April 1997

Abstract

We examine the extent to which an investor's tastes and beliefs can be recovered from knowledge of his/her consumption choice. More precisely, we assume that the investor's preferences admit an expected utility representation, but with subjective probabilities, and investigate what joint restrictions can be placed on utility functions and beliefs. If the investor has preferences for wealth or consumption at a single future date, the problem is clearly indeterminate, in the sense that, subject to minor technical conditions, for any given utility function and consumption policy, it is possible to construct a set of beliefs that would support the given consumption choice. On the other hand, if the investor draws utility from intertemporal consumption, we show that the set of utility functions and beliefs that are consistent with a given consumption choice can be characterized by a martingale condition. In the Markovian case, this characterization can be reexpressed in terms of an ordinary differential equation that must be satisfied by the investor's relative risk aversion function. To each solution of this differential equation is associated a unique set of beliefs.

*We are grateful to Abel Cadenillas, Guy Laroque (the editor), two anonymous referees and to participants in the Micro Lunch Seminar at the Wharton School, the 1996 CIRANO/CRM Workshop on the Mathematics of Finance and the 1997 Winter Meeting of the Econometric Society for their comments.

1. Introduction

Following the work of Mehra and Prescott (1985), there has been considerable empirical evidence that stock risk premia are too high relative to what would be implied by a Lucas (1978) model with a single representative agent having a constant relative risk aversion utility function, at least for "reasonable" levels of risk aversion. In the attempt of solving this "puzzle", various extensions of the basic model have been proposed, based on different preference assumptions for the representative agent or on the introduction of market frictions. Recently, several attempts have been made to explain the size of the equity premium puzzle by assuming that investors are heterogeneous, either in terms of their endowments (Constantinides and Duffie (1996)), or in terms of their information/beliefs (e.g., Abel (1990), Detemple and Murthy (1994)), or in terms of their risk aversion (Dumas (1989)). One of the motivations of this paper is to study under what circumstances heterogeneity in beliefs and risk aversion might be observationally equivalent.

The relevance and the apparent difficulty of this task were summarized by Kraus and Sick (1980, p. 335):

> Since restrictions on beliefs and restrictions on preferences have both shown themselves to be productive routes to theories of security market equilibrium, the question naturally arises as to whether one can distinguish the effects of beliefs and preferences in observed prices. It is certainly not obvious that this can ever be done. Since individual agent optimality conditions involve the product of probability and marginal utility, it may be that any set of equilibrium prices that are consistent with some combination of beliefs and preferences could also have resulted from different beliefs combined with different preferences.

Kraus and Sick approached this issue by considering a single-period model with a riskless asset, a single risky asset with a normally distributed payoff and a representative agent having constant absolute risk aversion and derived conditions under which both the mean of the risky asset's payoff and the representative agent's risk aversion coefficient can be uniquely inferred from the two assets' prices.

In this paper we consider a general dynamic model with von Neumann-Morgenstern utility functions and possibly non-Markovian price processes. By making use of martingale techniques and Girsanov's theorem, beliefs are represented in full generality as probability measures on the set of observable events. Because of this generality, it is easy to see that if the investor has a utility function for wealth or consumption at a single future date, then the problem is indeed indeterminate, in the sense that, subject to minor technical conditions, for any given utility function and consumption/investment policy, it is possible to construct a set of beliefs that would make the given policy optimal. On the other hand, if the investor draws utility from intertemporal consumption, we show in Section 3 that the set of preferences and beliefs that are consistent with a given consumption process can be characterized by a martingale condition. We formulate our dynamic model in continuous time for ease of notation, but our conclusions also apply to a discrete-time economy.

As an application of our characterization result, we consider a more general version of the "integrability problem" discussed by Wang (1993a,b) and He and Leland (1993) and relating to the possibility of recovering the representative agent's preferences from knowledge of the aggregate consumption process and the equilibrium prices. In Section 4,

we specialize our setting to a Markovian environment similar to that considered in the above papers and show that our martingale condition characterizing the set of preferences and beliefs consistent with a given consumption choice can be restated in terms of an ordinary differential equation. The problem of recovering *both* preferences and beliefs amounts to solving such a differential equation. If the beliefs are exogenously fixed, then our differential equation reduces to a condition on the equity risk premium that must be satisfied for the assumed price dynamics to be compatible with the existence of a representative agent having a von Neumann-Morgenstern utility function. This extends the result of He and Leland (1993) to a setting with intertemporal consumption and a general interest rate process.

2. The Economy

We consider a continuous-time complete-market economy on a finite time span $[0,T]$.[1]

Information structure. The uncertainty is represented by a filtered probability space $(\Omega,\mathcal{F},\mathbf{F},\bar{P})$, on which is defined an n-dimensional Brownian motion $w_{\bar{P}}$. The filtration $\mathbf{F}=\{\mathcal{F}_t\}$ is the augmentation under $\bar{P}$ of the filtration generated by $w_{\bar{P}}$. We assume that $\mathcal{F}=\mathcal{F}_T$, or that the true state of nature is completely determined by the sample paths of $w_{\bar{P}}$ on $[0,T]$.

Securities market. The investment opportunities are represented by $n+1$ long-lived securities. The first security is a locally riskless bond with price process

$$B(t,\omega)=\exp\left(\int_0^t r(\tau,\omega)\,d\tau\right). \tag{1}$$

The remaining n assets are risky. Letting $S=(S_1,\ldots,S_n)$ denote their price process and $D=(D_1,\ldots,D_n)$ their cumulative dividend process, we assume that $S+D$ is an Itô process:

$$S(t,\omega)+D(t,\omega)=S_0+\int_0^t I_S(\tau,\omega)\mu_{\bar{P}}(\tau,\omega)\,d\tau+\int_0^t I_S(\tau,\omega)\sigma(\tau,\omega)\,dw_{\bar{P}}(\tau,\omega), \tag{2}$$

where $I_S(t)$ denotes the $n\times n$ diagonal matrix with elements $S(t)$ and the diffusion matrix σ has full rank for almost all $(t,\omega)\in[0,T]\times\Omega$.

Beliefs. The investor's beliefs are represented by a probability measure P on $(\Omega,\mathcal{F})$. The only *a-priori* restriction placed on beliefs is that P be equivalent to $\bar{P}$.[2] We will denote by $\mathcal{P}$ the set of possible beliefs, i.e., the set of probability measures equivalent to $\bar{P}$.[3]

Letting

$$\eta_P(t)=\mathrm{E}_{\bar{P}}\left[\frac{dP}{d\bar{P}}\;\middle|\;\mathcal{F}_t\right] \tag{3}$$

denote the density process of a probability measure $P\in\mathcal{P}$ with respect to $\bar{P}$, it follows from the martingale representation and Girsanov's theorems that η_P has the stochastic

[1] The infinite-horizon case is identical, apart for the need to impose additional technical conditions.

[2] Two probability measures are *equivalent* if they attribute zero probability to the same events.

[3] It will become clear in the sequel that the only role played by the "reference" probability measure $\bar{P}$ is to identify the set $\mathcal{P}$ of beliefs. In particular, replacing $\bar{P}$ by any other probability measure in $\mathcal{P}$ would leave all of our results unchanged.

exponential representation

$$\eta_P(t) = \exp\Big(-\int_0^t \gamma_P(\tau)^\top dw_{\bar{P}}(\tau) - \frac{1}{2}\int_0^t |\gamma_P(\tau)|^2\, d\tau\Big)$$

for some n-dimensional process γ with with $\int_0^T |\gamma(t)|^2\, dt < \infty$ a.s., and that the process

$$w_P(t) = w_{\bar{P}}(t) + \int_0^t \gamma_P(\tau)\, d\tau$$

is a standard Brownian motion under P. We remark that under P, the gain process $S + D$ is still an Itô process, with representation

$$S(t,\omega) + D(t,\omega) = S(0) + \int_0^t I_S(\tau,\omega)\mu_P(\tau,\omega)\, d\tau + \int_0^t I_S(\tau,\omega)\sigma(\tau,\omega)\, dw_P(\tau,\omega),$$

where $\mu_P = \mu_{\bar{P}} - \sigma\gamma_P$.

Consumption Set. There is a single consumption good (the numeraire). A *consumption plan* is a strictly positive, progressively measurable consumption rate process c with $\int_0^T c(t)\, dt < \infty$ a.s..

Preferences. The investor's preferences are represented by an additive expected utility function U for intertemporal consumption,

$$U(c) = \mathrm{E}_P\Big[\int_0^T e^{-\rho t} u(c(t))\, dt\Big],$$

where E_P represents the expectation operator under P, $\rho \in \mathbb{R}$ and $u : (0,\infty)\times[0,T] \to \mathbb{R}$ is an increasing, strictly concave and three times continuously differentiable felicity function. We will denote by $\mathcal{U}$ the set of felicity functions u satisfying the stated assumptions.

3. Viable Consumption Plans

A consumption plan c^* will be said to be *viable* if there exist a set of beliefs $P \in \mathcal{P}$ and a felicity function $u \in \mathcal{U}$ such that c^* solves the following optimization problem (cf. Cox and Huang (1989) and Karatzas, Lehoczky and Shreve (1987)):

$$\max_{c\geq 0}\ \mathrm{E}_P\Big[\int_0^T e^{-\rho t} u(c(t))\, dt\Big]$$
$$\text{s.t. } \mathrm{E}_P\Big[\int_0^T \pi_P(t)c(t)\, dt\Big] = W_0,$$

where W_0 denotes the time 0 value of the investor's lifetime endowment,

$$\pi_P(t) = B(t)^{-1}\exp\Big(-\int_0^t \kappa_P(\tau)^\top dw_P(\tau) - \frac{1}{2}\int_0^t |\kappa_P(\tau)|^2\, d\tau\Big), \tag{4}$$

and $\kappa_P = \sigma^{-1}(\mu_P - r\bar{1})$. The process π_P identifies the state-price density with respect to the investor's beliefs P. This immediately implies that, if a consumption plan c^* is optimal for a pair (u, P) of preferences and beliefs, then it must satisfy the first-order condition

$$e^{-\rho t} u_c(c^*(t)) = \psi\pi_P(t) \tag{5}$$

for some Lagrangian multiplier $\psi > 0$. Defining the state-price density with respect to the "reference" probability measure $\bar{P}$ by

$$\pi_{\bar{P}} = B(t)^{-1} \exp\Big(- \int_0^t \kappa_{\bar{P}}(\tau)^\top d\bar{w}_{\bar{P}}(\tau) - \frac{1}{2} \int_0^t |\kappa_{\bar{P}}(\tau)|^2 \, d\tau \Big),$$

it can be easily verified that $\pi_P = \eta_P^{-1} \pi_{\bar{P}}$, where η_P is the process in (3). Thus, we can rewrite (5) as $e^{\rho t} \pi_{\bar{P}}(t) / u_c(c^*(t)) = \eta_P(t)/\psi$, and we have the following characterization of viable consumption plans.

Theorem 1. *Let c^* be a consumption plan. Then c^* is optimal for an investor with felicity function $u \in \mathcal{U}$, beliefs $P \in \mathcal{P}$ and endowment*

$$W_0 = \mathrm{E}_{\bar{P}} \Big[\int_0^T \pi_{\bar{P}}(t) c^*(t) \, dt \Big]$$

if and only if the process $e^{\rho t} \pi_{\bar{P}}(t) / u_c(c^*(t))$ is a $\bar{P}$-martingale, in which case

$$\frac{dP}{d\bar{P}} = \frac{u_c(c^*(0)) e^{\rho T} \pi_{\bar{P}}(T)}{u_c(c^*(T))}.$$

Proof. Necessity is immediate from the above discussion, the fact that η_P is a $\bar{P}$-martingale, and the fact that

$$\mathrm{E}_{\bar{P}} \Big[\int_0^T \pi_{\bar{P}}(t) c^*(t) \, dt \Big] = \mathrm{E}_{\bar{P}} \Big[\int_0^T \eta_P(t) \pi_P(t) c^*(t) \, dt \Big] = \mathrm{E}_P \Big[\int_0^T \pi_P(t) c^*(t) \, dt \Big].$$

Conversely, if $e^{\rho t} \pi_{\bar{P}} / u_c(c^*(t))$ is a $\bar{P}$-martingale, then

$$\eta_P(t) = \mathrm{E}_{\bar{P}} \Big[\frac{dP}{d\bar{P}} \;\Big|\; \mathcal{F}_t \Big] = \mathrm{E}_{\bar{P}} \Big[\frac{u_c(c^*(0)) e^{\rho T} \pi_{\bar{P}}(T)}{u_c(c^*(T))} \;\Big|\; \mathcal{F}_t \Big] = \frac{u_c(c^*(0)) e^{\rho t} \pi_{\bar{P}}(t)}{u_c(c^*(t))},$$

so that (5) is satisfied and c^* is optimal. □

While in the case of preferences for terminal wealth only it is easy to show that any consumption plan is viable (in fact, given any felicity function satisfying minor technical conditions, it is always possible to construct a set of beliefs that would make the given policy optimal for that particular felicity function), Theorem 1 shows that a similar result does not apply to the case of preferences for intertemporal consumption. A given consumption plan is not necessarily viable, as the above martingale condition is not necessarily satisfied for some $u \in \mathcal{U}$.

4. The Recoverability Problem

One application of the above result relates to the literature on the *recoverability* (or *integrability*) problem. This line of research is concerned with the possibility of recovering preferences of the investor by observing consumption/investment decisions. Preferences are assumed to have an expected utility representation and beliefs are treated as objective and

hence assumed to be known. Wang (1993a) studied the recoverability problem in a time-homogeneous model with intertemporal consumption and showed that the representative agent's felicity function can be recovered (although not uniquely) under mild technical conditions. On the other hand, this recoverability result was based on the assumption that the only traded asset was a claim to aggregate consumption and thus ignored the information contained in the interest rate process. In a discrete-time setting with a Markov output growth rate process and two traded assets (a bond and a claim to aggregate consumption), Wang (1993b) showed that recoverability is *generically* possible within the larger class of preferences introduced by Kreps and Porteus (1978). Finally, He and Leland (1993) considered a setting with a constant interest rate, a single risky asset which pays no dividends and follows a diffusion, and a representative agent that maximizes utility from terminal wealth. They showed that the representative agent's utility function can be recovered from knowledge of the equity risk premium at the terminal date and obtained a partial differential equation that must be satisfied by the relative risk premium for it to be consistent with the representative agent optimally holding the risky asset (see also Bick (1990)).

We investigate next to what extent simultaneous recovery of tastes and beliefs is possible. As in Wang (1993a), we consider a simple continuous-time time-homogeneous Lucas economy with a single representative agent. However, we assume that two assets are traded: a locally riskless bond and a risky asset. The supply of the risky asset is normalized to one share, while the riskless asset is taken to be in zero net supply. The representative agent is endowed with the share of the risky asset at time 0. The risky asset pays a strictly positive dividend rate process y, where

$$y(t) = y_0 + \int_0^t m_{\bar{P}}(y(\tau))y(\tau)\,d\tau + \int_0^t s(y(\tau))y(\tau)\,dw_{\bar{P}}(\tau) \tag{6}$$

for some continuous functions $m_{\bar{P}}, s : (0,\infty) \to \mathbb{R}$. We assume that for all $t \in (0,T)$ the random variable $y(t)$ has support $(0,\infty)$ and focus on Markovian equilibria in which

$$B(t) = \exp\left(\int_0^t \hat{r}(y(\tau))\,d\tau\right) \tag{7}$$

and

$$S(t) + \int_0^t y(\tau)\,d\tau = S_0 + \int_0^t \hat{\mu}_{\bar{P}}(y(\tau))S(\tau)\,d\tau + \int_0^t \hat{\sigma}(y(\tau))S(\tau)\,dw_{\bar{P}}(\tau) \tag{8}$$

for some continuous functions $\hat{r}, \hat{\mu}_{\bar{P}}, \hat{\sigma} : (0,\infty) \to \mathbb{R}$ with $\hat{\sigma}(y) \neq 0$ for all $y \in (0,\infty)$.[4] Clearly, (7) and (8) are a special case of (1) and (2), respectively. The equilibrium condition for this economy is that the representative agent optimally holds the share of the risky asset, consuming the dividend y. The following theorem completely characterizes the set of tastes and beliefs $(u,P) \in \mathcal{U} \times \mathcal{P}$ for the representative agent that are consistent with an equilibrium.

Theorem 2. *A pair $(u,P) \in \mathcal{U} \times \mathcal{P}$ of tastes and beliefs for the representative agent is consistent with an equilibrium for the above economy if and only if*

$$u(c) = \psi_1 + \psi_2 \int_{y_0}^{c} \exp\left(-\int_{y_0}^{x} \frac{R(y)}{y}\,dy\right)dx \tag{9}$$

[4] The case considered by Wang (1993a), in which the price of the risky asset at every time t is a smooth deterministic function of the contemporaneous dividend $y(t)$, is a special case of the setting we consider.

and

$$\frac{dP}{d\bar{P}} = \exp\Big(\int_0^T \hat{\gamma}_P(y(t))\, dw_{\bar{P}}(t) - \frac{1}{2}\int_0^T \hat{\gamma}_P(y(t))^2\, dt\Big), \tag{10}$$

where $\psi_1 \in \mathbb{R}$ and $\psi_2 > 0$ are arbitrary constants that do not affect preferences,

$$\begin{aligned} \hat{\gamma}_P(y) &= \hat{\kappa}_{\bar{P}}(y) - s(y)R(y), \\ \hat{\kappa}_{\bar{P}}(y) &= \Big(\hat{\mu}_{\bar{P}}(y) - \hat{r}(y)\Big)/\hat{\sigma}(y) \end{aligned} \tag{11}$$

and R is a strictly positive and continuously differentiable solution of the Riccati differential equation

$$\frac{s(y)^2}{2}\Big(yR'(y) + R(y)^2\Big) + \Big(m_{\bar{P}}(y) - \hat{\kappa}_{\bar{P}}(y)s(y) - \frac{s(y)^2}{2}\Big)R(y) - \hat{r}(y) + \rho = 0 \tag{12}$$

that satisfies the integrability condition

$$\mathrm{E}_{\bar{P}}\Big[\exp\Big(\int_0^T \Big(\hat{\kappa}_{\bar{P}}(y(t)) - s(y(t))R(y(t))\Big)dw_{\bar{P}}(t) - \frac{1}{2}\int_0^T \Big(\hat{\kappa}_{\bar{P}}(y(t)) - s(y(t))R(y(t))\Big)^2 dt\Big)\Big] = 1.$$

Proof. Suppose first that $(u, P) \in \mathcal{U} \times \mathcal{P}$ is a pair of preferences and beliefs for the representative agent that is consistent with an equilibrium. Let $\psi_1 = u(y_0)$, $\psi_2 = u_c(y_0)$, $v(c) = \log(u_c(c)/\psi_2)$ and $R(y) = -yv'(y)$. Then

$$v(c) = -\int_{y_0}^c \frac{R(y)}{y}\, dy$$

and

$$u_c(c) = \psi_2 \exp(v(c)) = \psi_2 \exp\Big(-\int_{y_0}^c \frac{R(y)}{y}\, dy\Big). \tag{13}$$

In analogy with (4), let

$$\pi_{\bar{P}}(t) = B(t)^{-1} \exp\Big(-\int_0^t \hat{\kappa}_{\bar{P}}(\tau)^\top dw_{\bar{P}}(\tau) - \frac{1}{2}\int_0^t \hat{\kappa}_{\bar{P}}(\tau)^2\, d\tau\Big). \tag{14}$$

By Theorem 1, the pair (u, P) is consistent with an equilibrium if and only if the process $e^{\rho t}\pi_{\bar{P}}(t)/u_c(y(t))$ is a $\bar{P}$-martingale, in which case

$$\frac{dP}{d\bar{P}} = \frac{\psi_2 e^{\rho T}\pi_{\bar{P}}(T)}{u_c(y(T))}. \tag{15}$$

By the martingale representation theorem, the above condition amounts to the existence of a process γ_P such that

$$\frac{\psi_2 e^{\rho t}\pi_{\bar{P}}(t)}{u_c(y(t))} = \exp\Big(-\int_0^t \gamma_P(\tau)\, dw_{\bar{P}}(\tau) - \frac{1}{2}\int_0^t \gamma_P(\tau)^2\, d\tau\Big), \tag{16}$$

or, equivalently (using (13) and (14) and taking logs),

$$\begin{aligned} -\int_0^t \hat{\kappa}_{\bar{P}}(y(\tau))\, dw_{\bar{P}}(\tau) - \int_0^t \Big(\hat{r}(y(\tau)) - \rho + \frac{1}{2}\hat{\kappa}_{\bar{P}}(y(\tau))^2\Big)\, d\tau - v(y(t)) \\ = -\int_0^t \gamma_P(\tau)\, dw_{\bar{P}}(\tau) - \frac{1}{2}\int_0^t \gamma_P(\tau)^2\, d\tau. \end{aligned}$$

Also, by Itô's lemma,

$$\begin{aligned} v(y(t)) &= \int_0^t s(y(\tau))y(\tau)v'(y(\tau))\,dw_{\bar{P}}(\tau) \\ &\quad + \int_0^t \left(\frac{s(y(\tau))^2}{2}\, y(\tau)^2 v''(y(\tau)) + m_{\bar{P}}(y(\tau))y(\tau)v'(y(\tau)) \right) d\tau \\ &= -\int_0^t s(y(\tau))R(y(\tau))\,dw_{\bar{P}}(\tau) \\ &\quad - \int_0^t \left(\frac{s(y(\tau),\tau)^2}{2} \Big(y(\tau)R'(y(\tau)) - R(y(\tau)) \Big) + m_{\bar{P}}(y(\tau))R(y(\tau)) \right) d\tau. \end{aligned}$$

The last two equations imply $\gamma_P(t) = \hat{\gamma}_P(y(t))$, where

$$\hat{\gamma}_P(y) = \hat{\kappa}_{\bar{P}}(y) - s(y)R(y)$$

and

$$\begin{aligned} &\frac{s(y)^2}{2}\Big(yR'(y) - R(y) \Big) + m_{\bar{P}}(y)R(y) - \hat{r}(y) + \rho \\ &= \frac{1}{2}\Big(\hat{\kappa}_{\bar{P}}(y)^2 - \hat{\gamma}_P(y)^2 \Big) \;=\; \hat{\kappa}_{\bar{P}}(y)s(y)R(y) - \frac{s(y,t)^2}{2}\, R(y,t)^2. \end{aligned}$$

The strict positivity of R follows from the definition and the fact that $u \in \mathcal{U}$, while the integrability condition follows from (16) and the martingale property of $e^{\rho t}\pi_{\bar{P}}(t)/u_c(y(t))$.

The proof of sufficiency is similar. □

Theorem 2 shows that recovering the set of pairs (u, P) consistent with the aggregate consumption process y in (6) and the equilibrium price process (B, S) in (7)–(8) amounts to determining the set of strictly positive solutions R to the ordinary differential equation (12) which in addition satisfy the integrability condition. To each such solution R corresponds a unique (up to a linear transformation that does not affect preferences) utility function $u \in \mathcal{U}$ through (9) and a unique set of beliefs $P \in \mathcal{P}$ through (10) that are consistent with an equilibrium. We remark that the function R corresponds to the Arrow-Pratt coefficient of relative risk aversion associated with u. The following proposition establishes the existence of a strictly positive solution to the Riccati equation (12) over any interval $[\epsilon, \infty)$ with $\epsilon > 0$.

Proposition 1. *Let $\epsilon > 0$ and suppose that $s(y) \neq 0$ and $\rho < \hat{r}(y)$ for all $y \in [\epsilon, \infty)$. Then the Riccati equation (12) with initial condition $R(\epsilon) = R_\epsilon > 0$ has a unique strictly positive and continuously differentiable solution on $[\epsilon, \infty)$.*

Proof. The claim follows from Theorems IV.5.2 (existence and strict positivity) and IV.5.1 and IV.4.4 (uniqueness) in Fleming and Rishel (1975). □

The set of solutions R to the ODE (12) is, as we would expect, unaffected by the particular probability measure $\bar{P}$ under which the functions $m_{\bar{P}}$ and $\hat{\kappa}_{\bar{P}}$ are determined, as we have

$$m_P + \hat{\kappa}_P s = m_{\bar{P}} + \hat{\gamma}_P s - \frac{\hat{\mu}_{\bar{P}} + \hat{\gamma}_P \sigma - \hat{r}}{\sigma} s = m_{\bar{P}} + \hat{\kappa}_{\bar{P}} s$$

for all $P \in \mathcal{P}$.

In the case of objective beliefs ($P = \bar{P}$), we have $\hat{\gamma}_P \equiv 0$, and thus, assuming that $s(y) \neq 0$ for all $y \in (0, \infty)$, (11) implies $R(y) = \hat{\kappa}_{\bar{P}}(y)/s(y)$, so that the representative agent's utility function is uniquely identified (up to a linear transformation), provided that $\hat{\kappa}_{\bar{P}}(y)/s(y) > 0$ for all $y \in (0, \infty)$. The ODE (12) then becomes a condition that must be satisfied by the coefficients of the price process for them to be consistent with an equilibrium with a single representative agent with a felicity function in $\mathcal{U}$. This generalizes the results of Bick (1990) and He and Leland (1993) to allow for intertemporal consumption and a nonconstant interest rate process.

The following example illustrates our approach and clarifies the implications of subjective rather than objective beliefs.

Example. Suppose that $\hat{\kappa}_{\bar{P}}$ and s are constant, with $s \neq 0$ and $\hat{\kappa}_{\bar{P}}/s > 0$. In the case of objective beliefs, (11) implies a constant coefficient of relative risk aversion, $R = \hat{\kappa}_{\bar{P}}/s$, which is consistent with the CCAPM. Equation (12) then becomes a restriction on the price coefficients,

$$\hat{r}(y) - \rho = m_{\bar{P}}(y)\,\frac{\hat{\kappa}_{\bar{P}}}{s} - \frac{1}{2}\,\hat{\kappa}_{\bar{P}} s \left(\frac{\hat{\kappa}_{\bar{P}}}{s} + 1\right), \tag{17}$$

which restates the well known relation between the interest rate and the rate of change in the marginal utility of the representative agent:

$$\hat{r}(y) - \rho = -\,\frac{m_{\bar{P}}(y) y u_{cc}(y) + \frac{1}{2} s^2 y^2 u_{ccc}(y)}{u_c(y)}.$$

Allowing for subjective beliefs, (11) gives

$$\hat{\gamma}_P(y) = \hat{\kappa}_{\bar{P}} - sR(y),$$

where R is any strictly positive solution of the ODE (12).

In particular, if $m_{\bar{P}}$ and $\hat{r} > \rho$ are constant, then the solution of (12) with an arbitrary initial condition $R(\epsilon) = R_\epsilon > 0$ for some $\epsilon > 0$ is given by

$$R(y) = \frac{\alpha(\alpha s^2 R_\epsilon + 2r) y^\beta + 2r(R_\epsilon - \alpha)\epsilon^\beta}{(\alpha s^2 R_\epsilon + 2r) y^\beta - \alpha s^2 (R_\epsilon - \alpha)\epsilon^\beta}, \tag{18}$$

where

$$\alpha = \frac{\frac{s^2}{2} - m_{\bar{P}} - \hat{\kappa}_{\bar{P}} s + \sqrt{\left(\frac{s^2}{2} - m_{\bar{P}} - \hat{\kappa}_{\bar{P}} s\right)^2 + 2s^2(r - \rho)}}{s^2} > 0$$

and

$$\beta = 2\,\frac{\sqrt{\left(\frac{s^2}{2} - m_{\bar{P}} - \hat{\kappa}_{\bar{P}} s\right)^2 + 2s^2(r - \rho)}}{s^2} > 0.$$

Taking $R_\epsilon = \alpha$, (18) gives $R(y) = \alpha$ for all y, which is the only solution that is defined and strictly positive on $(0, \infty)$.[5] Thus, both preferences and beliefs can be uniquely recovered in this case. Moreover, $\alpha = \hat{\kappa}_{\bar{P}}/s$ if and only if (17) is satisfied. □

[5] As implied by Proposition 1, all solutions are defined and strictly positive on $[\epsilon, \infty)$.

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